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Regularly and Strongly Decaying Solutions for Quasilinear Dynamic Equations[1]

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Abstract

We consider quasilinear dynamic equations whose solutions are classified into disjoint subsets by certain integral conditions. In particular, we investigate the asymptotic behavior of solutions approaching zero for quasilinear dynamic equations.

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1. Introduction and Preliminary Results

The theory of time scales, $\mathbb{T}$ (nonempty closed subsets of real numbers) was initiated by Stefan Hilger in his PhD dissertation [12] in 1988 in order to unify continuous and discrete analysis. The theory of dynamic equations on time scales helps us not only to avoid proving results twice but also to extend them for other time scales such as the set of all integer multiples of a number $h > 0$, the set of all integer powers of a number $q > 1$. We refer readers the books by Bohner and Peterson [4, 5] for an excellent introduction with applications and advances in dynamic equations.

In this paper, we consider a quasilinear dynamic equation

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)f(x^\sigma), \tag{1.1}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$, $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$, and a and b are real positive rd-continuous functions on $\mathbb{T}$. Throughout this paper

[1]Dedicated to my advisor, Professor Allan C. Peterson on the occasion of his 65th birthday.
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we assume that $\mathbb{T}$ is unbounded above. By a *solution* we mean a delta-differentiable function x satisfying equation (1.1) such that the delta derivative of $a\Phi_p(x^\Delta)$ is rd-continuous.

Such studies are essentially motivated by the dynamics of positive radial solutions of reaction-diffusion (flow through porous media, nonlinear elasticity) problems modelled by the nonlinear elliptic equation

$$-\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \lambda f(u) = 0, \tag{1.2}$$

where $\alpha : (0,\infty) \mapsto (0,\infty)$ is continuous and such that $\delta(v) := \alpha(|v|)v$ is an odd increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, λ is a positive constant (the Thiele modulus) and f presents the ratio of the reaction rate at concentration u to the reaction rate at concentration unity, see Diaz [10] and Grossinho and Omari [11]. If $\alpha(|v|) = |v|^{p-2}$, then the differential operator in equation (1.2) is the one dimensional analogue of the p-Laplacian $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and equation (1.2) leads to the quasilinear differential equation (i.e., when $\mathbb{T} = \mathbb{R}$)

$$[a(t)\Phi_p(x')]' = b(t)f(x).$$

It is shown in the paper by Akın-Bohner [2, Lemma 3.1] that any nontrivial solution of (1.1) is eventually monotone and belongs to one of the two classes:

$$\begin{aligned} M^+ &:= \{x \in S : \text{ there exists } T \geq t_0 \text{ such that } x(t)x^\Delta(t) > 0 \text{ for } t \geq T\} \\ M^- &:= \{x \in S : x(t)x^\Delta(t) < 0 \text{ on } [t_0,\infty)\}, \end{aligned}$$

where S is the set of nontrivial solutions of equation (1.1) on $[t_0,\infty)$. Asymptotic behavior of solutions in M^+ and M^- on $\mathbb{T}$ is considered in the papers by Akın-Bohner [2, 3], respectively.

The subclasses

$$\begin{aligned} M_B^- &= \{x \in M^- : \lim_{t\to\infty} x(t) = l \neq 0\}, \\ M_0^- &= \{x \in M^- : \lim_{t\to\infty} x(t) = 0\} \end{aligned}$$

are characterized by certain integrals

$$\begin{aligned} Y_1 &= \lim_{T\to\infty} \int_{t_0}^T \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t_0}^t b(s)\Delta s\right) \Delta t, \\ Y_2 &= \lim_{T\to\infty} \int_{t_0}^T \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_t^T b(s)\Delta s\right) \Delta t, \\ Y_3 &= \lim_{T\to\infty} \int_{t_0}^T \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Delta t, \\ Y_4 &= \lim_{T\to\infty} \int_{t_0}^T b(t))\Delta t, \end{aligned}$$

where Φ_{p^*} is the inverse of the map Φ_p, i.e., $\Phi_p(\Phi_{p^*}(u)) = \Phi_{p^*}(\Phi_p(u)) = u$. Then $\Phi_{p^*}(u) = |u|^{p^*-2}u$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. The following results are shown in [2, Lemma 3.2, Theorem 4.1 and Theorem 4.2].

Lemma 1.1.

(i) If $Y_1 < \infty$, then $Y_3 < \infty$.

(ii) If $Y_2 < \infty$, then $Y_4 < \infty$.

(iii) If $Y_1 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.

(iv) If $Y_2 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.

(v) $Y_1 < \infty$ and $Y_2 < \infty$ if and only if $Y_3 < \infty$ and $Y_4 < \infty$.

Theorem 1.2.

(i) $Y_2 = \infty$ if and only if $M_B^- = \emptyset$.

(ii) $Y_1 < \infty$ and $Y_2 < \infty$ implies $M_0^- \neq \emptyset$ and $M_B^- \neq \emptyset$, i.e., $M^- \neq \emptyset$.

Remark 1.3. When $\mathbb{T} = \mathbb{Z}$, it is shown in [7] that equation (1.1) has at least one solution in M^-. However, M^- can be empty when $\mathbb{T} = \mathbb{R}$, see [13].

Let $q > 1$ and consider the integral

$$Y_q = \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Phi_q\left(\int_{\sigma(t)}^{T} \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s\right)\Delta t.$$

The following lemma can be shown as Lemma 1.1.

Lemma 1.4.

(i) If $Y_q < \infty$, then $Y_3 < \infty$.

(ii) If $Y_q = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.

(iii) $Y_q < \infty$ and $Y_2 < \infty$ if and only if $Y_3 < \infty$ and $Y_4 < \infty$.

Since the quasiderivative

$$x^{[1]}(t) := a(t)\Phi_p(x^{\Delta}(t)) \tag{1.3}$$

of any solution $x \in M^-$ is either positive decreasing or negative increasing, $\lim_{t\to\infty} x^{[1]}(t)$ must be finite. Hence M_0^- can be divided into

$$M_{0R}^- = \left\{x \in M^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = c \neq 0\right\}$$
$$M_{0S}^- = \left\{x \in M^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = 0\right\}.$$

Solutions in M_{0R}^- and M_{0S}^- are called *regularly decaying solutions* and *strongly decaying solutions*, respectively. Clearly,

$$M^- = M_B^- \cup M_0^- = M_B^- \cup M_{0R}^- \cup M_{0S}^-.$$

The following is also shown in the paper by Akın-Bohner [2, Lemma 4.1].

Lemma 1.5. $Y_3 = \infty$ implies for any $x \in M^-$

$$\lim_{t\to\infty} x^{[1]}(t) = 0.$$

The purpose of this paper is to consider the asymptotic behavior of solutions in M_{0R}^- and M_{0S}^-. In the last section, we give necessary conditions to obtain the existence and uniqueness of solutions of boundary value problems in M_0^-. See Cecchi, Došlá and Marini [6] when $\mathbb{T} = \mathbb{R}$ and [7–9] when $\mathbb{T} = \mathbb{Z}$.

2. Limit Behavior

The next two results give conditions ensuring that M_{0R}^- and M_{0S}^- are empty.

Theorem 2.1. $M_{0R}^- = \emptyset$ if any of the following conditions holds:

(i) $Y_3 = \infty$;

(ii) there exists $q > 1$ such that

$$\liminf_{u\to 0} \frac{f(u)}{\Phi_q(u)} > 0 \tag{2.1}$$

and $Y_q = \infty$.

Proof. (i) follows from Lemma 1.5. To prove (ii) we assume that $Y_3 < \infty$. Otherwise, the assertion follows from (i). Without loss of generality, we assume x is a positive solution of (1.1) in M_{0R}^-, i.e., $x(t) \to 0$ and $x^{[1]}(t) \to c \neq 0$ as $t \to \infty$. Then there exists $l > 0$ such that $x^{[1]}(t) < -l$ for large t. Therefore, $x(t) \geq \Phi_{p^*}(l) \int_t^\infty \Phi_{p^*}\left(\frac{1}{a(\tau)}\right) \Delta\tau$. By (2.1), there exists a positive constant h such that for all $u \in (0, x(t_0)]$ we have

$$f(u) \geq h\Phi_q(u).$$

This implies that

$$f(x^\sigma(t)) \geq h\Phi_q\left(\Phi_{p^*}(l)\int_{\sigma(t)}^{\infty}\Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s\right)$$

and so

$$(x^{[1]})^\Delta(t) \geq h_1 b(t)\Phi_q\left(\int_{\sigma(t)}^{\infty}\Phi_{p*}\left(\frac{1}{a(s)}\right)\Delta s\right),$$

where $h_1 = h\Phi_q(\Phi_{p*}(l))$. By integrating from t to ∞ we have

$$-x^{[1]}(t) \geq h_1\int_t^\infty b(\tau)\Phi_q\left(\int_{\sigma(\tau)}^{\infty}\Phi_{p*}\left(\frac{1}{a(s)}\right)\Delta s\right)\Delta\tau.$$

But this contradicts to the boundedness of $x^{[1]}$. ■

Theorem 2.2. $M_{0S}^- = \emptyset$ if any of the following conditions holds:

(i) $Y_2 < \infty$ and

$$\limsup_{u\to 0}\frac{f(u)}{\Phi_p(u)} < \infty; \tag{2.2}$$

(ii) there exists $q \geq p$ such that

$$\limsup_{u\to 0}\frac{f(u)}{\Phi_q(u)} < \infty \tag{2.3}$$

and $Y_q < \infty$.

Proof. Let $x \in M_{0S}^-$ and without loss of generality, we assume that $x > 0$. (2.2) implies that there exists a positive constant H_1 such that for all $u \in (0, x(t_0)]$ we have

$$f(u) \leq H_1\Phi_p(u). \tag{2.4}$$

Integrating equation (1.1) twice from t to ∞ we obtain

$$\begin{aligned} x(t) &= \int_t^\infty \Phi_{p^*}\left(\frac{1}{a(\tau)}\int_\tau^\infty b(s)f(x^\sigma(s))\Delta s\right)\Delta\tau \\ &\leq H_1\int_t^\infty \Phi_{p*}\left(\frac{1}{a(\tau)}\int_\tau^\infty b(s)\Phi_p(x^\sigma(s))\Delta s\right)\Delta\tau \\ &\leq H_1\Phi_{p*}\left(\Phi_p(x^\sigma(t))\right)\int_t^\infty \Phi_{p^*}\left(\int_\tau^\infty \frac{b(s)}{a(\tau)}\Delta s\right)\Delta\tau, \end{aligned}$$

where we used (2.4) as well. But

$$1 < \frac{x(t)}{x^\sigma(t)} \leq H_1\int_t^\infty \Phi_{p*}\left(\int_\tau^\infty \frac{b(s)}{a(\tau)}\Delta s\right)\Delta\tau,$$

and we obtain a contradiction as $t \to \infty$. To prove (ii), we let $x \in M_{0S}^-$ and without loss of generality, we assume that $x > 0$. From (1.3), we have

$$\Phi_q(x^\sigma(t)) \leq \Phi_q\left(\Phi_{p^*}(-x^{[1]}(t))\right)\Phi_q\left(\int_{\sigma(t)}^\infty \Phi_{p^*}\left(\frac{1}{a(\tau)}\right)\Delta\tau\right).$$

From (2.3), there exists a positive constant H_2 such that for all $u \in (0, x(t_0)]$ we have

$$f(u) \leq H_2\Phi_q(u).$$

Therefore, we obtain that

$$(x^{[1]})^\Delta(t) \leq H_2 b(t)\Phi_q\left(\Phi_{p*}(-x^{[1]}(t))\right)\Phi_q\left(\int_{\sigma(t)}^\infty \Phi_{p*}\left(\frac{1}{a(\tau)}\right)\Delta\tau\right).$$

Integrating the above inequality from t to ∞ we have

$$\frac{-x^{[1]}(t)}{\Phi_q(\Phi_{p*}(-x^{[1]}(t)))} \leq H_2\int_t^\infty b(\tau)\Phi_q\left(\int_{\sigma(\tau)}^\infty \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s\right)\Delta\tau.$$

Since $Y_q < \infty$ and

$$\frac{u}{\Phi_q(\Phi_{p*}(u))} = \frac{1}{u^{\frac{q-1}{p-1}}},$$

we get a contradiction to the assumption that $x \in M_{0S}^-$. ■

The next two results give conditions ensuring that the classes M_{0R}^- and M_{0S}^- are not empty.

Theorem 2.3. $M_{0R}^- \neq \emptyset$ if any of following conditions holds.

(i) $Y_2 < \infty$, $Y_q < \infty$ and (2.2) holds;

(ii) there exists $q > 1$ such that (2.3) holds and $Y_q < \infty$.

Proof. (i) follows from Lemma 1.1 and 1.4, Theorem 1.2 (ii), and Theorem 2.2 (i). To prove (ii) we observe that (2.3) implies the existence of a positive constant M such that for all $u \in (0, 1]$

$$f(u) \leq M\Phi_q(u). \tag{2.5}$$

By Lemma 1.4, $Y_q < \infty$ implies $Y_3 < \infty$. Choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^\infty b(\tau)\Phi_q\left(\int_{\sigma(\tau)}^\infty \Phi_{p*}\left(\frac{1}{a(s)}\right)\Delta s\right)\Delta\tau < \frac{1}{2M}. \tag{2.6}$$

Define X to be the Banach space of all continuous functions on $[t_1, \infty)$ and endowed with the topology of the supremum norm. Let Ω be the nonempty subset of X given by

$$\Omega = \left\{u \in X : \int_t^\infty \Phi_{p^*}\left(\frac{1}{2a(\tau)}\right)\Delta\tau \leq u(t) \leq \int_t^\infty \Phi_{p*}\left(\frac{1}{a(\tau)}\right)\Delta\tau\right\}.$$

Clearly, Ω is bounded, closed and convex. Now we consider the operator $T : \Omega \mapsto X$ which assigns to any $u \in \Omega$

$$Tu(t) = \int_t^\infty \Phi_{p*}\left(\frac{1}{a(\tau)}\left(\frac{1}{2} + \int_\tau^\infty b(s)f(u^\sigma(s))\Delta s\right)\right)\Delta\tau \geq \int_t^\infty \frac{1}{\Phi_{p*}(2a(\tau))}\Delta\tau.$$

By (2.5) and (2.6), we obtain

$$\begin{aligned}\int_{t_1}^\infty b(\tau)f(u^\sigma(\tau))\Delta\tau &\leq \int_{t_1}^\infty Mb(\tau)\Phi_q(u^\sigma(\tau))\Delta\tau \\ &\leq M\int_{t_1}^\infty b(\tau)\Phi_q\left(\int_{\sigma(\tau)}^\infty \Phi_{p*}\left(\frac{1}{a(s)}\right)\Delta s\right)\Delta\tau \\ &< \frac{1}{2}.\end{aligned}$$

Therefore,

$$Tu(t) \leq \int_t^\infty \Phi_{p*}\left(\frac{1}{a(\tau)}\right)\Delta\tau$$

and so $T(\Omega) \subset \Omega$. To complete the proof, it is enough to show the relative compactness of $T(\Omega)$, the continuity of T in Ω to apply the Schauder fixed point theorem. The details are left to the reader. ■

Remark 2.4. When $\mathbb{T} = \mathbb{Z}$, (2.2) in Theorem 2.3 (i) is not assumed to show that $M_{0R}^- \neq \emptyset$, see [8].

Theorem 2.5. Every solution in M^- is in M_{0S}^- if $Y_2 = Y_q = \infty$ and there exists $q > 1$ such that (2.1) holds.

Proof. We use Theorem 1.2 (i) and Theorem 2.1 (ii). ■

3. Boundary Value Problems in M_0^-

In this section, we assume that f is nondecreasing and consider two boundary problems in M_0^-. The existence and uniqueness of solutions of boundary value problems are shown in [9] for half-linear difference equations, where $f(x) = \Phi_p(x)$. We now start with the following lemma which is essential for the last two theorems.

Lemma 3.1. If x and y are solutions of equation (1.1) such that

$$x(t_0) \geq y(t_0) > 0, \qquad x^\Delta(t_0) > y^\Delta(t_0),$$

then for $t > t_0 \in \mathbb{T}$

$$x(t) > y(t), \qquad x^\Delta(t) > y^\Delta(t).$$

Proof. Let x and y be two differentiable solutions of equation (1.1) such that

$$x(t_0) \geq y(t_0) > 0 \quad \text{and} \quad x^\Delta(t_0) > y^\Delta(t_0).$$

Set $d(t) := x(t) - y(t)$. Then we have

$$d(t_0) \geq 0, \quad d^{\Delta}(t_0) > 0.$$

It is enough to show that d does not have a positive maximum. Assume not, then there exists $t_1 \geq t_0$ such that

$$d(t_1) = \max\{d(t) : t \in [t_0, \infty)\} > 0$$

and

$$d(t) < d(t_1) \text{ for } t > t_1.$$

One can show as in the proof of [1, Theorem 5] that t_1 cannot be left-dense and right-scattered at the same time and for other cases we obtain

$$d^{\Delta}(t_1) \leq 0, \quad d^{\Delta}(\rho(t_1)) \leq 0.$$

Define

$$G(t) = a(t)\left[\Phi_p(y^{\Delta}(t)) - \Phi_p(x^{\Delta}(t))\right].$$

Monotonicity of f and the positivity of a and b implies

$$G^{\Delta}(t) = b(t)\left[f(y^{\sigma}(t)) - f(x^{\sigma}(t))\right].$$

Since $d(t_1) > 0$, $G^{\Delta}(\rho(t_1)) < 0$.

Case 1: $\rho(t_1) < t_1 < \sigma(t_1)$ and $\rho(t_1) < t_1 = \sigma(t_1)$. In this case, we obtain $G(t_1) \geq 0$ and $G(\rho(t_1)) < 0$. Therefore

$$G^{\sigma}(\rho(t_1)) = G(\rho(t_1)) + \mu(\rho(t_1))G^{\Delta}(\rho(t_1)) < 0.$$

Since t_1 cannot be left-dense and right-scattered, $\sigma(\rho(t_1)) = t_1$. Therefore, this gives us a contradiction.

Case 2: $\rho(t_1) = t_1 = \sigma(t_1)$. In this case, $G^{\Delta}(t_1) < 0$ and $G(t_1) = 0$. This implies that $G(t) < 0$ in a right neighborhood I of t_1, i.e., $d^{\Delta}(t) > 0$ for $t \in I$, which gives us a contradiction.

Therefore, d does not have a positive maximum. This completes the proof. ■

Theorem 3.2. Assume conditions in Theorem 2.3 (i) or (ii). Then for any $\nu \neq 0$ the boundary value problem of equation (1.1) with

$$x(t_0) = \nu, \quad x \in M_{0R}^{-} \tag{3.1}$$

has a unique solution.

Proof. The existence of solutions of boundary value problem (1.1) with (3.1) for any $\nu \neq 0$ follows from Theorem 2.3. Without loss of generality, assume that there exist solutions x and y of the boundary value problem of equation (1.1) with $x(t_0) = y(t_0) =$

$\nu > 0$ and $y^{\Delta}(t_0) < x^{\Delta}(t_0)$. By the proof of Lemma 3.1, $d(t) > 0$ and is increasing for $t > t_0$. But this contradicts the fact that $\lim_{t\to\infty} d(t) = 0$. ■

Theorem 3.3. Assume $Y_1 < \infty$ and $Y_2 < \infty$. Then for any $\nu \neq 0$ the boundary value problem of equation (1.1) with

$$x(t_0) = \nu, \quad x \in M_0^- \tag{3.2}$$

has a unique solution.

Proof. The existence of solutions of boundary value problem (1.1) with (3.2) for any $\nu \neq 0$ follows from Theorem 1.2. The uniqueness can be shown as in the proof of Theorem 3.2. ■

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