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RECESSIVE SOLUTIONS FOR HALF-LINEAR DYNAMIC EQUATIONS

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Abstract

We study recessive solutions of nonoscillatory half-linear dynamic equations. Recessive solutions are characterized using limit and integral properties.

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1 Introduction

In this paper, we consider the half-linear dynamic equation

$$[a(t)\Phi(x^{\Delta}(t))]^{\Delta}+b(t)\Phi(x^{\sigma}(t))=0, \quad t\geq t_0,\ t_0\in\mathbb{T} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, a closed subset of real numbers. We assume that $\sup\mathbb{T}=\infty$, $\Phi(u)=|u|^{p-2}u$, $p>1$, and $\dfrac{1}{a}, b$ are positive right-dense continuous functions on $\mathbb{T}$.

Throughout this paper, by a solution of (1.1) we mean a nontrivial solution of (1.1).

The recessive solution for the linear dynamic equation

$$[a(t)x^{\Delta}(t)]^{\Delta}+b(t)x^{\sigma}(t)=0$$

has been characterized by the similar way as in the continuous case (i.e. $\mathbb{T}=\mathbb{R}$) and the discrete case (i.e. $\mathbb{T}=\mathbb{N}$) , see [4, Theorem 4.61].

The extension of the notion of a recessive solution to the half-linear differential equation and difference equation is in general difficult problem and only partial results have been obtained, see e.g. [6, 7, 9, 10, 11] and [8, 12], respectively. We refer [15] for the discussion why $\frac{1}{a}$ has to be a right-dense continuous function on $\mathbb{T}$.

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In this paper we study recessive solutions for (1.1) under the assumption

$$\int_{t_0}^{\infty} b(\tau)\Phi\left(\int_{\sigma(\tau)}^{\infty} \frac{\Delta s}{\Phi^*(a(s))}\right)\Delta\tau < \infty, \qquad t_0 \in \mathbb{T}, \tag{1.2}$$

where Φ^* is the inverse function of Φ.

In the next section, we give a brief introduction to a time scale. In the third section, we obtain some essential results on solutions of half-linear dynamic equations and the Gronwall inequality on time scales. In the last two sections, we consider recessive solutions of half-linear dynamic equations.

2 Time Scale Calculus

In this section, we only mention preliminary results on time scales. More details of dynamic equations with applications can be found in [4, 5].

The *forward jump operator* $\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}$, for all $t \in \mathbb{T}$ gives us the next point in $\mathbb{T}$ while the *backward jump operator* $\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$ for all $t \in \mathbb{T}$ gives us the previous point in $\mathbb{T}$. The *graininess function* $\mu : \mathbb{T} \mapsto [0,\infty)$ is the distance between two consecutive points in $\mathbb{T}$, i.e., $\mu(t) := \sigma(t) - t$.

We define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ if m is a left-scattered maximum, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. We assume $f, g : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. *Delta derivative* $f^\Delta(t)$ of $f(t)$ at t is defined to be the number (provided it exists) if for given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon\,|\sigma(t) - s|$$

for all $s \in U$. The delta derivative f^Δ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

For right-scattered points ($\sigma(t) > t$) in $\mathbb{T}$, we have $f^\Delta(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$ while $f^\Delta(t) = \lim\limits_{s\to t} \dfrac{f(t) - f(s)}{t - s}$ for right-dense points ($\sigma(t) = t$) in $\mathbb{T}$ if the limit exists. For any $t \in \mathbb{T}$, we have $f^\sigma(t) = f(t) + \mu(t) f^\Delta(t)$, where $f^\sigma = f \circ \sigma$. The product and quotient rules are given by

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f^\Delta g^\sigma + f g^\Delta, \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}$$

if $gg^\sigma \neq 0$.

If $f : \mathbb{T} \mapsto \mathbb{R}$ is continuous at each right-dense point $t \in \mathbb{T}$ and its left sided limits exist as a finite number at all left-dense points ($\rho(t) = t$) on $\mathbb{T}$, then it is called a *right-dense continuous* (rd-continuous) function. For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(s)\Delta s = f(b) - f(a).$$

The *exponential function* $e_p(t,t_0)$ on $\mathbb{T}$ is for each fixed $t_0 \in \mathbb{T}$ the unique solution of the initial value problem

$$x^\Delta = p(t)x, \;\; x(t_0) = 1,$$

where $p:\mathbb{T}\mapsto\mathbb{R}$ is regressive ($1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^\kappa$) and rd-continuous. Following properties of exponential functions in time scales are important to prove Gronwall's inequality, see [2] and [3], respectively. All the other properties of exponential functions in time scales can be found in [4].

Theorem 2.1. *Let $p:\mathbb{T}\mapsto\mathbb{R}$ be positively regressive, i.e., $1+\mu(t)p(t)>0$ for all $\in\mathbb{T}^\kappa$ and rd-continuous. Then we have*

1. $e_p(t,t_0)>0$ *for all* $t\in\mathbb{T}$,
2. $e_p(t,s)\leq e^{\int_s^t p(\tau)\Delta\tau}$.

The time scale version of L'Hopital's Rule can be found in [1] and is useful to classify nonoscillatory solutions of equation (1.1).

Theorem 2.2. *Assume f and g are differentiable functions on $\mathbb{T}$ with*

$$\lim_{t\to t_0^-} f(t)=\lim_{t\to t_0^-} g(t)=0 \quad \textit{for some left-dense } t_0\in\bar{\mathbb{T}}, \quad \bar{T}=\mathbb{T}\cup\sup\mathbb{T}\cup\inf\mathbb{T}$$

Suppose there exists $\varepsilon>0$ with

$$g(t)>0, \quad g^\Delta(t)<0 \quad \textit{for all } t\in L_\varepsilon(t_0), \quad L_\varepsilon(t_0)=\{t\in\mathbb{T}:0<t_0-t<\varepsilon\}.$$

Then we have

$$\liminf_{t\to t_0^-}\frac{f^\Delta(t)}{g^\Delta(t)}\leq\liminf_{t\to t_0^-}\frac{f(t)}{g(t)}\leq\limsup_{t\to t_0^-}\frac{f(t)}{g(t)}\leq\limsup_{t\to t_0^-}\frac{f^\Delta(t)}{g^\Delta(t)}.$$

The chain rule on time scales by Pötzsche ([16]) plays an important role for integral characterization of recessive solutions.

Theorem 2.3. *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable and the formula*

$$(f\circ g)^\Delta(t)=g^\Delta(t)\int_0^1 f'\left(g(t)+h\mu(t)g^\Delta(t)\right)dh$$

holds.

3 Nonoscillatory solutions

The half-linear equation (1.1) has the homogeneity property, i.e., if x is a solution of (1.1), then λx is also a solution of (1.1), where $\lambda\in\mathbb{R}$. If x is a solution of (1.1), then

$$x^{[1]}(t)=a(t)\Phi(x^\Delta(t)), \quad t\in\mathbb{T}$$

is called the quasi-derivative of x.

A solution x of (1.1) is said to be *nonoscillatory* if there exists $T\geq t_0$, $T\in\mathbb{T}$ such that $x(t)x^\sigma(t)>0$ for $t\geq T$. Equation (1.1) is called *nonoscillatory* if it has a nonoscillatory solution. Due to the Sturm-theory ([14]), if (1.1) has a nonoscillatory solution, then all its solutions are nonoscillatory.

The condition (1.2) implies

$$\int_{t_0}^{\infty} \frac{\Delta\tau}{\Phi^*(a(\tau))} < \infty, \quad t_0 \in \mathbb{T}. \tag{3.1}$$

Denote

$$A(t) = \int_{t}^{\infty} \frac{\Delta\tau}{\Phi^*(a(\tau))}. \tag{3.2}$$

We start with the properties of nonoscillatory solutions of (1.1).

Lemma 3.1. *Assume that* (3.1) *holds. If x is a nonoscillatory solution of* (1.1), *then*

1. *x and $x^{[1]}$ are eventually strongly monotone;*
2. *x is bounded;*
3. *if $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} x^{[1]}(t) = c$, where $-\infty \le c < 0$ or $0 < c \le \infty$ according to whether $x(t) > 0$ or $x(t) < 0$ for large $t \in \mathbb{T}$, respectively.*

Proof. Without loss of generality, we assume that $x(t) > 0$ for $t \ge t_0$, $t_0 \in \mathbb{T}$.

From (1.1), $(x^{[1]})^{\Delta}(t) < 0$ for large $t \in \mathbb{T}$, thus $x^{[1]}$ is eventually decreasing and x^{Δ} is eventually positive or negative, i.e., x is eventually strongly monotone.

Since $x^{[1]}$ is eventually decreasing, $x^{[1]}(t) \le x^{[1]}(t_0)$ for $t \ge t_0$. This implies that

$$x(t) \le x(t_0) + \Phi^*(x^{[1]}(t_0)) \int_{t_0}^{t} \frac{\Delta\tau}{\Phi^*(a(\tau))}.$$

Since (3.1) holds, x is bounded. This completes the proof of the second part.

Finally, since x is eventually strongly monotone, positive and $\lim_{t\to\infty} x(t) = 0$, x is eventually decreasing and so $x^{\Delta}(t) < 0$ for large $t \in \mathbb{T}$. This implies that $x^{[1]}(t) < 0$ for large $t \in \mathbb{T}$. If $\lim_{t\to\infty} x^{[1]}(t) = 0$, then integrating (1.1) yields that $x^{[1]}(t) > 0$ for large t, which gives a contradiction. □

From Lemma 3.1 and Theorem 2.2 it follows that if (3.1) holds, then any nonoscillatory solution of (1.1) is bounded and is one the following types:

$$\text{Type (a)}: \quad \lim_{t\to\infty} \frac{x(t)}{A(t)} = c \quad 0 < |c| < \infty,$$
$$\text{Type (b)}: \quad \lim_{t\to\infty} \frac{x(t)}{A(t)} = \infty.$$

The following Gronwall Inequality plays an important role to obtain the uniqueness result on solutions of Type (a). It is an extension of [13, Lemma 4.1] for the continuous case and [8, Lemma 2.3] for the discrete case.

Lemma 3.2. *Let z, w be two nonnegative rd-continuous functions on $\mathbb{T}$ such that*

$$\int_T^\infty w(\tau)z^\sigma(\tau)\Delta\tau < \infty$$

and

$$\int_T^\infty w(\tau)\Delta\tau < \infty$$

for $T \in \mathbb{T}$. If $z(t) \le \int_t^\infty w(\tau)z^\sigma(\tau)\Delta\tau$, $t \ge T$, then $z(t) = 0$ for all $t \ge T$.

Proof. Define $v(t) := \int_t^\infty w(\tau)z^\sigma(\tau)\Delta\tau$. This implies that $(0 \le)z(t) \le v(t)$, $t \ge T$.

From here and the fact $v^\Delta(t) = -w(t)z^\sigma(t)$, we obtain

$$v^\Delta(t) + w(t)v^\sigma(t) \ge 0.$$

Since $w(t) \ge 0$ and $\int_T^\infty w(\tau)\Delta\tau < \infty$, $e_w(t,T)$ is bounded and positive by Theorem 2.1. By the product rule on $\mathbb{T}$,

$$(e_w(t,T)v(t))^\Delta = (v^\Delta(t) + w(t)v^\sigma(t))e_w(t,T) \ge 0.$$

Since $\lim_{t\to\infty} v(t) = 0$ and $e_w(t,T)$ is bounded, $e_w(t,T)v(t) \le 0$ for $t \ge T$ and so $v(t) = 0$ for $t \ge T$. This implies that $z(t) = 0$ for $t \ge T$. □

Now, we can prove the existence of a unique (up to a multiplicity constant) vanishing solution.

Theorem 3.3. *Assume* (1.2). *For any $c \ne 0$ equation* (1.1) *has a unique (nonoscillatory) solution u of Type (a), i. e.,*

$$\lim_{t\to\infty} u(t) = 0, \qquad \lim_{t\to\infty} u^{[1]}(t) = c, \quad c \in \mathbb{R} - \{0\}. \tag{3.3}$$

Proof. The existence. We choose $t_1 \ge t_0$, $t_1 \in \mathbb{T}$ such that

$$\int_{t_1}^\infty b(t)\Phi\left(\int_{\sigma(t)}^\infty \frac{\Delta\tau}{\Phi^*(a(\tau))}\right)\Delta t < 1 - \frac{1}{\Phi(2)}$$

and denote X as a Banach space of all bounded and continuous functions defined for every $t \ge t_1$ with the supremum norm. Define the set $\Omega \subset X$ by

$$\Omega = \left\{u \in X : \frac{1}{2}A(t) \le u(t) \le A(t), t \ge t_1\right\}.$$

Obviously, Ω is bounded, closed and convex. We now consider the operator $K : \Omega \mapsto X$ defined by

$$Ku(t) = \int_t^\infty \frac{1}{\Phi^*(a(s))}\Phi^*(1 - \int_s^\infty b(\tau)\Phi(u^\sigma(\tau))\Delta\tau)\Delta s.$$

Then

$$\frac{1}{2}A(t) \le Ku(t) \le A(t)$$

so that $K(\Omega) \subset \Omega$. Obviously, $K(\Omega)$ is relatively compact in X, and K is continuous in X. By Schauder fixed point theorem, there exists u in Ω such that $Ku = u$. Therefore, (3.3) holds.

The uniqueness. Without loss of generality, assume x and z are two positive solutions of (1.1) for $t \geq T$, $T \in \mathbb{T}$ such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0,$$

$$\lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} z^{[1]}(t) = c < 0.$$

Since $x^{[1]}$ and $z^{[1]}$ are eventually decreasing, we can assume that

$$0 < -\frac{c}{2} < -x^{[1]} < -c, \qquad 0 < -\frac{c}{2} < -z^{[1]} < -c. \tag{3.4}$$

We have

$$\int_t^\infty \frac{1}{\Phi^*(a(\tau))} \Phi^*(-x^{[1]}(\tau))\Delta\tau = x(t) \tag{3.5}$$

and

$$\int_t^\infty \frac{1}{\Phi^*(a(\tau))} \Phi^*(-z^{[1]}(\tau))\Delta\tau = z(t) \tag{3.6}$$

from the quasi-derivative of x and z, respectively. From (3.4), we have

$$-\Phi^*(\frac{c}{2})A(t) < x(t) < -\Phi^*(c)A(t) \tag{3.7}$$

and

$$-\Phi^*(\frac{c}{2})A(t) < z(t) < -\Phi^*(c)A(t). \tag{3.8}$$

Since $\Phi(r) = r^{p-1}$ for $r > 0$, by the mean value theorem we obtain

$$|\Phi(x(t)) - \Phi(z(t))| \leq (p-1)(w(t))^{p-2}|x(t) - z(t)|,$$

where $w(t) = \max\{x(t), z(t)\}$ if $p > 2$, $w(t) = \min\{x(t), z(t)\}$ if $1 < p < 2$, and $w(t) = 1$ if $p = 2$. Then for any $p > 1$, there exists a positive constant L such that

$$(p-1)(w(t))^{p-2} \leq L(A(t))^{p-2}$$

by (3.7) and (3.8). By (3.5) and (3.6), we have

$$|\Phi(x(t)) - \Phi(z(t))| \leq L(A(t))^{p-2}|x(t) - z(t)|$$

and so

$$|\Phi(x(t)) - \Phi(z(t))| \leq L(A(t))^{p-2} \int_t^\infty \frac{1}{\Phi^*(a(\tau))} |\Phi^*(-x^{[1]})(\tau) - \Phi^*(-z^{[1]})(\tau)|\Delta\tau. \tag{3.9}$$

Similarly, since

$$\lim_{t\to\infty} \Phi^*(x^{[1]})(t) = \lim_{t\to\infty} \Phi^*(z^{[1]})(t) = \Phi^*(c) < 0,$$

there exists a positive constant H such that

$$|\Phi^*(x^{[1]}(t)) - \Phi^*(z^{[1]}(t))| \le H|x^{[1]}(t) - z^{[1]}(t)|. \tag{3.10}$$

Integrating (1.1) from t to ∞, we have

$$x^{[1]}(t) = c + \int_t^\infty b(\tau)\Phi(x^\sigma(t))\Delta\tau$$

and

$$z^{[1]}(t) = c + \int_t^\infty b(\tau)\Phi(z^\sigma(t))\Delta\tau.$$

From (3.9) and (3.10), we have

$$\begin{aligned}|\Phi^*(x^{[1]}(t)) - \Phi^*(z^{[1]}(t))| &\le H\int_t^\infty b(\tau)|\Phi(x^\sigma(\tau)) - \Phi(z^\sigma(\tau))|\Delta\tau \\ &\le HL\int_t^\infty b(\tau))(A^\sigma(\tau))^{p-2}\int_{\sigma(\tau)}^\infty \frac{1}{\Phi^*(a(s))}|\Phi^*(-x^{[1]}(s)) - \Phi^*(-z^{[1]}(s))|\Delta s\Delta\tau.\end{aligned}$$

Put $u(t) = \sup_{t\ge T}|\Phi^*(x^{[1]}(t)) - \Phi^*(z^{[1]}(t))|$. Then

$$\begin{aligned}u(t) &\le HL\int_t^\infty b(\tau)(A^\sigma(\tau))^{p-2}\int_{\sigma(\tau)}^\infty \frac{1}{\Phi^*(a(s))}u(s)\Delta s\Delta\tau \\ &\le HL\int_t^\infty b(\tau)(A^\sigma(\tau))^{p-2}u^\sigma(\tau)\int_{\sigma(\tau)}^\infty \frac{1}{\Phi^*(a(s))}\Delta s\Delta\tau \\ &= HL\int_t^\infty b(\tau)\Phi(A^\sigma(\tau))u^\sigma(\tau)\Delta\tau.\end{aligned}$$

By (1.2), and Lemma 3.2, $u(t) = 0$ for $t \ge T$. This implies that $x^{[1]}(t) = z^{[1]}(t)$ for all $t \ge T$, and this completes the proof. □

Corollary 3.4. *Assume* (1.2) *and u is a solution of Type (a). Then any solution x of* (1.1) *linearly independent of u is of Type (b).*

Proof. We will prove that if u and w are two solutions of Type (a), then u, w are linearly dependent, i.e., there exists $\lambda \in \mathbb{R} - \{0\}$ such that $u = \lambda w$.

Let

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} w(t) = 0,$$

$$\lim_{t\to\infty} u^{[1]}(t) = c \text{ and } \lim_{t\to\infty} w^{[1]}(t) = d,$$

where $c, d \in \mathbb{R} - \{0\}$, and let z be the solution of (1.1) such that

$$z = \Phi^*(\frac{c}{d})w.$$

Then $\lim_{t\to\infty} z(t) = 0$ and because

$$z^{[1]} = \frac{c}{d}w^{[1]},$$

we have

$$\lim_{t\to\infty} z^{[1]}(t) = \frac{c}{d}\lim_{t\to\infty} w^{[1]}(t) = c.$$

By Theorem 3.3, $z = u$. Consequently, any solution x linearly independent of u must be of Type (b). □

If $\mathbb{T} = \mathbb{N}$, then Theorem 3.3 gives [8, Theorem 3.4] and if $\mathbb{T} = \mathbb{R}$, then it gives [9, Theorem A and Theorem B].

4 Recessive Solutions

Our main result is the following characterization of solutions of Type (a). The continuous case of the part of the following theorem can be found in [9, Theorem 1].

Theorem 4.1. *Assume* (1.2)*. The following statements are equivalent:*

(a) Solution u is of Type (a).

(b) Solution u satisfies the limit property, i.e.,

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0 \tag{4.1}$$

for any solution x linearly independent of u.

(c) Solution u satisfies the Riccati property, i.e.,

$$\frac{u^{\Delta}(t)}{u(t)} < \frac{x^{\Delta}(t)}{x(t)} \quad \text{for large } t \in \mathbb{T} \tag{4.2}$$

for any solution x linearly independent of u.

Proof. Without loss of generality, assume u and x are eventually positive solutions.

"(a)⟹(b)": If x is a solution of (1.1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R} - \{0\}$, then by Corollary 3.4 x is of Type (b). Consequently, (4.1) holds.

"(b)⟹(c)": From (4.1) we have that the function $\frac{u}{x}$ is eventually positive and eventually decreasing. Thus

$$\left(\frac{u}{x}\right)^{\Delta}(t) = \frac{u^{\Delta}(t)x(t) - x^{\Delta}(t)u(t)}{x(t)x^{\sigma}(t)} < 0 \text{ for large } t \in \mathbb{T},$$

from where (4.2) follows.

"(c)⟹(a)": Let there exist a solution u satisfying (4.2) for any x linearly independent of u. Assume that u is of Type (b), i.e.

$$\lim_{t\to\infty} \frac{u(t)}{A(t)} = \infty. \tag{4.3}$$

By Theorem 3.3, there exists a unique solution z such that $\lim_{t\to\infty} \frac{z(t)}{A(t)} = 1$. Obviously, z is linearly independent of u and we have

$$\frac{u^{\Delta}(t)}{u(t)} < \frac{z^{\Delta}(t)}{z(t)} \quad \text{for large } t \in \mathbb{T}.$$

This implies that

$$\left(\frac{u}{z}\right)^{\Delta}(t)=\frac{u^{\Delta}(t)z(t)-z^{\Delta}(t)u(t)}{z(t)z^{\sigma}(t)}<0 \ \text{ for large } t\in\mathbb{T}$$

and so $\dfrac{u}{z}$ is eventually decreasing and eventually positive. Therefore,

$$\lim_{t\to\infty}\frac{u(t)}{z(t)}=c, \quad 0\le c<\infty,$$

which gives a contradiction with (4.3). □

By Theorems 3.3 and 4.1, equation (1.1) possesses a unique (up to a nonzero multiplicative factor) solution u with the property (4.2). In accordance with the discrete case, such a solution is called a *recessive solution* of (1.1) and every solution of (1.1), which is not a recessive solution is called a *dominant solution* of (1.1).

Remark 4.2. The property (4.2) is closely related with the minimal solution of the generalized Riccati dynamic equation ([14])

$$R[w]:=w^{\Delta}+b(t)+S[w,a](t)=0, \tag{4.4}$$

where

$$S[w,a](t)=\begin{cases}\frac{p-1}{\Phi^*(a(t))}|w(t)|^{\beta} & \text{at right-dense } t\\ \frac{w(t)}{\mu(t)}\left(1-\frac{a(t)}{\Phi(\Phi^*(a(t))+\mu(t)\Phi^*(w(t)))}\right) & \text{at right-scattered } t.\end{cases}$$

Indeed, if x is a solution of (1.1) with $(axx^{\sigma})(t)>0$ for $t\ge t_0$, $t_0\in\mathbb{T}$, then

$$w(t)=\frac{x^{[1]}(t)}{\Phi(x(t))}$$

is a solution of (4.4) satisfying

$$\Big(\Phi^*(a)+\mu\Phi^*(w)\Big)(t)>0 \quad \text{for } t\ge t_0,$$

see [14]. Thus, the property (4.2), or equivalently

$$\frac{u^{[1]}(t)}{\Phi(u(t))}<\frac{x^{[1]}(t)}{\Phi(x(t))} \quad \text{for large } t\in\mathbb{T},$$

means that the solution w_u of (4.4) corresponding to the recessive solution u of (1.1) is smaller than any other solution of (4.4) for large t.

It is an open problem whether the minimal solution of (4.4) and recessive solution of (1.1) exist without assuming $b(t)>0$ and (1.2).

5 Integral characterization of recessive solutions

The integral and summation characterization of recessive solutions for differential and difference equations has been investigated in [6, 10, 11] and [8, 12], respectively. In this section, we extend some of these results to the dynamic equations.

Lemma 5.1. *Suppose* (3.1) *and A is defined as in* (3.2). *Then*

$$\int_T^\infty \frac{-A^\Delta(\tau)}{A^m(\tau)A(\sigma(\tau))}\Delta\tau = \infty,\ T \in \mathbb{T} \tag{5.1}$$

if $0 < m \le 1$, *and*

$$\int_T^\infty \frac{-A^\Delta(\tau)}{A(\tau)A^m(\sigma(\tau))}\Delta\tau = \infty,\ T \in \mathbb{T} \tag{5.2}$$

if $m \ge 1$.

Proof. By the quotient rule, we have

$$\left(\frac{1}{A^m(t)}\right)^\Delta = \frac{-(A^m)^\Delta(t)}{A^m(t)A^m(\sigma(t))}.$$

By Theorem 2.3, we have

$$(A^m)^\Delta(t) = (t^m \circ A)^\Delta = mA^\Delta(t)\int_0^1 \left(A(t) + h\mu(t)A^\Delta(t)\right)^{m-1} dh.$$

Obviously,

$$0 < A^\sigma \le A + h\mu A^\Delta \le A.$$

If $0 < m \le 1$, then

$$(A^m)^\Delta(t) \ge mA^\Delta(t)\int_0^1 A^{m-1}(\sigma(t))dh = mA^\Delta(t)A^{m-1}(\sigma(t))$$

and so

$$\left(\frac{1}{A^m(t)}\right)^\Delta \le \frac{-mA^\Delta(t)}{A^m(t)A(\sigma(t))}.$$

Integrating above from T to t, $T \in \mathbb{T}$ yields

$$\frac{1}{A^m(t)} - \frac{1}{A^m(T)} \le \int_T^t \frac{-mA^\Delta(\tau)}{A^m(\tau)A(\sigma(\tau))}\Delta\tau.$$

Since $\lim_{t\to\infty} A(t) = 0$, we obtain (5.1).

Similarly, if $m \ge 1$, then

$$(A^m)^\Delta(t) \ge mA^\Delta(t)\int_0^1 A^{m-1}(t)dh = mA^\Delta(t)A^{m-1}(t),$$

which implies that

$$\left(\frac{1}{A^m(t)}\right)^\Delta \le \frac{-mA^\Delta(t)}{A(t)A^m(\sigma(t))} \tag{5.3}$$

and integrating above from T to t, $T \in \mathbb{T}$ yields

$$\frac{1}{A^m(t)} - \frac{1}{A^m(T)} \leq \int_T^t \frac{-mA^\Delta(\tau)}{A(\tau)A^m(\sigma(\tau))}\Delta\tau.$$

Since $\lim_{t\to\infty} A(t) = 0$, we obtain (5.2). □

The following theorem is new for discrete case while the continuous version of it can be found in [11, Proposition 7].

Theorem 5.2. *Assume* (1.2) *holds. If u is a recessive solution of* (1.1)*, then there exists $T \in \mathbb{T}$, $T \geq t_0$ such that*

$$I := \int_T^\infty \frac{\Delta\tau}{\Phi^*(a(\tau))\big(u(\tau)u^\sigma(\tau)\big)^m} = \infty \quad \textit{for any } m \geq 1, \tag{5.4}$$

$$J := \int_T^\infty \frac{\Delta\tau}{\Phi^*(a(\tau))u^m(\tau)u^\sigma(\tau)} = \infty \quad \textit{for any } m \in (0,1], \tag{5.5}$$

and

$$S := \int_T^\infty \frac{u^\Delta(\tau)}{u^{[1]}(\tau)u(\tau)u^\sigma(\tau)}\Delta\tau = \infty. \tag{5.6}$$

Proof. Let u be a recessive solution. By Theorem 4.1, u is of Type (a). Without loss of generality, we assume u is eventually positive satisfying

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u^{[1]}(t) = c < 0.$$

By Theorem 2.2, there exists $T \in \mathbb{T}$, $T \geq t_0$ such that

$$u(t) < -2cA(t) \text{ for } t \geq T.$$

So since A is decreasing, and (5.3) holds, we have

$$\begin{aligned} \int_T^t \frac{\Delta\tau}{\Phi^*(a(\tau))(u(\tau)u^\sigma(\tau))^m} &> \frac{1}{4^m c^{2m}} \int_T^t \frac{-A^\Delta(\tau)\Delta\tau}{A^m(\tau)A^m(\sigma(\tau))} \\ &> \frac{1}{4^m c^{2m} A^{m-1}(T)} \int_T^t \frac{-A^\Delta(\tau)\Delta\tau}{A(\tau)A^m(\sigma(\tau))}. \end{aligned}$$

Passing $t \to \infty$ and applying Lemma 5.1 we get (5.4). By the same argument we get (5.5).

Similarly, integrals S and

$$\int_T^\infty \frac{-u^\Delta(\tau)}{u(\tau)u^\sigma(\tau)}\Delta\tau$$

have the same character, i.e., they are either both convergent or both divergent. Since

$$\int_T^\infty \left(\frac{1}{A(\tau)}\right)^\Delta \Delta\tau = \infty,$$

we have

$$\int_T^\infty \frac{-u^\Delta(\tau)}{u(\tau)u^\sigma(\tau)}\Delta\tau = \int_T^\infty \left(\frac{1}{u(\tau)}\right)^\Delta \Delta\tau = \infty,$$

and so (5.6) holds. □

It is an open problem whether $I=\infty$, $J=\infty$ or $S=\infty$ implies that u is a recessive solution. A partial answer gives the following theorem.

Corollary 5.3. *Assume* (3.1) *and*

$$\int_{t_0}^{\infty} b(\tau)\Delta\tau < \infty. \tag{5.7}$$

Then the following statements are equivalent:

(a) u is a recessive solution of (1.1).

(b) (5.4) *holds.*

(c) (5.5) *holds.*

(d) (5.6) *holds.*

Proof. First we prove that any solution is bounded together with its quasiderivative. By Lemma 3.1 (ii), any solution x of (1.1) is bounded. Integrating (1.1) from t to ∞ and using (5.7) the boundedness of $x^{[1]}$ follows. Hence, a solution is of Type (b) if and only if $\lim_{t\to\infty} u(t)=c$, $0<|c|<\infty$.

Now by Theorem 5.2, it is enough to prove that if (5.4), (5.5) or (5.6) holds, then u is a recessive solution of (1.1).

Assume $I=\infty$ or $J=\infty$. Then, in view of (3.1), we get $\lim_{t\to\infty} u(t)=0$. Since $u^{[1]}$ is bounded, u is of Type (a) and by Theorem 4.1 solution u is recessive.

Assume $S=\infty$. If u is dominant, then $\lim_{t\to\infty} u(t)=c$, $0<|c|<\infty$. Because $u^{[1]}$ is bounded and

$$\int_{T}^{\infty} u^{\Delta}(\tau)\Delta\tau < \infty,$$

we have $S<\infty$, a contradiction. □

Lemma 5.4. *Assume* (1.2) *and*

$$\int_{t_0}^{\infty} b(\tau)\Delta\tau = \infty,\ t_0\in\mathbb{T} \tag{5.8}$$

Then any solution x of (1.1) *satisfies $x(t)x^{[1]}(t)<0$ for $t\in\mathbb{T}$.*

Proof. By Theorem 3.3, equation (1.1) is nonoscillatory. Without loss of generality, we assume that $x(t)>0$ for $t\geq t_0$, $t_0\in\mathbb{T}$. Then $x^{[1]}(t)$ is decreasing for $t\geq t_0$. Assume that $x^{[1]}(t)>0$ for $t\geq t_1\geq t_0$, $t_1\in\mathbb{T}$. This implies that $x(t)$ is increasing $t\geq t_1$. Integrating equation (1.1) from t_1 to t yields

$$x^{[1]}(t)\leq x^{[1]}(t_1)-\Phi(x(t_0))\int_{t_1}^{t} b(s)\Delta s,$$

where we also use the monotonicity of x. But this contradicts with the positivity of $x^{[1]}$ as $t\to\infty$. □

Moreover, the following result holds for the special choice of $\mathbb{T}=\mathbb{N}$.

Theorem 5.5. *[12, Theorem 1] Let $\mathbb{T}=\mathbb{N}$, $p\in(1,2]$ and let $u(t)u^{\Delta}(t)<0$ for large t. If* (5.6) *holds, then u is a recessive solution of* (1.1).

By Theorem 5.2, Lemma 5.4, and Theorem 5.5 we improve the previous result.

Corollary 5.6. *Let* $\mathbb{T}=\mathbb{N}$, $p\in(1,2]$, (1.2) *and* (5.8) *hold. Then u is a recessive solution if and only if* (5.6) *holds.*

Concluding remarks.

(1) Does exist the minimal solution of the Riccati dynamical equation (4.4) without assuming (1.2)?

(2) Theorem 3.3 can be extended for b which can change sign replacing (1.2) by

$$\int_{t_0}^{\infty} |b(\tau)|\Phi\left(\int_{\sigma(\tau)}^{\infty} \frac{\Delta s}{\Phi^*(a(s))}\right)\Delta\tau < \infty. \tag{5.9}$$

The proof is a similar as that one in [7] where the continuous case is treated.

(3) Does it hold Corollary 5.6 for any time scales?

(4) Let $p>2$, (5.9) holds and

$$\int_{t_0}^{\infty} \Phi^*\left(\frac{1}{a(s)}\int_{t_0}^{t} |b(s)|\Delta s\right)\Delta\tau = \infty, \qquad t_0\in\mathbb{T}.$$

We conjecture that in this case the integral characterization (5.6) can fail.

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