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Oscillatory properties for three-dimensional dynamic systems

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Abstract

In this paper, we investigate oscillation and asymptotic properties for three-dimensional systems of dynamic equations.

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1. Introduction

In this paper, we investigate three-dimensional dynamic systems of the form

$$\begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = -c(t)x^{\gamma}(t) \end{cases} \tag{1}$$

on a time scale $\mathbb{T}$ which is unbounded from above. We assume that $a, b : [0, \infty) \mapsto [0, \infty)$ (not identically zero) and $c : [0, \infty) \mapsto (0, \infty)$ are rd-continuous functions such that

$$\int_0^{\infty} a(\tau)\Delta\tau = \int_0^{\infty} b(\tau)\Delta\tau = \infty, \tag{2}$$

and α, β, γ are ratios of odd positive integers.
A solution of system (1) is denoted by (x, y, z). Solution (x, y, z) defined for $t \geq t_0$, $t_0 \in [0, \infty)$ is said to be *proper* if

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)\} > 0 \quad \text{for } t \geq t_0.$$

A proper solution of system (1) is said to be *oscillatory* if all of its component x, y, z are oscillatory, i.e. neither eventually positive nor eventually negative. Otherwise, a proper solution is said to be *nonoscillatory*.

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Definition 1.1. System (1) is said to have *Property A*, or to be *almost oscillatory*, if every proper solution (x, y, z) of system (1) is either oscillatory or

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0. \tag{3}$$

Originally, Property A has been introduced for differential systems and equations, see [1,2] and the monography [3]. Here together with Property A, so called Property B was investigated in the case when c is negative.

As it is noticed in [4, p. 126] in a picturesque way, Property A and Property B state that every solution which may oscillate, does oscillate. Some authors use a different terminology — the system or higher order equation is *almost oscillatory* [5], or *strongly oscillatory* [4] in order not to separate system (1) with c positive or negative into subtypes.

Dynamic system (1) includes linear and nonlinear systems and third order differential and difference equations, which were deeply investigated in the literature.

When $\mathbb{T} = \mathbb{R}$ and $a(t), b(t), c(t)$ are real continuous positive functions for $t \geq 0$, then (1) is equivalent to the system of differential equations

$$\begin{cases} x'(t) = a(t)y^{\alpha}(t) \\ y'(t) = b(t)z^{\beta}(t) \\ z'(t) = -c(t)x^{\gamma}(t). \end{cases} \tag{4}$$

Properties of higher order nonlinear systems have been deeply investigated by Chanturia [2]. If $\alpha = \beta = 1$, then system (1) is equivalent to the third order differential equation

$$\left(\frac{1}{b(t)}\left(\frac{1}{a(t)}x'(t)\right)'\right)' + c(t)|x(t)|^{\gamma}\operatorname{sgn} x(t) = 0. \tag{5}$$

Third order differential equations have a long history concerning linear equations. We refer to the pioneering works of Birkhoff [6], Sansone [7] and Villari [8]. Further references can be found in the papers [9–11].

When $\mathbb{T} = \mathbb{N}$ and $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences of positive real numbers, then (1) is equivalent to the system of first order difference equations

$$\begin{cases} \Delta x(n) = a(n)y^{\alpha}(n) \\ \Delta y(n) = b(n)z^{\beta}(n) \\ \Delta z(n) = -c(n)x^{\gamma}(n). \end{cases} \tag{6}$$

These properties have been investigated in [5]. Asymptotic and oscillatory properties of the third order difference equation

$$\Delta\left(\frac{1}{b_n}\Delta\left(\frac{1}{a_n}\Delta x_n\right)\right) \pm c_n f(x_{n+\tau}) = 0, \quad \tau \in \{0, 1, 2\}, \tag{7}$$

has been studied in [12–15]. Observe that Eq. (7) with $\tau = 0$ is equivalent to system (6) with $\alpha = \beta = 1$.

In our paper we use a unified approach to systems (4) and (6) and we study oscillatory and asymptotic properties of solutions of (1). The global asymptotic properties and oscillation for linear dynamic systems were investigated in the last section.

2. Calculus on time scale

In this section, we only give preliminary results on a time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of real numbers. An introduction with applications and advances in dynamic equations are given in [16,17].

We have two jump operators on a time scale. The *forward jump operator* and the *backward jump operator* on $\mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \quad \text{and} \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$, respectively. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. The *graininess* function $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. We define the interval $[t_0, \infty)$ in $\mathbb{T}$ by $[t_0, \infty) := \{t \in \mathbb{T} : t \geq t_0\}$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Assume $f : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of $f(t)$ at t, and it turns out that $^\Delta$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the usual forward difference operator Δ if $\mathbb{T} = \mathbb{Z}$.

If $f : \mathbb{T} \mapsto \mathbb{R}$ is continuous at $t \in \mathbb{T}$, then the delta-derivative of f at t equals to $\frac{f(\sigma(t)) - f(t)}{\mu(t)}$ for $\sigma(t) > t$, and $\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$ for $\sigma(t) = t$ if the limit exists. If f is differentiable at t, then

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \text{where } f^\sigma = f \circ \sigma. \tag{8}$$

If $f, g : \mathbb{T} \mapsto \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then the product and quotient rules are as follows:

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t)$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g^\sigma(t)} \quad \text{if } g(t) g^\sigma(t) \neq 0.$$

We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense $\lim_{s \to t^-} f(s)$ exists as a finite number.

A function $F : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \mapsto \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In this case we define the integral of f by

$$\int_a^t f(s) \Delta s = F(t) - F(a) \quad \text{for } t \in \mathbb{T}.$$

If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and $f \in \mathrm{C}_{\mathrm{rd}}$ on $[a, \infty)$, then we define the *improper integral* by

$$\int_a^\infty f(t) \Delta t := \lim_{b \to \infty} \int_a^b f(t) \Delta t$$

provided this limit exists, and we say the improper integral converges in this case.

The following result is the chain rule on $\mathbb{T}$, which is an essential to determine whether system (1) has Property A, see [16, Theorem 1.90].

Theorem 2.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t) g^\Delta(t)\right) \mathrm{d}h \right\} g^\Delta(t)$$

holds.

An $n \times n$ matrix valued function A on $\mathbb{T}$ is called *regressive* provided

$$I + \mu(t) A(t) \text{ is invertible for all } t \in \mathbb{T}^\kappa.$$

The existence and uniqueness theorem for initial value problems for linear systems is proven in [16, Theorem 5.8].

Theorem 2.2. *Let A be a rd-continuous and regressive $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \mapsto \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^\Delta = A(t) y + f(t), \quad y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \mapsto \mathbb{R}^n$.

3. Nonoscillatory solutions of system (1)

In this section, we study asymptotic properties of nonoscillatory solutions which we use in the next section.

Lemma 3.1. *Let (x, y, z) be a solution of system (1) with $x(t) \neq 0$ for $t \geq t_0$, $t_0 \in [0, \infty)$. Then (x, y, z) is nonoscillatory and x, y, z are monotone for large t.*

Proof. Let (x, y, z) be a solution of system (1) with $x(t)$ nonoscillatory for $t \geq t_0$, $t_0 \in [0, \infty)$. Then without loss of generality assume that $x(t) > 0$ for $t \geq T \geq t_0$, $T \in [0, \infty)$ and so from the third equation of system (1) we have

$$z^{\Delta}(t) = -c(t)x^{\gamma}(t) < 0 \quad \text{for all } t \geq T \geq t_0.$$

This implies that $z(t)$ is strictly decreasing for $t \geq T$, and eventually of one sign for $t \geq T$. Since the first and second equations of system (1) are either nonnegative or nonpositive for all $t \geq T$, $y(t)$ and $x(t)$ are monotone for $t \geq T$ and eventually of one sign for $t \geq T$. Therefore, (x, y, z) is nonoscillatory. $\square$

Lemma 3.2. *Any nonoscillatory solution (x, y, z) of system (1) is one of the following types:*

$$\begin{aligned} &\textit{Type (a)}: \quad \operatorname{sgn} x(t) = \operatorname{sgn} y(t) = \operatorname{sgn} z(t) \quad \textit{for all large } t \in [0, \infty), \\ &\textit{Type (b)}: \quad \operatorname{sgn} x(t) = \operatorname{sgn} z(t) \neq \operatorname{sgn} y(t) \quad \textit{for all large } t \in [0, \infty). \end{aligned}$$

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of generality, we assume that $x(t) > 0$ for $t \geq T$, $T \in [0, \infty)$. Then from Lemma 3.1, we have $y(t)$ and $z(t)$ are monotone for $t \geq T$. Since y is monotonic, we have either $y(t) < 0$ or $y(t) > 0$ for all $t \geq T$. Similarly, either $z(t) < 0$ or $z(t) > 0$ for all $t \geq T$. We now show that $z(t) < 0$ cannot hold. Suppose it holds, then there is $T_1 \geq T$, $T_1 \in [0, \infty)$ and a constant $d > 0$ such that $z(t) < -d < 0$ for $t \geq T_1$. Therefore, from the second equation of system (1) we have

$$y^{\Delta}(t) = b(t)z^{\beta}(t) < -d^{\beta}b(t), \quad t \geq T_1.$$

In view of (2), $y(t) \to -\infty$ as $t \to \infty$. Therefore, there is $T_2 \geq T_1$, $T_2 \in [0, \infty)$ and a constant ν such that $y(t) < \nu < 0$ for $t \geq T_2$. From the first equation of system (1), we have

$$x^{\Delta}(t) < \nu^{\alpha}a(t) \quad \text{for } t \geq T_2$$

and hence $x(t) \to -\infty$ as $t \to \infty$. But this contradicts the fact that $x(t) > 0$ for all $t \geq T$. This implies that $z(t) > 0$ for all $t \geq T$. One can show similarly the proof for the case when $x(t) < 0$ eventually for $t \geq T$. $\square$

Solutions of Type (b) are sometimes called *Kneser solutions* (see e.g. [3]). In the following lemma, we describe asymptotic properties of Kneser solutions:

Lemma 3.3. *Let (x, y, z) be a Kneser solution of (1). Then*

$$\lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} z(t) = 0.$$

Proof. Let (x, y, z) be a Kneser solution of system (1). Assume that $\lim_{t\to\infty} y(t) \neq 0$. Then $\lim_{t\to\infty} y(t) = l < 0$. Therefore, there exists $T_1 \in [0, \infty)$ such that $y(t) \geq \frac{l}{2}$ for all $t \geq T_1$. From the first equation of system (1)

$$x^{\Delta}(t) = a(t)y^{\alpha}(t) \geq \left(\frac{l}{2}\right)^{\alpha} a(t).$$

This implies that

$$\lim_{t\to\infty} (x(t) - x(T_1)) \geq \left(\frac{l}{2}\right)^{\alpha} \int_{T_1}^{\infty} a(\tau)\Delta\tau = \infty,$$

and so $\lim_{t\to\infty} x(t) = \infty$. But this gives us a contradiction. Similarly one can show that $\lim_{t\to\infty} z(t) = 0$. $\square$

Theorem 3.1. *Assume that one of the following conditions is satisfied:*

$$\int_0^\infty c(t)\Delta t = \infty, \tag{9}$$

or

$$\int_0^\infty b(t)\left(\int_t^\infty c(s)\Delta s\right)^\beta \Delta t = \infty, \tag{10}$$

or

$$\int_0^\infty a(t)\left(\int_t^\infty b(s)\left(\int_s^\infty c(\tau)\Delta\tau\right)^\beta \Delta s\right)^\alpha \Delta t = \infty. \tag{11}$$

Then every Kneser solution of (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Without loss of generality, assume that (x, y, z) is a Kneser solution of system (1) such that $x(t) > 0$ for $t \geq t_0$, $t_0 \in [0, \infty)$. From the first equation of system (1), x is nonincreasing, and $\lim_{t\to\infty} x(t) = L_1 < \infty$. By Lemma 3.3, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. We now show that $\lim_{t\to\infty} x(t) = 0$, and so we let $\lim_{t\to\infty} x(t) = L_1 > 0$. Then there exists $T_1 \geq t_0$, $T_1 \in [0, \infty)$ such that $x(t) \geq L_1 > 0$ for $t \geq T_1$.

Assume (9). Integrating the third equation from t_0 to t and passing $t \to \infty$ we get a contradiction with the fact that $\lim_{t\to\infty} z(t) = 0$. Now assume (10). From the third equation of (1), we obtain

$$z^\beta(t) \geq L_1^{\beta\gamma}\left(\int_t^\infty c(\tau)\Delta\tau\right)^\beta.$$

Hence, integrating the second equation of system (1) from t $(t \geq T_1)$ to ∞ and using the fact $\lim_{t\to\infty} y(t) = 0$, we have

$$-y(t) \geq L_1^{\beta\gamma}\int_t^\infty b(\tau)\left(\int_\tau^\infty c(s)\Delta s\right)^\beta \Delta\tau. \tag{12}$$

This implies that $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction. Finally assume (11). Integrating the first equation of (1) from T_1 to t we get

$$-x(t) + x(T_1) \geq \int_{T_1}^t a(s)(-y)^\alpha(s)\Delta s$$

and using (12) we have

$$-x(t) + x(T_1) \geq L_1^{\alpha\beta\gamma}\int_{T_1}^t a(s)\left(\int_s^\infty b(\tau)\left(\int_\tau^\infty c(u)\Delta u\right)^\beta \Delta\tau\right)^\alpha \Delta s. \tag{13}$$

As $t \to \infty$ we get $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. $\square$

4. Almost oscillatory solutions of system (1)

In this section, we give sufficient conditions for the oscillatory behavior of solutions of system (1).

Theorem 4.1. *If* $\alpha\beta\gamma < 1$ *and*

$$\int_0^\infty c(t)\left(\int_0^t a(s)\left(\int_0^s b(\tau)\Delta\tau\right)^\alpha \Delta s\right)^\gamma \Delta t = \infty, \tag{14}$$

then every nonoscillatory solution of (1) *is a Kneser solution.*

In addition, if any of conditions (9), (10) *or* (11) *is satisfied, then system* (1) *has Property A.*

Proof. By Lemma 3.2 any nonoscillatory solution of system (1) is either of Type (a) or (b) for $t \geq T$, $T \in [0, \infty)$. We now show that a nonoscillatory solution of system (1) of Type (a) cannot occur. Assume that there exists a nonoscillatory solution (x, y, z) of system (1) of Type (a) for $t \geq T$. Without loss of generality we assume that $x(t) > 0$ for $t \geq T$. From the second equation of system (1) we have

$$y(t) - y(T) = \int_T^t b(\tau) z^\beta(\tau) \Delta\tau, \quad t \geq T$$

and so

$$y(t) \geq \int_T^t b(\tau) z^\beta(\tau) \Delta\tau, \quad t \geq T. \tag{15}$$

Using the monotonicity of z in (15), we have

$$y(t) \geq z^\beta(t) \int_T^t b(\tau) \Delta\tau, \quad t \geq T$$

and so

$$y^\alpha(t) \geq z^{\alpha\beta}(t) \left(\int_T^t b(\tau) \Delta\tau \right)^\alpha, \quad t \geq T.$$

From the previous inequality and the first equation of system (1), we have

$$\begin{aligned} x(t) \geq x(t) - x(T_1) &= \int_{T_1}^t a(\tau) y^\alpha(\tau) \Delta\tau \\ &\geq \int_{T_1}^t a(\tau) z^{\alpha\beta}(\tau) \left(\int_T^\tau b(s) \Delta s \right)^\alpha \Delta\tau, \end{aligned}$$

for $t \geq T_1 \geq T$, $T_1 \in [0, \infty)$. The above calculation and the monotonicity of z imply that

$$x(t) \geq z^{\alpha\beta}(t) \int_{T_1}^t a(\tau) \left(\int_T^\tau b(s) \Delta s \right)^\alpha \Delta\tau, \quad t \geq T_1 \geq T.$$

And so

$$x^\gamma(t) \geq z^{\alpha\beta\gamma}(t) \left(\int_{T_1}^t a(\tau) \left(\int_T^\tau b(s) \Delta s \right)^\alpha \Delta\tau \right)^\gamma, \quad t \geq T_1 \geq T. \tag{16}$$

Multiplying inequality (16) by $\frac{c(t)}{z^{\alpha\beta\gamma}(t)}$ yields

$$\frac{x^\gamma(t) c(t)}{z^{\alpha\beta\gamma}(t)} \geq c(t) \left(\int_{T_1}^t a(\tau) \left(\int_T^\tau b(s) \Delta s \right)^\alpha \Delta\tau \right)^\gamma, \quad t \geq T_1 \geq T,$$

and so

$$\int_{T_1}^t \frac{-z^\Delta(\tau)}{z^{\alpha\beta\gamma}(\tau)} \Delta\tau \geq \int_{T_1}^t c(s) \left(\int_{T_1}^s a(\tau) \left(\int_T^\tau b(m) \Delta m \right)^\alpha \Delta\tau \right)^\gamma \Delta s, \quad t \geq T_1 \geq T. \tag{17}$$

Note that from Theorem 2.1, we obtain

$$\left(\frac{1}{z^{\alpha\beta\gamma-1}(t)} \right)^\Delta = (-\alpha\beta\gamma + 1) z^\Delta(t) \int_0^1 \frac{1}{(z(t) + \mu(t) h z^\Delta(t))^{\alpha\beta\gamma}} \mathrm{d}h. \tag{18}$$

Moreover, since $z(t) \geq z(t) + \mu(t) h z^\Delta(t) \geq z^\sigma(t) > 0$ by Eq. (8), and z is monotonic, we have

$$-\frac{z^\Delta(t)}{z^{\alpha\beta\gamma}(t)} \leq \frac{1}{\alpha\beta\gamma - 1} \left(\frac{1}{z^{\alpha\beta\gamma-1}(t)} \right)^\Delta.$$

Integrating above inequality from T_1 to t we obtain

$$\int_{T_1}^{t}\frac{-z^{\Delta}(\tau)}{z^{\alpha\beta\gamma}(\tau)}\Delta\tau \le \frac{1}{\alpha\beta\gamma-1}\int_{T_1}^{t}\left(\frac{1}{z^{\alpha\beta\gamma-1}(\tau)}\right)^{\Delta}\Delta\tau = \frac{1}{\alpha\beta\gamma-1}\left[\frac{1}{z^{\alpha\beta\gamma-1}(t)}-\frac{1}{z^{\alpha\beta\gamma-1}(T_1)}\right].$$

This implies that $\int_{T_1}^{\infty}\frac{-z^{\Delta}(\tau)}{z^{\alpha\beta\gamma}(t)}\Delta\tau<\infty$. But this contradicts inequality (17) as $t\to\infty$. Hence, a nonoscillatory solution of system (1) of Type (a) cannot occur.

Finally, if any of conditions (9), (10) or (11) is satisfied, then by Lemma 3.3 and Theorem 3.1 every Kneser solution satisfies (3), and so (1) has Property A. □

Remark 4.1. If $\alpha\beta\gamma<1$ and (x,y,z) is a solution of system (1) of Type (a), then

$$\int_{T}^{\infty}\frac{-z^{\Delta}(\tau)}{z^{\alpha\beta\gamma}(\tau)}\Delta\tau<\infty,\qquad T\in[0,\infty).$$

Theorem 4.2. *Let* $\alpha\beta\gamma=1$ *and*

$$\int_{0}^{\infty}c(t)\left(\int_{0}^{t}a(s)\left(\int_{0}^{s}b(\tau)\Delta\tau\right)^{\alpha}\Delta s\right)^{\gamma(1-\epsilon)}\Delta t=\infty,\tag{19}$$

where $0<\epsilon<1$. *Then every nonoscillatory solution of* (1) *is a Kneser solution.*

In addition, if any of conditions (9), (10) *or* (11) *is satisfied, then* (1) *has Property A.*

Proof. By Lemma 3.2 any nonoscillatory solution of system (1) is either of Type (a) or (b) for $t\ge T$, $T\in[0,\infty)$. We now show that a nonoscillatory solution of system (1) of Type (a) cannot occur. Assume that there exists a nonoscillatory solution (x,y,z) of system (1) of Type (a) for $t\ge T$. Without loss of generality assume $x(t)>0$ for $t\ge T$. In this case, we have inequality (16) for $t\ge T_1\ge T$, where $\alpha\beta\gamma=1$. Raising this inequality with $\alpha\beta\gamma=1$ to $(1-\epsilon)$-th power we get

$$x^{\gamma(1-\epsilon)}(t)\ge z^{1-\epsilon}(t)\left(\int_{T_1}^{t}a(\tau)\left(\int_{T}^{\tau}b(s)\Delta s\right)^{\alpha}\Delta\tau\right)^{\gamma(1-\epsilon)},\quad t\ge T_1.$$

Since x is nondecreasing, there exists $T_2\ge T_1$, $T_2\in[0,\infty)$ and a constant $d_1>0$ such that $x^{\gamma}(t)\ge d_1$ for $t\ge T_2$. This implies that $(x^{\gamma})^{1-\epsilon}(t)\le\frac{1}{d_1^{\epsilon}}x^{\gamma}(t)$ for $t\ge T_2$. Combining these inequalities we obtain

$$\frac{1}{d_1^{\epsilon}}x^{\gamma}(t)\ge z^{1-\epsilon}(t)\left(\int_{T_1}^{t}a(\tau)\left(\int_{T}^{\tau}b(s)\Delta s\right)^{\alpha}\Delta\tau\right)^{\gamma(1-\epsilon)},\quad t\ge T_2.\tag{20}$$

Multiplying inequality (20) by $cz^{\epsilon-1}$, integrating the third equation of system (1) from T_2 to t and using the fact that z is positive and nonincreasing, we have

$$\int_{T_2}^{t}c(s)\left[\int_{T_1}^{s}a(\tau)\left(\int_{T}^{\tau}b(i)\Delta i\right)^{\alpha}\Delta\tau\right]^{\gamma(1-\epsilon)}\Delta s\le\frac{1}{d_1^{\epsilon}}\int_{T_2}^{t}\frac{-z^{\Delta}(s)}{z^{1-\epsilon}(s)}\Delta s,\quad t\ge T_2.$$

Since $0<1-\epsilon<1$, we have by a direct computation that $\int_{T_2}^{\infty}\frac{-z^{\Delta}(s)}{z^{1-\epsilon}(s)}\Delta s<\infty$, which yields a contradiction with (19). Hence, a nonoscillatory solution of system (1) of Type (a) cannot occur.

Finally, if any of conditions (9), (10) or (11) is satisfied, then by Lemma 3.3 and Theorem 3.1 every Kneser solution satisfies (3), and so (1) has Property A. □

Theorem 4.3. *If* $\alpha\beta\gamma>1$ *and*

$$\int_{0}^{\infty}a(t)\left(\int_{0}^{t}b(s)\Delta s\right)^{\alpha}\left(\int_{\sigma(t)}^{\infty}c(s)\Delta s\right)^{\alpha\beta}\Delta t=\infty,\tag{21}$$

then every nonoscillatory solution of (1) is a Kneser solution.

In addition, if any of conditions (9)–(11) is satisfied, then system (1) has Property A.

Proof. By Lemma 3.2 any nonoscillatory solution of system (1) is either of Type (a) or Type (b) for $t \geq T$, $T \in [0, \infty)$. We now show that a nonoscillatory solution of system (1) of Type (a) cannot occur. Assume that there exists a nonoscillatory solution (x, y, z) of system (1) of Type (a) for $t \geq T$. Without loss of generality we assume that $x(t) > 0$ for $t \geq T$. From the third equation of system (1) and the nondecreasing nature of x we have

$$z(t) \geq x^{\gamma}(t) \int_t^{\infty} c(\tau)\Delta\tau, \quad t \geq T. \tag{22}$$

One can show that

$$y(t) \geq z^{\beta}(\sigma(t)) \int_T^t b(\tau)\Delta\tau, \quad t \geq T, \tag{23}$$

by integrating the second equation of system (1) from T to t and using the fact that z is nonincreasing. By (22) and (23) and from the first equation of system (1), we have

$$\begin{aligned} x^{\Delta}(t) = a(t)y^{\alpha}(t) &\geq a(t)\left(\int_T^t b(\tau)\Delta\tau\right)^{\alpha} z^{\alpha\beta}(\sigma(t)) \\ &\geq a(t)\left(\int_T^t b(\tau)\Delta\tau\right)^{\alpha} x^{\alpha\beta\gamma}(\sigma(t))\left(\int_{\sigma(t)}^{\infty} c(\tau)\Delta\tau\right)^{\alpha\beta}. \end{aligned}$$

So this implies that

$$\int_T^t \frac{x^{\Delta}(s)}{x^{\alpha\beta\gamma}(\sigma(s))}\Delta s \geq \int_T^t a(s)\left(\int_T^s b(\tau)\Delta\tau\right)^{\alpha}\left(\int_{\sigma(s)}^{\infty} c(i)\Delta i\right)^{\alpha\beta}\Delta s, \quad t \geq T. \tag{24}$$

From condition (21), the right-hand side of inequality (24) is divergent for large t. We will show that the right-hand side is convergent for large t to get a contradiction. Since $0 < x + h\mu x^{\Delta} \leq x^{\sigma}$ by Eq. (8), we have

$$x^{\Delta}(t)\int_0^1 \frac{1}{(x(t) + h\mu(t)x^{\Delta}(t))^{\alpha\beta\gamma}}\mathrm{d}h \geq \frac{x^{\Delta}(t)}{x^{\alpha\beta\gamma}(\sigma(t))}$$

and so

$$(1 - \alpha\beta\gamma)x^{\Delta}(t)\int_0^1 \frac{1}{(x(t) + h\mu(t)x^{\Delta}(t))^{\alpha\beta\gamma}}\mathrm{d}h \leq (1 - \alpha\beta\gamma)\frac{x^{\Delta}(t)}{x^{\alpha\beta\gamma}(\sigma(t))}.$$

Since by Theorem 2.1

$$\left(\frac{1}{x^{\alpha\beta\gamma-1}(t)}\right)^{\Delta} = (1 - \alpha\beta\gamma)x^{\Delta}(t)\int_0^1 \frac{1}{(x(t) + h\mu(t)x^{\Delta}(t))^{\alpha\beta\gamma}}\mathrm{d}h,$$

we obtain

$$\left(\frac{1}{x^{\alpha\beta\gamma-1}(t)}\right)^{\Delta} \leq (1 - \alpha\beta\gamma)\frac{x^{\Delta}(t)}{x^{\alpha\beta\gamma}(\sigma(t))}.$$

This implies that

$$\int_T^{\infty} \frac{x^{\Delta}(\tau)}{x^{\alpha\beta\gamma}(\sigma(\tau))}\Delta\tau \leq \frac{1}{1 - \alpha\beta\gamma}\left[\frac{1}{x^{\alpha\beta\gamma-1}(\infty)} - \frac{1}{x^{\alpha\beta\gamma-1}(T)}\right] < \infty.$$

But this contradicts with inequality (24), and so a nonoscillatory solution of system (1) of Type (a) cannot occur.

Finally, by Lemma 3.3 and Theorem 3.1 every Kneser solution satisfies Eq. (3) and this completes the proof. □

Remark 4.2. If $\alpha\beta\gamma > 1$ and (x, y, z) is a solution of system (1) of Type (a) of system (1), then

$$\int_T^{\infty} \frac{x^{\Delta}(\tau)}{x^{\alpha\beta\gamma}(\sigma(\tau))}\Delta\tau < \infty, \quad T \in [0, \infty).$$

Theorem 4.4. *If*

$$\int_0^\infty c(t)\left(\int_0^t a(s)\Delta s\right)^\gamma \Delta t=\infty, \tag{25}$$

then every nonoscillatory solution of (1) is a Kneser solution.

Proof. Let (x, y, z) be a nonoscillatory solution of system (1) of Type (a) such that $x(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T$, $T \in [0, \infty)$. Integrating the third equation of (1) we have

$$z(t)-z(T)+\int_T^t c(s)x^\gamma(s)\Delta s=0.$$

Because y is increasing, we have $x(t) > d\int_T^t a(t)\Delta t$, where $d = y^\alpha(T)$. Since $z(t) > 0$

$$z(T)>d^\gamma\int_T^t c(s)\left(\int_T^s a(\tau)\Delta\tau\right)^\gamma\Delta s$$

and as $t \to \infty$ we get a contradiction. The rest of the proof is the same as in proof of Theorem 4.1. □

Remark 4.3. If $\mathbb{T} = \mathbb{R}$, Theorems 4.1 and 4.3 coincide with Theorem 2.3 of [2] for system (4), and Theorem 4.4 generalizes Theorem 8 of [9] for linear equations of the form (5).

If $\mathbb{T} = \mathbb{N}$, Theorems 4.1–4.3 generalize Theorems 1–3 of [5] and Theorem 4.4 agrees with [13, Theorem 2] and [12, Theorem 2.6] for third order difference equations.

Remark 4.4. Let us compare conditions in Theorems 4.1–4.4 under assumption (2).

In the sublinear case $\alpha\beta\gamma < 1$, if (25) holds, then (14) holds as well, but not vice versa. Thus Theorem 4.1 yields a better condition than Theorem 4.4.

In the superlinear case $\alpha\beta\gamma > 1$, Theorems 4.3 and 4.4 are different because conditions (21) and (25) are independent.

Also in the case $\alpha\beta\gamma = 1$ Theorems 4.2 and 4.4 assume different conditions. Similar to the superlinear case, one can easily see that it can happen that (19) holds and (25) not. To show vice versa, let $\mathbb{T} = \mathbb{R} \cap [2, \infty)$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and

$$a(t)=\ln t,\quad b(t)=\frac{1}{t},\quad c(t)=\frac{1}{t^2(\ln t)^{1+\varepsilon}},\quad \varepsilon\in(0,1).$$

Then (2) holds and because

$$\int_2^t a(s)\mathrm{d}s=t(\ln t-1)+c_1,\ \int_2^t a(s)\int_2^s b(u)\mathrm{d}u\mathrm{d}s\le\int_2^t(\ln s)^2\mathrm{d}s\le t\ln^2 t+c_2,$$

we have

$$\begin{aligned}\int_2^\infty c(t)\left(\int_2^t a(s)\int_2^s b(\tau)\mathrm{d}\tau\mathrm{d}s\right)^{(1-\epsilon)}\mathrm{d}t &\le\int_2^\infty\frac{(t\ln^2 t)^{(1-\varepsilon)}}{t^2(\ln t)^{1+\varepsilon}}\mathrm{d}t\\&=\int_2^\infty\frac{(\ln t)^{(1-\varepsilon)}}{t^{1+\varepsilon}}\mathrm{d}t<\infty\end{aligned}$$

for any $\varepsilon \in (0, 1)$. On the other hand, (25) is satisfied because

$$\int_2^\infty c(t)\int_2^t a(s)\mathrm{d}s\mathrm{d}t\ge\int_2^\infty\frac{\ln t-1}{t(\ln t)^{1+\varepsilon}}\mathrm{d}t=\int_2^\infty\frac{\mathrm{d}t}{t(\ln t)^\varepsilon}=\int_{\ln 2}^\infty\frac{\mathrm{d}s}{s^\varepsilon}=\infty.$$

Example 4.1. Consider the superlinear differential system

$$x'(t)=y(t),\quad y'(t)=z(t),\quad z'(t)=-\frac{6}{t(t+1)^3}x^3(t).$$

By Theorem 4.4 every nonoscillatory solution of this system is Kneser. However, the system does not have Property A, because it has a solution $(\frac{t+1}{t}, -\frac{1}{t^2}, \frac{2}{t^3})$ satisfying $\lim_{t\to\infty} x(t) = 1$. Note that none of assumptions of Theorem 3.1 is satisfied and Theorem 4.3 is not applicable in this case.

We close this section with a discussion on the global existence and uniqueness. It is well known that the existence and uniqueness for nonlinear systems (1) does not hold. In the continuous case, this means that system (4) can have, so-called singular solutions of the first kind, i.e. solutions are trivial for large t but not identically zero, see [3].

In general, as it is noticed in [16], "going forward" in time is easy to do under the Lipschitz continuity (see [16, Theorem 8.16]), but for going "backward" in time, this condition is not sufficient and the regressivity is required (see [16, Theorems 8.18, 8.20]). In [5] the existence and uniqueness "going forward" for (6) are considered. Similarly as in the continuous case, the uniqueness going "backward" for (6) in general does not hold as the following example illustrates:

Example 4.2. Consider the difference system

$$\Delta x(n) = y(n), \quad \Delta y(n) = z(n), \quad \Delta z(n) = -x^3(n).$$

By Theorem 4.3 this system has Property A and has a (not identically zero) solution (x, y, z) which is not proper. In fact, (x, y, z), where $x_1 = -y_1 = z_1 = 1$ and $x_n = y_n = z_n = 0$ for $n > 1$ is such solution.

5. Linear dynamic systems

This section is devoted to the linear dynamic system

$$\begin{cases} x^\Delta(t) = a(t)y(t) \\ y^\Delta(t) = b(t)z(t) \\ z^\Delta(t) = -c(t)x(t), \end{cases} \tag{26}$$

where a, b, c are rd-continuous and positive functions for $t \ge 0$ satisfying (2) and the regressivity condition

$$\mu^3(t)a(t)b(t)c(t) \neq 1 \quad \text{for } t \ge 0. \tag{27}$$

We start with the existence and uniqueness result for initial value problems for system (26).

Lemma 5.1. *If (27) holds, then for any $t_0 \ge 0$ and $(x_0, y_0, z_0) \in \mathbb{R}^3$ the initial value problem (26), $x(t_0) = x_0$, $y(t_0) = y_0$, $z(t_0) = z_0$ has a unique solution (x, y, z) defined for $t \ge 0$.*

Proof. Denote the matrix of (26) by $A(t)$. Obviously, $A(t)$ is rd-continuous on $[0, \infty)$. By a direct computation we have

$$\det(I + \mu(t)A(t)) = 1 - \mu^3(t)a(t)b(t)c(t) \neq 0,$$

i.e. the matrix $A(t)$ is regressive from (27). Now the result follows from Theorem 2.2. □

Remark 5.1. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and the existence and uniqueness for the linear system of first order differential equations hold.

If $\mathbb{T} = \mathbb{N}$, then $\mu(t) = 1$ and the existence and uniqueness for the linear system of first order difference equations hold provided $a(n)b(n)c(n) \neq 1$ for $n \in \mathbb{N}$.

The next lemma shows that the property of Kneser solutions remains valid going "backward" in time.

Lemma 5.2. *Assume (27). If a solution (x, y, z) of (26) satisfies for some $T > 0$*

$$x(T) \ge 0, \quad y(T) \le 0, \quad z(T) > 0,$$

then

$$x(t) > 0, \quad y(t) < 0, \quad z(t) > 0 \quad \textit{for } t \ge 0. \tag{28}$$

Proof. By Lemma 5.1 every solution of (26) is defined on $[0, \infty)$ and any solution which is not identically zero on $[0, \infty)$, is proper.

Since $x(T) \geq 0$, we have from the last equation of (26) that $z^{\Delta}(T) \leq 0$. And in view of $z(T) > 0$, it follows that there exists $T_1 \in [0, \infty)$ such that $z(t) > z(T) > 0$ for $t \in (T_1, T)$. From this and the second equation of (26) we have $y(t) < y(T) \leq 0$ for $t \in (T_1, T)$ and so, from the first equation, $x(t) > x(T) \geq 0$ for $t \in (T_1, T)$. Repeating this argument, (28) follows. □

Now we are ready to prove the existence of Kneser and oscillatory solutions of linear system (26).

Theorem 5.1. *Assume (27) holds. Then (26) has Kneser solutions with the property*

$$x(t)y(t) < 0, \quad x(t)z(t) > 0 \quad \text{for } t \geq 0. \tag{29}$$

Proof. Let $X^i(t) = \left(x^i(t), y^i(t), z^i(t)\right)$, $i = 1, 2, 3$, be a basis of the solution space of (26). We consider a sequence of solutions $w_k(t) = (x_k(t), y_k(t), z_k(t))$, $k \in \mathbb{N}$ such that

$$w_k(t) = c_k X^1(t) + d_k X^2(t) + e_k X^3(t),$$

where $x_k(T) = 0$, $y_k(T) = 0$ and $c_k^2 + d_k^2 + e_k^2 = 1$. Then by Lemma 5.1 we have $z_k(T) \neq 0$. Without loss of generality assume that $z_k(T) > 0$. By Lemma 5.2

$$x_k(t) > 0, \quad y_k(t) < 0, \quad z_k(t) > 0, \quad \text{for } 0 \leq t < T,\ t \geq 0. \tag{30}$$

Put $A_k = (c_k, d_k, e_k)$. Then $\|A_k\| = 1$ for each k. The unit ball is compact in $\mathbb{R}^3$, so (A_k) has a convergent subsequence $\left(A_{k_i}\right)$. Denote

$$A = \lim_{i\to\infty} A_{k_i} = (c, d, e).$$

Then $c^2 + d^2 + e^2 = 1$ and $w(t)$, defined by

$$\omega(t) = (x(t), y(t), z(t)) = \lim_{i\to\infty} (c_{k_i} X^1(t) + d_{k_i} X^2(t) + e_{k_i} X^3(t)),$$

is a nontrivial solution of (26). In view of (30) and the fact that k is an arbitrary integer, we have

$$x(t) \geq 0, \quad y(t) \leq 0, \quad z(t) \geq 0 \quad \text{for } t \geq 0.$$

If $x(t_0) = 0$ for some t_0, then from (26) we have $x(t) = 0$, $y(t) = 0$ and $z(t) = 0$ for $t \geq t_0$, and because ω is a nontrivial solution, this contradicts the uniqueness of the initial value problem (26), $x(t_0) = y(t_0) = z(t_0) = 0$ stated in Lemma 5.1. Thus $x(t) > 0$ for $t \geq 0$ and from (26) and positivity of functions a, b, c we have $z(t) > 0$ and $y(t) < 0$ for $t \geq 0$. □

Theorem 5.2. *Assume (27) and either (19) with $0 < \epsilon < 1$ or (25). Then (26) has an oscillatory solution.*

Proof. By Lemma 3.2 every nonoscillatory solution of system (1) is either of Type (a) or (b). By Theorems 4.2 and 4.4, solution of Type (a) does not exist. By Lemma 5.2 Kneser solutions satisfy $x(t)y(t) < 0$ and $x(t)z(t) > 0$ for $t \geq 0$. Thus any solution satisfying $x(t_0)y(t_0)z(t_0) = 0$ is oscillatory. □

Example 5.1. The linear difference system

$$\Delta x(n) = \frac{1}{n+1} y(n),$$
$$\Delta y(n) = \frac{1}{n+1} z(n),$$
$$\Delta z(n) = -\frac{1}{(n+1)^2} x(n)$$

does not have Property A because it possesses a Kneser solution $(x, y, z) = \left(\frac{n+1}{n}, -\frac{1}{n}, \frac{1}{n}\right)$. Theorem 5.2 is not applicable and it is an open problem whether this system is oscillatory.

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