COMPARISON CRITERIA FOR THIRD ORDER FUNCTIONAL DYNAMIC EQUATIONS WITH MIXED NONLINEARITIES

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Abstract. In this paper, we investigate comparison criteria for third order nonlinear dynamic equations with mixed nonlinearities on time scales. Our results are essentially new. Some applications illustrating the importance of our results are included and these applications solve a problem posed in [2, Remark 3.3].

1. Introduction

We investigate comparison criteria for the third order nonlinear dynamic equation with mixed nonlinearities on time scales of the form

$$[r_2(t)\phi_{\gamma_2}([r_1(t)\phi_{\gamma_1}(x^{\Delta}(t))]^{\Delta})]^{\Delta}+p(t)\phi_{\gamma_2}([r_1(t)\phi_{\gamma_1}(x^{\Delta}(t))]^{\Delta^{\sigma}})+f(t,x(t))=0, \tag{1.1}$$

where

$$\begin{aligned} f(t,x(t)) \quad : &= A(t)\phi_{\gamma}(x(h_1(t)))+B(t)\phi_{\beta}(x(h_2(t))) \\ &+\int_a^b q(t,s)\phi_{\alpha(s)}(x(h(t,s)))\Delta\zeta(s), \end{aligned} \tag{1.2}$$

on a time scale $\mathbb{T}$ which is unbounded above, where $-\infty < a < b \leq \infty$ and $r_i \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $i=1,2$, where C_{rd} is the space of right-dense continuous functions; $\phi_\theta(u) := |u|^{\theta-1}u$, $\theta>0$; $\alpha \in C_{rd}\left([a,b)_{\hat{\mathbb{T}}},\mathbb{R}^+\right)$ is strictly increasing such that $0 \leq \alpha(a) < \lambda < \alpha(b-)$ with $\beta > \gamma := \gamma_1\gamma_2 > \lambda > 0$, where $\hat{\mathbb{T}}$ is a time scale; $\zeta \in C_{rd}\left([a,b)_{\hat{\mathbb{T}}},\mathbb{R}\right)$ is nondecreasing; $p \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$; and $A,\ B \in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$ and also $q \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\hat{\mathbb{T}}},[0,\infty)\right)$. The functions $h_1,h_2:\mathbb{T}\to\mathbb{T}$ and $h:\mathbb{T}\times\hat{\mathbb{T}}\to\mathbb{T}$ are rd-continuous functions such that

$$\lim_{t\to\infty}h_1(t)=\lim_{t\to\infty}h_2(t)=\lim_{t\to\infty}h(t,s)=\infty \quad \text{for } s\in\hat{\mathbb{T}}.$$

Here $\int_a^b f(s)\,\Delta\zeta(s)$ denotes the Riemann-Stieltjes integral of the function f on $[a,b)_{\hat{\mathbb{T}}}$ with respect to ζ. We note that as special cases, the integral term in the equation becomes a finite sum when $\zeta(s)$ is a step function and a Riemann integral when $\zeta(s)=s$. For $\hat{\mathbb{T}}=\mathbb{R}$, $n\in\mathbb{N}$, and $s\in[0,n+1)$, we assume that

$$\zeta(s)=\sum_{j=1}^{n}\chi(s-j) \quad \text{with} \quad \chi(s)=\begin{cases}1, & s\geq 0\\ 0, & s<0;\end{cases}$$

$\alpha\in C\,[0,n+1)$ such that $\alpha(j)=\alpha_j$, $j=1,...,n$,

$$\alpha_j<\lambda,\ j=1,2,...,l, \quad \alpha_j>\lambda,\ j=l+1,l+2,...,n; \tag{1.3}$$

$q(t,j)=q_j(t)$ and $h(t,j)=\bar{h}_j(t)$ for $j=1,...,n$. In this case, mixed nonlinearities $f(t,x(t))$ can be written as

$$f(t,x(t))=A(t)\phi_\gamma(x(h_1(t)))+B(t)\phi_\beta(x(h_2(t)))+\sum_{j=1}^{n}q_j(t)\phi_{\alpha_j}(x(\bar{h}_j(t))).$$

Note that we can get that all terms are sublinear, or superlinear, or a combination of sublinear and superlinear depending on different choices of α_i. For more details, see [3, 25]. Throughout this paper, we let

$$x^{[i]}:=r_i\phi_{\gamma_i}([x^{[i-1]}]^\Delta),\ i=1,2,\ \text{with } x^{[0]}=x.$$

In this case, equation (1.1) becomes

$$\left[x^{[2]}(t)\right]^\Delta+p(t)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta^\sigma}\right)+f(t,x(t))=0, \tag{1.4}$$

where $f(t,x(t))$ is defined by (1.2).

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD dissertation written under the direction of Bernd Aulbach (see [23]). Since then a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Recall that a time scale $\mathbb{T}$ is a nonempty, closed subset of the reals, and the cases when this time scale is the reals or the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [7]). Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T}=q^{\mathbb{N}_0}$ (which has important applications in quantum theory (see [24])) and can be applied on different types of time scales such as $\mathbb{T}=h\mathbb{Z}$, $\mathbb{T}=\mathbb{N}_0^2$ and $\mathbb{T}=\mathbb{H}_n$ (the space of harmonic numbers). For an excellent introduction to the calculus on time scales, see Bohner and Peterson [7] and [8].

Although not all solutions of equation (1.4) exist on the whole time scale $\mathbb{T}$ for the asymptotic and oscillation purpose, we are only interested in the solutions that are extendable to ∞. Thus, we use the following definition of solutions.

Definition. By a solution of Eq. (1.4) we mean a nontrivial real–valued function $x \in C_{rd}^{1}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ for some $T_x \geq t_0$ such that $x^{[1]}$, $x^{[2]} \in C_{rd}^{1}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, and $x(t)$ satisfies Eq. (1.4) on $[T_x,\infty)_{\mathbb{T}}$.

Recently, there has been an increasing interest in studying the oscillatory behavior of all order dynamic equations on time scales, we refer the reader to the papers [1,5,6,9–16,18–22,26–34] and the references contained therein.

The study content on the oscillatory and asymptotic behavior of second order dynamic equations on time scales is very rich. In contrast, the study of oscillation criteria of fourth order dynamic equations is relatively less. To the best of our knowledge, the oscillatory behavior of fourth order nonlinear dynamic equations with nonlinear middle term has not been studied till now. Our aim here is to initiate such a study by establishing some new criteria for the oscillation of equation (1.4) and some related equations. Our approach is to reduce the problem in such a way that specific oscillation results for first and second order equations can be adapted for the third order case.

2. Main Results

In the following, we denote by $L_\zeta(a,b)_{\hat{\mathbb{T}}}$ the set of Riemann-Stieltjes integrable functions on $[a,b)_{\hat{\mathbb{T}}}$ with respect to ζ. Let $c \in [a,b)_{\hat{\mathbb{T}}}$ such that $\alpha(c)=\lambda$. We further assume that $\alpha^{-1} \in L_\zeta(a,b)_{\hat{\mathbb{T}}}$ such that

$$0 \leq \alpha(a) < \lambda < \alpha(b-), \ \int_a^c \Delta\zeta(s) > 0 \text{ and } \int_c^b \Delta\zeta(s) > 0.$$

We start with the following two lemmas which generalize Lemma 2.1 and Lemma 2.2 in [20,30].

Lemma 2.1. *There exists $\eta \in L_\zeta(a,b)_{\hat{\mathbb{T}}}$ such that $\eta(s)>0$ on $[a,b)_{\hat{\mathbb{T}}}$,*

$$\int_a^b \alpha(s)\eta(s)\ \Delta\zeta(s) = \lambda \quad \textit{and} \quad \int_a^b \eta(s)\ \Delta\zeta(s) = 1. \tag{2.1}$$

Proof. Let

$$m := \lambda\left(\int_c^b \Delta\zeta(s)\right)^{-1} \int_c^b \alpha^{-1}(s)\ \Delta\zeta(s);$$

$$n := \lambda\left(\int_a^c \Delta\zeta(s)\right)^{-1} \int_a^c \alpha^{-1}(s)\ \Delta\zeta(s);$$

$$\eta_1(s) := \begin{cases} 0, & s \in [a,c)_{\hat{\mathbb{T}}}, \\ \lambda\alpha^{-1}(s)\left(\int_c^b \Delta\zeta(s)\right)^{-1}, & s \in [c,b)_{\hat{\mathbb{T}}}; \end{cases}$$

and

$$\eta_2(s) := \begin{cases} \lambda\alpha^{-1}(s)\left(\int_a^c \Delta\zeta(s)\right)^{-1}, & s \in [a,c)_{\hat{\mathbb{T}}}, \\ 0, & s \in [c,b)_{\hat{\mathbb{T}}}. \end{cases}$$

Clearly for $i=1,2$, $\eta_i \in L_\zeta(a,b)_{\hat{\mathbb{T}}}$ and

$$\int_a^b \alpha(s)\eta_i(s)\ \Delta\zeta(s) = \lambda.$$

Moreover,

$$\int_a^b \eta_1(s)\ \Delta\zeta(s) = m = \lambda \int_c^b \alpha^{-1}(s)\ \Delta\zeta(s) \left(\int_c^b \Delta\zeta(s)\right)^{-1} < 1,$$

and

$$\int_a^b \eta_2(s)\ \Delta\zeta(s) = n = \lambda \int_a^c \alpha^{-1}(s)\ \Delta\zeta(s) \left(\int_a^c \Delta\zeta(s)\right)^{-1} > 1.$$

For $k \in [0,1]$ let

$$\eta(s,k) := (1-k)\,\eta_1(s) + k\eta_2(s), \quad s \in [a,b)_{\hat{\mathbb{T}}}.$$

Then it is easy to see that

$$\int_a^b \alpha(s)\,\eta(s,k)\ \Delta\zeta(s) = \lambda.$$

Furthermore, since $\eta(s,0) = \eta_1(s)$ and $\eta(s,1) = \eta_2(s)$, we have

$$\int_a^b \eta(s,0)\ \Delta\zeta(s) = m \quad \text{and} \quad \int_a^b \eta(s,1)\ \Delta\zeta(s) = n.$$

By the continuous dependence of $\eta(s,k)$ on k there exists $k^* \in (0,1)$ such that $\eta(s) := \eta(s,k^*)$ satisfies

$$\int_a^b \eta(s)\ \Delta\zeta(s) = 1.$$

Note that $\eta(s) > 0$ for $s \in [a,b)_{\hat{\mathbb{T}}}$ and $\int_a^b \alpha(s)\,\eta(s)\ \Delta\zeta(s) = \lambda$. □

Lemma 2.2. *Let* $u \in C_{rd}\left([a,b)_{\hat{\mathbb{T}}}, \mathbb{R}\right)$ *and* $\eta \in L_\zeta(a,b)_{\hat{\mathbb{T}}}$ *satisfy* $u \geq 0$, $\eta > 0$ *on* $[a,b)_{\hat{\mathbb{T}}}$ *and* $\int_a^b \eta(s)\,\Delta\zeta(s) = 1$. *Then*

$$\int_a^b \eta(s)\,u(s)\ \Delta\zeta(s) \geq \exp\left(\int_a^b \eta(s)\ln[u(s)]\ \Delta\zeta(s)\right),$$

where we use the convention that $\ln 0 = -\infty$ *and* $e^{-\infty} = 0$.

Proof. Define an operator L as follows:

$$L(u) := \int_a^b \eta(s)\,u(s)\ \Delta\zeta(s).$$

It is easy to show that L is a linear operator satisfying $L(1) = 1$ and $L(u) > 0$. Since $\ln\theta \leq \theta - 1$ for $\theta > 0$. Then for $t \in [a,b)_{\hat{\mathbb{T}}}$ we obtain

$$\ln\left[\frac{u(s)}{L(u)}\right] \leq \frac{u(s)}{L(u)} - 1,$$

which implies

$$\ln(u(s)) - \ln(L(u)) \leq \frac{u(s)}{L(u)} - 1.$$

It follows that

$$\begin{aligned} L\left[\ln\left(u\left(s\right)\right)-\ln\left(L\left(u\right)\right)\right] &\leq L\left[\frac{u\left(s\right)}{L\left(u\right)}-1\right] \\ &= L\left[\frac{u\left(s\right)}{L\left(u\right)}\right]-L\left(1\right)=1-1=0, \end{aligned}$$

which implies

$$L\left[\ln\left(u\left(s\right)\right)\right]-\ln\left(L\left(u\right)\right)\leq 0,$$

and so

$$L\left(u\right)\geq \exp\left(L\left[\ln\left(u\left(s\right)\right)\right]\right).$$

This completes the proof. □

We will use the following notation:

$$h^*(t) := \sup_{s\in[a,b]_{\hat{\mathbb{T}}}}\{h_1(t),h_2(t),h(t,s)\},\ \ h_*(t) := \inf_{s\in[a,b]_{\hat{\mathbb{T}}}}\{h_1(t),h_2(t),h(t,s)\},$$

and

$$A_1(t) := A(t)R^{\gamma}(h_1(t),h_*(t)),\ B_1(t) := B(t)R^{\beta}(h_2(t),h_*(t)),$$
$$q_1\left(t,s\right) := q\left(t,s\right)R^{\alpha(s)}(h\left(t,s\right),h_*(t)),$$

$$A_2(t) := A(t)\Lambda^{\gamma}\left(h_1(t)\right),\ B_2(t) := B(t)\Lambda^{\beta}\left(h_2(t)\right),$$
$$q_2\left(t,s\right) := q\left(t,s\right)\Lambda^{\alpha(s)}(h\left(t,s\right)),$$

$$A_3(t) := A(t)R^{\gamma}\left(h^*\left(t\right),h_1\left(t\right)\right),\ B_3(t) := B(t)R^{\beta}\left(h^*\left(t\right),h_2\left(t\right)\right),$$
$$q_3\left(t,s\right) := q\left(t,s\right)R^{\alpha(s)}\left(h^*\left(t\right),h\left(t,s\right)\right),$$

$$A_4(t) := A(t)R^{\gamma}\left(h_1(t),T_1\right),\ B_4(t) := B(t)R^{\beta}\left(h_2(t),T_1\right),$$
$$q_4\left(t,s\right) := q\left(t,s\right)R^{\alpha(s)}\left(h(t,s),T_1\right),$$

with

$$R(v,u) := \int_u^v r_1^{-\frac{1}{\gamma_1}}(u)\ \ \Delta u \text{ and } \Lambda(u) := \int_u^{\infty} r_1^{-\frac{1}{\gamma_1}}(u)\ \ \Delta u,$$

and where

$$C_i(t) := \exp\left(\int_a^b \eta(s)\ln\left[\frac{q_i\left(t,s\right)}{\eta(s)}\right]\ \ \Delta\zeta(s)\right),\ \ i=1,2,3,4.$$

First, we use second order dynamic inequalities in order to obtain oscillatory solutions for (1.4).

Theorem 2.1. *If the second order dynamic inequalities*

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)+Q_1(t)\phi_{\gamma_2}\left(y\left(h_*\left(t\right)\right)\right)\leq 0; \quad (2.2)$$

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-Q_2(t)\phi_{\gamma_2}\left(y\left(h_*\left(t\right)\right)\right)\geq 0; \quad (2.3)$$

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-Q_3(t)\phi_{\gamma_2}\left(y\left(h_*\left(t\right)\right)\right)\geq 0; \quad (2.4)$$

and

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)+Q_4(t)\phi_{\gamma_2}\left(y\left(h_*\left(t\right)\right)\right)\leq 0, \quad (2.5)$$

where

$$Q_i(t) := A_i(t) + \delta B_i^{(\gamma_2-\hat{\lambda})/(\hat{\beta}-\hat{\lambda})}(t) C_i^{(\hat{\beta}-\gamma_2)/(\hat{\beta}-\hat{\lambda})}(t), \ i = 1, 2, 3, 4,$$

with $\hat{\beta} := \frac{\beta}{\gamma_1}, \ \hat{\lambda} := \frac{\lambda}{\gamma_1}$ *and*

$$\delta := (\hat{\beta} - \hat{\lambda})(\hat{\beta} - \gamma_2)^{(\gamma_2-\hat{\beta})/(\hat{\beta}-\hat{\lambda})}(\gamma_2 - \hat{\lambda})^{(\hat{\lambda}-\gamma_2)/(\hat{\beta}-\hat{\lambda})},$$

have no eventually positive solutions, then every solution of equation (1.4) *is oscillatory.*

Proof. Assume (1.4) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$, $x(h_i(t)) > 0$ on $[T, \infty)_{\mathbb{T}}$, $i = 1, 2$, and $x(h(t, s)) > 0$ on $[T, \infty)_{\mathbb{T}} \times [a, b]_{\hat{\mathbb{T}}}$. From (1.4), we have for $t \in [T, \infty)_{\mathbb{T}}$,

$$\begin{gathered}\left[x^{[2]}(t)\right]^{\Delta} + p(t)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta^\sigma}\right) = -A(t)\phi_\gamma x(h_1(t)) \\ -B(t)\phi_\beta(x(h_2(t))) - \int_a^b q(t, s)\,\phi_{\alpha(s)}(x(h(t, s)))\ \Delta\zeta(s) \le 0.\end{gathered}$$

Then

$$\begin{aligned}\left(e_{\frac{p}{r_2^\sigma}}(t, t_0)x^{[2]}(t)\right)^{\Delta} &= e_{\frac{p}{r_2^\sigma}}(t, t_0)\left[x^{[2]}(t)\right]^{\Delta} + e_{\frac{p}{r_2^\sigma}}(t, t_0)\frac{p(t)}{r_2^\sigma(t)}x^{[2]}(\sigma(t)) \\ &= e_{\frac{p}{r_2^\sigma}}(t, t_0)\left\{\left[x^{[2]}(t)\right]^{\Delta} + p(t)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta^\sigma}\right)\right\} \le 0.\end{aligned}$$

Then $e_{\frac{p}{r_2^\sigma}}(t, t_0)x^{[2]}(t)$ is nonincreasing on $[T, \infty)_{\mathbb{T}}$ and $x^{[2]}$ is eventually of one sign. Therefore $\left[x^{[0]}\right]^{\Delta}$ and $\left[x^{[1]}\right]^{\Delta}$ are eventually of one sign. Therefore, we consider the following cases:

(I) $\left[x^{[0]}\right]^{\Delta} > 0$ and $\left[x^{[1]}\right]^{\Delta} > 0$ eventually. Then there exists $T_1 \ge T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta} > 0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta} > 0 \qquad \text{for } t \ge T_1.$$

Then for $\tau \ge h_*(t)$,

$$\begin{aligned}x(\tau) &\ge x(\tau) - x(h_*(t)) = \int_{h_*(t)}^{\tau} x^{\Delta}(u)\Delta u \\ &= \int_{h_*(t)}^{\tau} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]\ r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &\ge \phi_{\gamma_1}^{-1}\left[x^{[1]}(h_*(t))\right]\int_{h_*(t)}^{\tau} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}(h_*(t))\right] R(\tau, h_*(t)).\end{aligned}$$

By using this and (1.4), we get

$$\begin{aligned}
&\left\{r_2\left(t\right)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta^{\sigma}}\right)\\
= \;& -A(t)\phi_{\gamma}\left(x(h_1\left(t\right))\right)-B(t)\phi_{\beta}\left(x(h_2\left(t\right))\right)\\
&-\int_a^b q\left(t,s\right)\phi_{\alpha(s)}\left(x(h\left(t,s\right))\right)\ \Delta\zeta\left(s\right)\\
\leq \;& -A_1(t)\phi_{\gamma}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h_*(t)\right)\right]\right]-B_1(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h_*(t)\right)\right]\right]\\
&-\int_a^b q_1\left(t,s\right)\ \phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h_*\left(t\right)\right)\right]\right]\ \Delta\zeta\left(s\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\left\{r_2\left(t\right)\phi_{\gamma_2}\left(\left[y(t)\right]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left[y^{\Delta^{\sigma}}(t)\right]+A_1(t)\phi_{\gamma}\left[\phi_{\gamma_1}^{-1}\left[y\left(h_*(t)\right)\right]\right]\\
&+B_1(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[y\left(h_*(t)\right)\right]\right]+\int_a^b q_1\left(t,s\right)\ \phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y\left(h_*\left(t\right)\right)\right]\right]\ \Delta\zeta\left(s\right)\leq 0,
\end{aligned}$$

or

$$\begin{aligned}
&\left\{r_2\left(t\right)\phi_{\gamma_2}\left(\left[y(t)\right]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left[y^{\Delta^{\sigma}}(t)\right]+A_1(t)y^{\gamma_2}\left(h_*\left(t\right)\right)\\
&+B_1(t)y^{\hat{\beta}}\left(h_*\left(t\right)\right)+y^{\hat{\lambda}}\left(h_*\left(t\right)\right)\int_a^b q_1\left(t,s\right)\ \left[y\left(h_*\left(t\right)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta\left(s\right)\leq 0,
\end{aligned} \tag{2.6}$$

where $y(t)=x^{[1]}\left(t\right)>0$ for $t\in[T_1,\infty)_{\mathbb{T}}$. Now let $\eta\in L_{\zeta}(a,b)_{\hat{\mathbb{T}}}$ be defined as in Lemma 2.1. Then η satisfies (2.1). It follows that

$$\int_a^b \eta(s)\left[\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}\right]\ \Delta\zeta(s)=0.$$

From Lemma 2.2 we get

$$\begin{aligned}
&\int_a^b q_1\left(t,s\right)\ \left[y\left(h_*\left(t\right)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta\left(s\right)\\
= \;& \int_a^b \eta(s)\frac{q_1\left(t,s\right)}{\eta(s)}\left[y\left(h_*\left(t\right)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta\left(s\right)\\
\geq \;& \exp\left(\int_a^b \eta(s)\ln\left(\frac{q_1\left(t,s\right)}{\eta(s)}\left[y\left(h_*\left(t\right)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\right)\ \Delta\zeta(s)\right)\\
= \;& \exp\left(\begin{array}{c}\int_a^b \eta(s)\ln\left[\frac{q_1(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\\ +\ln\left(y\left(h_*\left(t\right)\right)\right)\int_a^b \eta(s)\left[\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}\right]\ \Delta\zeta(s)\end{array}\right)\\
= \;& \exp\left(\int_a^b \eta(s)\ln\left[\frac{q_1\left(t,s\right)}{\eta(s)}\right]\ \Delta\zeta(s)\right)=C_1(t).
\end{aligned} \tag{2.7}$$

This together with (2.6) shows that

$$\begin{aligned}
&\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)+A_1(t)y^{\gamma_2}\left(h_*\left(t\right)\right)\\
&+B_1(t)y^{\hat{\beta}}\left(h_*\left(t\right)\right)+C_1(t)y^{\hat{\lambda}}\left(h_*\left(t\right)\right)\leq 0.
\end{aligned} \tag{2.8}$$

By using the inequality [16, Lemma 2.1] for all $a>0$ and $b\geq 0$,

$$a^{\hat{\beta}-\gamma_2}+ba^{\hat{\lambda}-\gamma_2}\geq \delta b^{(\hat{\beta}-\gamma_2)/(\hat{\beta}-\hat{\lambda})} \quad \text{for all } \hat{\beta}>\gamma_2>\hat{\lambda}>0. \tag{2.9}$$

Define

$$a:=B_1^{1/(\hat{\beta}-\gamma_2)}y \quad \text{and} \quad b:=B_1^{(\gamma_2-\hat{\lambda})/(\hat{\beta}-\gamma_2)}C_1.$$

Then

$$B_1y^{\hat{\beta}-\gamma_2}+C_1y^{\hat{\lambda}-\gamma_2}\geq \delta B_1^{(\gamma_2-\hat{\lambda})/(\hat{\beta}-\hat{\lambda})}C_1^{(\hat{\beta}-\gamma_2)/(\hat{\beta}-\hat{\lambda})}.$$

Therefore (2.8) becomes

$$\left\{r_2\left(t\right)\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)+Q_1(t)\phi_{\gamma_2}\left(y\left(h_*\left(t\right)\right)\right)\leq 0,$$

where $y(t)$ is a solution of the above inequality, which is a contradiction.
(II) $\left[x^{[0]}\right]^{\Delta}<0$ and $\left[x^{[1]}\right]^{\Delta}<0$ eventually. Then there exists $T_1\geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta}<0 \text{ and } \left[x^{[1]}\left(t\right)\right]^{\Delta}<0 \qquad \text{for } t\geq T_1.$$

Then for $\tau\geq T_1$,

$$\begin{aligned} -x\left(\tau\right) &\leq \int_{\tau}^{\infty}x^{\Delta}\left(u\right)\ \Delta u \\ &= \int_{\tau}^{\infty}\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(u\right)\right]\ r_1^{-\frac{1}{\gamma_1}}\left(u\right)\ \Delta u \\ &\leq \phi_{\gamma_1}^{-1}\left[x^{[1]}\left(\tau\right)\right]\ \int_{\tau}^{\infty}r_1^{-\frac{1}{\gamma_1}}\left(u\right)\ \Delta u \\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}\left(\tau\right)\right]\ \Lambda\left(\tau\right). \end{aligned}$$

From this and equation (1.4), we obtain

$$\begin{aligned} &\left\{r_2\left(t\right)\phi_{\gamma_2}\left(\left[-x^{[1]}(t)\right]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(\left[-x^{[1]}(t)\right]^{\Delta^{\sigma}}\right) \\ =\ & -A(t)\phi_{\gamma}\left(-x(h_1\left(t\right))\right)-B(t)\phi_{\beta}\left(-x(h_2\left(t\right))\right) \\ & -\int_a^b q\left(t,s\right)\phi_{\alpha(s)}\left(-x(h\left(t,s\right))\right)\Delta\zeta\left(s\right) \\ \geq\ & -A_2(t)\phi_{\gamma}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h_1\left(t\right)\right)\right]\right]-B_2(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h_2(t)\right)\right]\right] \\ & -\int_a^b q_2\left(t,s\right)\ \phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h\left(t,s\right)\right)\right]\right]\Delta\zeta\left(s\right) \\ =\ & A_2(t)\phi_{\gamma}\left[\phi_{\gamma_1}^{-1}\left[-x^{[1]}\left(h_1\left(t\right)\right)\right]\right]+B_2(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[-x^{[1]}\left(h_2(t)\right)\right]\right] \\ & +\int_a^b q_2\left(t,s\right)\ \phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[-x^{[1]}\left(h\left(t,s\right)\right)\right]\right]\Delta\zeta\left(s\right), \end{aligned}$$

which yields

$$\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^{\sigma}}\right)-A_2(t)\phi_{\gamma_2}\left[y\left(h_1(t)\right)\right]$$
$$-B_2(t)\phi_{\hat{\beta}}\left(y(h_2(t))\right)-\int_a^b q_2(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y\left(h\left(t,s\right)\right)\right]\right]\ \Delta\zeta(s)\geq 0,$$

where $y(t)=-x^{[1]}(t)>0$ for $t\in[T_1,\infty)_{\mathbb{T}}$. By using the fact that y is increasing on $[T_1,\infty)_{\mathbb{T}}$, we get

$$\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^{\sigma}}\right)-A_2(t)\phi_{\gamma_2}\left[y\left(h_*(t)\right)\right]$$
$$-B_2(t)\phi_{\hat{\beta}}\left(y(h_*(t))\right)-\int_a^b q_2(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y\left(h_*(t)\right)\right]\right]\ \Delta\zeta(s)\geq 0.$$

or

$$\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^{\sigma}}\right)-A_2(t)\phi_{\gamma_2}\left[y\left(h_*(t)\right)\right]$$
$$-B_2(t)y^{\hat{\beta}}\left(h_*(t)\right)-y^{\hat{\lambda}}\left(h_*(t)\right)\int_a^b q_2(t,s)\left[y\left(h_*(t)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta(s)\geq 0. \tag{2.10}$$

Then, from (2.7) with q_1 is replaced by q_2, we have

$$\int_a^b q_2(t,s)\left[y\left(h_*(t)\right)\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta(s)\geq C_2(t).$$

Therefore

$$\begin{aligned}\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-A_2(t)\phi_{\gamma_2}\left[y\left(h_*(t)\right)\right]\\ -B_2(t)y^{\hat{\beta}}\left(h_*(t)\right)-C_2(t)y^{\hat{\lambda}}\left(h_*(t)\right)\geq 0.\end{aligned} \tag{2.11}$$

Also, by using the inequality (2.9), (2.11) becomes

$$\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-Q_2(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right)\geq 0,$$

which has an eventually positive solution $y(t)$, which has a contradiction.
(III) $\left[x^{[0]}\right]^{\Delta}<0$ and $\left[x^{[1]}\right]^{\Delta}>0$ eventually. Then there exists $T_1\geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta}<0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta}>0, \qquad \text{for } t\geq T_1.$$

Then for $\tau\leq h^*(t)$,

$$\begin{aligned}-x(\tau) &\leq x\left(h^*(t)\right)-x(\tau)\\ &= \int_{\tau}^{h^*(t)}\phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]\ r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\\ &\leq \phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h^*(t)\right)\right]\int_{\tau}^{h^*(t)} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}\left(h^*(t)\right)\right]\ R\left(h^*(t),\tau\right).\end{aligned}$$

From this and equation (1.4), we have

$$\begin{aligned}
&\left\{r_2(t)\,\phi_{\gamma_2}\left(\left[-x^{[1]}(t)\right]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(\left[-x^{[1]}(t)\right]^{\Delta^{\sigma}}\right)\\
= \;& -A(t)\phi_{\gamma}\left(-x(h_1(t))\right)-B(t)\phi_{\beta}\left(-x(h_2(t))\right)\\
&-\int_a^b q(t,s)\,\phi_{\alpha(s)}\left(-x(h(t,s))\right)\Delta\zeta(s)\\
\geq \;& -A_3(t)\phi_{\gamma}\left(\phi_{\gamma_1}^{-1}\left[x^{[1]}(h^*(t))\right]\right)-B_3(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}(h^*(t))\right]\right]\\
&-\int_a^b q_3(t,s)\;\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}(h^*(t))\right]\right]\;\Delta\zeta(s)\\
= \;& A_3(t)\phi_{\gamma}\left(\phi_{\gamma_1}^{-1}\left[-x^{[1]}(h^*(t))\right]\right)+B_3(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[-x^{[1]}(h^*(t))\right]\right]\\
&+\int_a^b q_3(t,s)\;\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[-x^{[1]}(h^*(t))\right]\right]\;\Delta\zeta(s),
\end{aligned}$$

which yields

$$\begin{aligned}
&\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^{\sigma}}\right)-A_3(t)\phi_{\gamma}\left[\phi_{\gamma_1}^{-1}\left[y(h^*(t))\right]\right]\\
&-B_3(t)\phi_{\beta}\left[\phi_{\gamma_1}^{-1}\left[y(h^*(t))\right]\right]-\int_a^b q_3(t,s)\;\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y(h^*(t))\right]\right]\;\Delta\zeta(s)\geq 0,
\end{aligned}$$

or

$$\begin{aligned}
&\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^{\sigma}}\right)-A_3(t)\phi_{\gamma_2}\left[y(h^*(t))\right]\\
&-B_3(t)\left[y(h^*(t))\right]^{\hat{\beta}}-\left[y(h^*(t))\right]^{\hat{\lambda}}\int_a^b q_3(t,s)\left[y(h^*(t))\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\;\Delta\zeta(s)\geq 0.
\end{aligned} \tag{2.12}$$

where $y(t)=-x^{[1]}(t)>0$ for $t\in[T_1,\infty)_{\mathbb{T}}$. Then, from (2.7) with q_1 is replaced by q_3, we have

$$\int_a^b q_3(t,s)\left[y(h_*(t))\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\;\Delta\zeta(s)\geq C_3(t). \tag{2.13}$$

Then, from (2.12) and (2.13), we get

$$\begin{aligned}
\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-A_3(t)\phi_{\gamma_2}\left[y(h^*(t))\right]\\
-B_3(t)y^{\hat{\beta}}(h^*(t))-C_3(t)y^{\hat{\lambda}}(h_*(t))\geq 0.
\end{aligned}$$

In view of (2.9), we get

$$\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)-Q_3(t)\phi_{\gamma_2}\left(y(h_*(t))\right)\geq 0,$$

which has an eventually positive solution $y(t)$, which is a contradiction.
(IV) $\left[x^{[0]}\right]^{\Delta}>0$ and $\left[x^{[1]}\right]^{\Delta}<0$ eventually. Then there exists $T_1\geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta}>0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta}<0 \qquad \text{for } t\geq T_1.$$

Then for $\tau \geq T_1$,

$$\begin{aligned} x(\tau) &\geq x(\tau) - x(T_1) \\ &= \int_{T_1}^{\tau} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right] r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &\geq \phi_{\gamma_1}^{-1}\left[x^{[1]}(\tau)\right] \int_{T_1}^{\tau} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &= \phi_{\gamma_1}^{-1}\left[x^{[1]}(\tau)\right] R(\tau, T_1). \end{aligned}$$

From this and equation (1.4), we have

$$\begin{aligned} &\left\{r_2(t)\,\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta}\right)\right\}^{\Delta} + p(t)\phi_{\gamma_2}\left(\left[x^{[1]}(t)\right]^{\Delta^\sigma}\right) \\ = &\ -A(t)\phi_\gamma\left(x(h_1(t))\right) - B(t)\phi_\beta\left(x(h_2(t))\right) \\ &\ -\int_a^b q(t,s)\,\phi_{\alpha(s)}\left(x(h(t,s))\right)\Delta\zeta(s) \\ \leq &\ -A_4(t)\phi_\gamma\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}(h_1(t))\right]\right] - B_4(t)\phi_\beta\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}(h_2(t))\right]\right] \\ &\ -\int_a^b q_4(t,s)\ \phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[x^{[1]}(h(t,s))\right]\right]\ \Delta\zeta(s), \end{aligned}$$

which yields

$$\begin{aligned} &\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta} + p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^\sigma}\right) + A_4(t)\phi_\gamma\left[\phi_{\gamma_1}^{-1}\left[y(h_2(t))\right]\right] \\ &+B_4(t)\phi_\beta\left[\phi_{\gamma_1}^{-1}\left[y(h_2(t))\right]\right] + \int_a^b q_4(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y(h(t,s))\right]\right]\ \Delta\zeta(s) \leq 0, \end{aligned}$$

where $y(t) = x^{[1]}(t) > 0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. By using the fact that y is decreasing on $[T_1, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} &\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta} + p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^\sigma}\right) + A_4(t)\phi_\gamma\left[\phi_{\gamma_1}^{-1}\left[y(h_*(t))\right]\right] \\ &+B_4(t)\phi_\beta\left[\phi_{\gamma_1}^{-1}\left[y(h_*(t))\right]\right] + \int_a^b q_4(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma_1}^{-1}\left[y(h_*(t))\right]\right]\ \Delta\zeta(s) \leq 0, \end{aligned}$$

or

$$\begin{aligned} &\left\{r_2(t)\,\phi_{\gamma_2}\left([y(t)]^{\Delta}\right)\right\}^{\Delta} + p(t)\phi_{\gamma_2}\left([y(t)]^{\Delta^\sigma}\right) + A_4(t)\phi_{\gamma_2}\left[y(h_*(t))\right] \\ &+B_4(t)y^{\hat{\beta}}(h_*(t)) + y^{\hat{\lambda}}(h_*(t)) \int_a^b q_4(t,s)\left[y(h_*(t))\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta(s) \leq 0. \end{aligned} \tag{2.14}$$

Again, from (2.7) with q_1 replaced by q_4, we have

$$\int_a^b q_4(t,s)\left[y(h_*(t))\right]^{\frac{\alpha(s)}{\gamma_1}-\hat{\lambda}}\ \Delta\zeta(s) \geq C_4(t).$$

Then

$$\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^\sigma}(t)\right)+A_4(t)\phi_{\gamma_2}\left[y\left(h_*(t)\right)\right]$$
$$+B_4(t)y^{\hat{\beta}}\left(h_*(t)\right)+C_4(t)y^{\hat{\lambda}}\left(h_*(t)\right)\leq 0,$$

which yields, from inequality (2.9),

$$\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+p(t)\phi_{\gamma_2}\left(y^{\Delta^\sigma}(t)\right)+Q_4(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right)\leq 0$$

which has an eventually positive solution $y(t)$, which is a contradiction. This completes the proof. □

Next, we will reduce the equation (1.4) with $p\equiv 0$ to the following first order linear dynamic equation

$$\left[x^{[2]}(t)\right]^{\Delta}+f(t,x(t))=0. \tag{2.15}$$

Then we use first order dynamic inequalities in order to obtain oscillatory solutions for equation (2.15). For simplicity, we will use the following notations: for any $T_1\in[T,\infty)_{\mathbb{T}}$,

$$\bar{A}_1(t):=A(t)R_1^{\gamma}\left(h_1(t),T_1\right),\ \bar{B}_1(t):=B(t)R_1^{\beta}\left(h_2(t),T_1\right)$$
$$\bar{q}_1(t,s):=q(t,s)R_1^{\alpha(s)}\left(h(t,s),T_1\right);$$

$$\bar{A}_2(t):=A(t)R_2^{\gamma}\left(h_1(t),h_*(t)\right),\ \bar{B}_2(t):=B(t)R_2^{\beta}\left(h_2(t),h_*(t)\right)$$
$$\bar{q}_2(t,s):=q(t,s)R_2^{\alpha(s)}\left(h(t,s),h_*(t)\right);$$

$$\bar{A}_3(t):=A(t)R_3^{\gamma}(h_1(t),\bar{h}(t)),\ \bar{B}_3(t):=B(t)R_3^{\beta}(h_2(t),\bar{h}(t))$$
$$\bar{q}_3(t,s):=q(t,s)\,R_3^{\alpha(s)}(h(t,s),\bar{h}(t));$$

$$\bar{A}_4(t):=A(t)R_4^{\gamma}(h_1(t),T_1),\ \bar{B}_4(t):=B(t)R_4^{\beta}(h_2(t),T_1),$$
$$\bar{q}_4(t,s):=q(t,s)\,R_4^{\alpha(s)}(h(t,s),T_1)$$

where $\bar{h}$ is a function such that $\bar{h}(t)\geq h^*(t)$ for $t\in[T,\infty)_{\mathbb{T}}$ and

$$R_1(\tau,T_1):=\int_{T_1}^{\tau}\left[\frac{1}{r_1(v)}\int_{T_1}^{v}r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}}\ \Delta v;$$

$$R_2(\tau,h_*(t)):=\int_{\tau}^{\infty}r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{h_*(t)}^{\tau}r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}};$$

$$R_3(\tau,\bar{h}(t)):=\int_{\tau}^{h^*(t)}r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{h^*(t)}^{\bar{h}(t)}r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}};$$

$$R_4(\tau,T_1):=\int_{T_1}^{\tau}r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{\tau}^{\infty}r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}},$$

and where

$$\bar{C}_i(t):=\exp\left(\int_a^b\eta(s)\ln\left[\frac{\bar{q}_i(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\right),\ i=1,2,3,4.$$

Theorem 2.2. *If the first order dynamic inequalities*

$$z^{\Delta}(t)+\bar{Q}_1(t)z(h^*(t))\leq 0; \tag{2.16}$$

$$z^{\Delta}(t)-\bar{Q}_2(t)z\left(h_*(t)\right)\geq 0; \tag{2.17}$$

$$z^{\Delta}(t)+\bar{Q}_3(t)z\left(\bar{h}(t)\right)\leq 0; \tag{2.18}$$

and

$$z^{\Delta}(t)-\bar{Q}_4(t)z\left(h_*(t)\right)\geq 0, \tag{2.19}$$

where

$$\bar{Q}_i(t):=\bar{A}_i(t)+\bar{\delta}\bar{B}_i^{(1-\bar{\lambda})/(\bar{\beta}-\bar{\lambda})}\bar{C}_i^{(\bar{\beta}-1)/(\bar{\beta}-\bar{\lambda})},\ i=1,2,3,4, \tag{2.20}$$

with $\bar{\beta}:=\frac{\beta}{\gamma},\ \bar{\lambda}:=\frac{\lambda}{\gamma}$ *and*

$$\bar{\delta}:=(\bar{\beta}-\bar{\lambda})(\bar{\beta}-1)^{(1-\bar{\beta})/(\bar{\beta}-\bar{\lambda})}(1-\bar{\lambda})^{(\bar{\lambda}-1)/(\bar{\beta}-\bar{\lambda})},$$

have no eventually positive solutions, then equation (2.15) *is oscillatory.*

Proof. Assume (2.15) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t)>0,\ x(h_i(t))>0$ on $[T,\infty)_{\mathbb{T}},\ i=1,2,$ and $x\left(h\left(t,s\right)\right)>0$ on $[T,\infty)_{\mathbb{T}}\times[a,b]_{\hat{\mathbb{T}}}$. As seen in the proof of Theorem 2.1, we have $x^{[2]}(t)$ is nonincreasing on $[T,\infty)_{\mathbb{T}}$. Notice that $e_{\frac{p}{r^{\sigma}}}(t,t_0)$ disappears since $p\equiv 0$, and also $\left[x^{[0]}\right]^{\Delta}$ and $\left[x^{[1]}\right]^{\Delta}$ are eventually of one sign. Therefore, we consider the following cases:

(I) $\left[x^{[0]}\right]^{\Delta}>0$ and $\left[x^{[1]}\right]^{\Delta}>0$ eventually. Then there exists $T_1\geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta}>0 \text{ and } \left[x^{[1]}\left(t\right)\right]^{\Delta}>0 \qquad \text{for } t\geq T_1.$$

Then for $\tau\geq T_1$

$$\begin{aligned} x^{[1]}\left(\tau\right) &\geq x^{[1]}\left(\tau\right)-x^{[1]}\left(T_1\right)=\int_{T_1}^{\tau}\left(x^{[1]}\left(u\right)\right)^{\Delta}\Delta u \\ &= \int_{T_1}^{\tau}\phi_{\gamma_2}^{-1}\left[x^{[2]}\left(u\right)\right]\ r_2^{-\frac{1}{\gamma_2}}\left(u\right)\ \Delta u \\ &\geq \phi_{\gamma_2}^{-1}\left[x^{[2]}\left(\tau\right)\right]\int_{T_1}^{\tau}r_2^{-\frac{1}{\gamma_2}}\left(u\right)\ \Delta u. \end{aligned}$$

Hence,

$$x^{\Delta}(\tau)\geq\phi_{\gamma}^{-1}\left[x^{[2]}\left(\tau\right)\right]\ \left[\frac{1}{r_1\left(\tau\right)}\int_{T_1}^{\tau}r_2^{-\frac{1}{\gamma_2}}\left(u\right)\ \Delta u\right]^{\frac{1}{\gamma_1}}.$$

Similarly, we see

$$\begin{aligned} x(\tau) &\geq \phi_{\gamma}^{-1}\left[x^{[2]}\left(\tau\right)\right]\ \int_{T_1}^{\tau}\left[\frac{1}{r_1\left(v\right)}\int_{T_1}^{v}r_2^{-\frac{1}{\gamma_2}}\left(u\right)\ \Delta u\right]^{\frac{1}{\gamma_1}}\ \Delta v \\ &= \phi_{\gamma}^{-1}\left[x^{[2]}\left(\tau\right)\right]\ R_1\left(\tau,T_1\right). \end{aligned}$$

Let for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $h_i(t) \geq T_1,\ i = 1, 2$ and $h(t,s) \geq T_1$ for $t \geq T_2$ and $s \in [a,b]_{\hat{\mathbb{T}}}$. Then for $t \geq T_2$,

$$x(h_i(t)) \geq \phi_\gamma^{-1}\left[x^{[2]}(h_i(t))\right] R_1(h_i(t), T_1),\ i = 1, 2,$$

and

$$x(h(t,s)) \geq \phi_\gamma^{-1}\left[x^{[2]}(h(t,s))\right] R_1(h(t,s), T_1).$$

By using (2.15), we get

$$\begin{aligned}
\left[x^{[2]}(t)\right]^\Delta &= -A(t)\phi_\gamma(x(h_1(t))) - B(t)\phi_\beta(x(h_2(t))) \\
&\quad - \int_a^b q(t,s)\,\phi_{\alpha(s)}(x(h(t,s)))\,\Delta\zeta(s) \\
&\leq -\bar{A}_1(t)x^{[2]}(h_1(t)) - \bar{B}_1(t)\phi_\beta\left[\phi_\gamma^{-1}\left[x^{[2]}(h_2(t))\right]\right] \\
&\quad - \int_a^b \bar{q}_1(t,s)\,\phi_{\alpha(s)}\left[\phi_\gamma^{-1}\left[x^{[2]}(h(t,s))\right]\right]\Delta\zeta(s) \\
&\leq -\bar{A}_1(t)x^{[2]}(h^*(t)) - \bar{B}_1(t)\phi_\beta\left[\phi_\gamma^{-1}\left[x^{[2]}(h^*(t))\right]\right] \\
&\quad - \int_a^b \bar{q}_1(t,s)\,\phi_{\alpha(s)}\left[\phi_\gamma^{-1}\left[x^{[2]}(h^*(t))\right]\right]\Delta\zeta(s),
\end{aligned}$$

which yields

$$\begin{aligned}
&z^\Delta(t) + \bar{A}_1(t)z(h^*(t)) + \bar{B}_1(t)\phi_\beta\left[\phi_\gamma^{-1}[z(h^*(t))]\right] \\
&+ \int_a^b \bar{q}_1(t,s)\ \phi_{\alpha(s)}\left[\phi_\gamma^{-1}[z(h^*(t))]\right]\ \Delta\zeta(s) \leq 0,
\end{aligned}$$

where $z(t) = x^{[2]}(t) > 0$ for $t \in [T_2, \infty)_{\mathbb{T}}$. As seen in the proof of Theorem 2.1, we obtain

$$\begin{aligned}
\int_a^b \bar{q}_1(t,s)\ [z(h^*(t))]^{\frac{\alpha(s)}{\gamma} - \bar{\lambda}}\ \Delta\zeta(s) &\geq \exp\left(\int_a^b \eta(s)\ln\left[\frac{\bar{q}_1(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\right) \\
&= \bar{C}_1(t). \qquad (2.21)
\end{aligned}$$

Then

$$z^\Delta(t) + \bar{A}_1(t)z(h^*(t)) + \bar{B}_1(t)z^{\bar{\beta}}(h^*(t)) + \bar{C}_1(t)z^{\bar{\lambda}}(h^*(t)) \leq 0. \qquad (2.22)$$

By using inequality (2.9), we get for $\bar{\beta} > 1 > \bar{\lambda} > 0$,

$$\bar{B}_1(t)z^{\bar{\beta}-1}(h^*(t)) + \bar{C}_1(t)z^{\bar{\lambda}-1}(h^*(t)) \geq \bar{\delta}\bar{B}_1^{(1-\bar{\lambda})/(\bar{\beta}-\bar{\lambda})}\bar{C}_1^{(\bar{\beta}-1)/(\bar{\beta}-\bar{\lambda})}.$$

Therefore (2.22) becomes

$$z^\Delta(t) + \bar{Q}_1(t)z(h^*(t)) \leq 0 \qquad \text{for } t \in [T_2, \infty)_{\mathbb{T}}.$$

We have shown that the above inequality has an eventually positive solution, which is a contradiction.

(II) $\left[x^{[0]}\right]^\Delta < 0$ and $\left[x^{[1]}\right]^\Delta < 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^\Delta < 0 \text{ and } \left[x^{[1]}(t)\right]^\Delta < 0 \qquad \text{for } t \geq T_1.$$

Then for $\tau \geq T_1$, we have

$$\begin{aligned} -x(\tau) &\leq \int_\tau^\infty x^\Delta(u)\ \Delta u \\ &= \int_\tau^\infty \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]\ r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &< \phi_{\gamma_1}^{-1}\left[x^{[1]}(\tau)\right]\ \int_\tau^\infty r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u. \end{aligned} \tag{2.23}$$

Also for $\tau \geq h_*(t)$,

$$\begin{aligned} x^{[1]}(\tau) &\leq x^{[1]}(\tau) - x^{[1]}(h_*(t)) \\ &= \int_{h_*(t)}^\tau \phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right]\ r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u \\ &\leq \phi_{\gamma_2}^{-1}\left[x^{[2]}(h_*(t))\right]\int_{h_*(t)}^\tau r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u. \end{aligned} \tag{2.24}$$

By (2.23) and (2.24), we find

$$\begin{aligned} -x(\tau) &\leq \phi_\gamma^{-1}\left[x^{[2]}(h_*(t))\right]\int_\tau^\infty r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{h_*(t)}^\tau r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}} \\ &= \phi_\gamma^{-1}\left[x^{[2]}(h_*(t))\right]R_2(\tau, h_*(t)). \end{aligned}$$

From this and equation (2.15), we have

$$\begin{aligned} &\left[-x^{[2]}(t)\right]^\Delta = -A(t)\phi_\gamma(-x(h_1(t))) - B(t)\phi_\beta(-x(h_2(t))) \\ &\quad - \int_a^b q(t,s)\,\phi_{\alpha(s)}(-x(h(t,s)))\,\Delta\zeta(s) \\ \geq\ & -\bar{A}_2(t)x^{[2]}(h_*(t)) - \bar{B}_2(t)\phi_\beta\left[\phi_\gamma^{-1}\left[x^{[2]}(h_*(t))\right]\right] \\ &\quad - \int_a^b \bar{q}_2(t,s)\,\phi_{\alpha(s)}\left[\phi_\gamma^{-1}\left[x^{[2]}(h_*(t))\right]\right]\Delta\zeta(s) \\ =\ & \bar{A}_2(t)\left(-x^{[2]}(h_*(t))\right) + \bar{B}_2(t)\phi_\beta\left[\phi_\gamma^{-1}\left[-x^{[2]}(h_*(t))\right]\right] \\ &\quad + \int_a^b \bar{q}_2(t,s)\,\phi_{\alpha(s)}\left[\phi_\gamma^{-1}\left[-x^{[2]}(h_*(t))\right]\right]\Delta\zeta(s), \end{aligned}$$

which yields

$$\begin{gathered} z^\Delta(t) - \bar{A}_2(t)z(h_*(t)) - \bar{B}_2(t)\phi_\beta\left[\phi_\gamma^{-1}\left[z(h_*(t))\right]\right] \\ - \int_a^b \bar{q}_2(t,s)\,\phi_{\alpha(s)}\left[\phi_\gamma^{-1}\left[z(h_*(t))\right]\right]\Delta\zeta \geq 0, \end{gathered}$$

where $z(t) = -x^{[2]}(t) > 0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. As shown in the proof of Theorem 2.1, we have

$$\begin{aligned}\int_a^b \bar{q}_2(t,s)\ [z(h_*(t))]^{\frac{\alpha(s)}{\gamma}-\bar{\lambda}}\ \Delta\zeta(s) &\geq \exp\left(\int_a^b \eta(s)\ln\left[\frac{\bar{q}_2(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\right) \\ &= \bar{C}_2(t),\end{aligned}$$

and so

$$z^{\Delta}(t) - \bar{A}_2(t)z(h_*(t)) - \bar{B}_2(t)z^{\bar{\beta}}(h_*(t)) - \bar{C}_2(t)z^{\bar{\lambda}}(h_*(t)) \geq 0.$$

By using the inequality (2.9), we get

$$z^{\Delta}(t) - \bar{Q}_2(t)z(h_*(t)) \geq 0,$$

where $z(t)$ is a positive solution of the above inequality, which is a contradiction.

(III) $\left[x^{[0]}\right]^{\Delta} < 0$ and $\left[x^{[1]}\right]^{\Delta} > 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta} < 0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta} > 0 \qquad \text{for } t \geq T_1.$$

Then for $\tau \leq h^*(t)$,

$$\begin{aligned}-x(\tau) &\leq x(h^*(t)) - x(\tau) \\ &= \int_\tau^{h^*(t)} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]\ r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &\leq \phi_{\gamma_1}^{-1}\left[x^{[1]}(h^*(t))\right]\int_\tau^{h^*(t)} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u. \end{aligned} \tag{2.25}$$

Also

$$\begin{aligned}-x^{[1]}(h^*(t)) &> x^{[1]}\left(\bar{h}(t)\right) - x^{[1]}(h^*(t)) \\ &= \int_{h^*(t)}^{\bar{h}(t)} \phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right]\ r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u \\ &\geq \phi_{\gamma_2}^{-1}\left[x^{[2]}\left(\bar{h}(t)\right)\right]\int_{h^*(t)}^{\bar{h}(t)} r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u. \end{aligned} \tag{2.26}$$

By (2.25) and (2.26), we find

$$\begin{aligned}x(\tau) &> \phi_{\gamma}^{-1}\left[x^{[2]}\left(\bar{h}(t)\right)\right]\int_\tau^{h^*(t)} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{h^*(t)}^{\bar{h}(t)} r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}} \\ &= \phi_{\gamma}^{-1}\left[x^{[2]}\left(\bar{h}(t)\right)\right]\ R_3(\tau, \bar{h}(t)).\end{aligned}$$

From this and equation (2.15), we have

$$\begin{aligned}\left[x^{[2]}(t)\right]^{\Delta} &= -A(t)\phi_{\gamma}\left(x(h_1(t))\right) - B(t)\phi_{\beta}\left(x(h_2(t))\right) \\ &\quad -\int_a^b q(t,s)\,\phi_{\alpha(s)}\left(x(h(t,s))\right)\Delta\zeta(s) \\ &\leq -\bar{A}_3(t)x^{[2]}\left(\bar{h}(t)\right) - \bar{B}_3(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[x^{[2]}\left(\bar{h}(t)\right)\right]\right] \\ &\quad -\int_a^b \bar{q}_3(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma}^{-1}\left[x^{[2]}\left(\bar{h}(t)\right)\right]\right]\Delta\zeta(s),\end{aligned}$$

which yields

$$\begin{array}{c} z^{\Delta}(t) + \bar{A}_3(t)z\left(\bar{h}(t)\right) + \bar{B}_3(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[z\left(\bar{h}(t)\right)\right]\right] \\ +\int_a^b \bar{q}_3(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma}^{-1}\left[z\left(\bar{h}(t)\right)\right]\right]\Delta\zeta(s) \leq 0, \end{array}$$

where $z(t) = x^{[2]}(t) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. As seen in the proof of Theorem 2.1, we have

$$\begin{aligned}\int_a^b \bar{q}_3(t,s)\ \left[z\left(\bar{h}(t)\right)\right]^{\frac{\alpha(s)}{\gamma}-\bar{\lambda}}\ \Delta\zeta(s) &\geq \exp\left(\int_a^b \eta(s)\ln\left[\frac{\bar{q}_3(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\right) \\ &= \,= \bar{C}_3(t),\end{aligned}$$

which yields

$$z^{\Delta}(t) + \bar{A}_3(t)z\left(\bar{h}(t)\right) + \bar{B}_3(t)z^{\bar{\beta}}(\bar{h}(t)) + \bar{C}_3(t)z^{\bar{\lambda}}\left(\bar{h}(t)\right) \leq 0.$$

By using inequality (2.9), we get

$$z^{\Delta}(t) + \bar{Q}_3(t)z\left(\bar{h}(t)\right) \leq 0,$$

where $z(t)$ is a positive solution of the above inequality, which is a contradiction.
(IV) $\left[x^{[0]}\right]^{\Delta} > 0$ and $\left[x^{[1]}\right]^{\Delta} < 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta} > 0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta} < 0 \qquad \text{for } t \geq T_1.$$

Then for $\tau \geq T_1$,

$$\begin{aligned} x(\tau) &> x(\tau) - x(T_1) \\ &= \int_{T_1}^{\tau} \phi_{\gamma_1}^{-1}\left[x^{[1]}(u)\right]\ r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u \\ &> \phi_{\gamma_1}^{-1}\left[x^{[1]}(\tau)\right]\int_{T_1}^{\tau} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u. \end{aligned} \tag{2.27}$$

Also

$$\begin{aligned}-x^{[1]}(\tau) &\leq \int_{\tau}^{\infty}\left(x^{[1]}(u)\right)^{\Delta}\ \Delta u\\ &= \int_{\tau}^{\infty}\phi_{\gamma_2}^{-1}\left[x^{[2]}(u)\right]\ r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\\ &< \phi_{\gamma_2}^{-1}\left[x^{[2]}(\tau)\right]\int_{\tau}^{\infty} r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u. \end{aligned} \tag{2.28}$$

By (2.27) and (2.28), we find

$$\begin{aligned} x(\tau) &> -\phi_{\gamma}^{-1}\left[x^{[2]}(\tau)\right]\int_{T_1}^{\tau} r_1^{-\frac{1}{\gamma_1}}(u)\ \Delta u\ \left[\int_{\tau}^{\infty} r_2^{-\frac{1}{\gamma_2}}(u)\ \Delta u\right]^{\frac{1}{\gamma_1}}\\ &= -\phi_{\gamma}^{-1}\left[x^{[2]}(\tau)\right]R_4(\tau,T_1).\end{aligned}$$

Let for sufficiently large $T_2\in[T_1,\infty)_{\mathbb{T}}$ such that $h_i(t)\geq T_1,\ i=1,2$ and $h(t,s)\geq T_1$ for $t\geq T_2$ and $s\in[a,b]_{\hat{\mathbb{T}}}$. From this and equation (2.15), we have

$$\begin{aligned} &\left[-x^{[2]}(t)\right]^{\Delta} = A(t)\phi_{\gamma}\left(x(h_1(t))\right)+B(t)\phi_{\beta}\left(x(h_2(t))\right)\\ &\quad+\int_a^b q(t,s)\,\phi_{\alpha(s)}\left(x(h(t,s))\right)\Delta\zeta(s)\\ \geq\ & -\bar{A}_4(t)x^{[2]}(h_1(t))+\bar{B}_4(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[-x^{[2]}(h_2(t))\right]\right]\\ &\quad+\int_a^b \bar{q}_4(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma}^{-1}\left[-x^{[2]}(h(t,s))\right]\right]\Delta\zeta(s)\\ \geq\ & -\bar{A}_4(t)x^{[2]}(h_*(t))+\bar{B}_4(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[-x^{[2]}(h_*(t))\right]\right]\\ &\quad+\int_a^b \bar{q}_4(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma}^{-1}\left[-x^{[2]}(h_*(t))\right]\right]\Delta\zeta(s),\end{aligned}$$

which yields

$$\begin{aligned} z^{\Delta}(t)-\bar{A}_4(t)z(h_*(t))-\bar{B}_4(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[z(h_*(t))\right]\right]\\ -\textstyle\int_a^b \bar{q}_4(t,s)\,\phi_{\alpha(s)}\left[\phi_{\gamma}^{-1}\left[z(h_*(t))\right]\right]\Delta\zeta(s)\geq 0,\end{aligned}$$

where $z(t)=-x^{[2]}(t)>0$ for $t\in[T_2,\infty)_{\mathbb{T}}$. Again as shown in the proof of Theorem 2.1, we have

$$\int_a^b \bar{q}_4(t,s)\ \left[z(h_*(t))\right]^{\frac{\alpha(s)}{\gamma}-\bar{\lambda}}\ \Delta\zeta(s)\geq\exp\left(\int_a^b \eta(s)\ln\left[\frac{\bar{q}_4(t,s)}{\eta(s)}\right]\ \Delta\zeta(s)\right)=\bar{C}_4(t),$$

which implies

$$z^{\Delta}(t)-\bar{A}_4(t)z(h_*(t))-\bar{B}_4(t)\phi_{\beta}\left[\phi_{\gamma}^{-1}\left[z(h_*(t))\right]\right]-\bar{C}_4(t)z^{\bar{\lambda}}(h_*(t))\geq 0.$$

By using the inequality (2.9), we get

$$z^{\Delta}(t)-\bar{Q}_4(t)z(h_*(t))\geq 0,$$

where $z(t)$ is a solution of the above inequality, which is a contradiction. This completes the proof. $\square$

3. Applications

In this section, we highlight the importance of our main results obtained in the previous section. The following results are new and solve an open problem posed in [2, Remark 3.3] when $h_*(t) < t$ for $t \geq t_0 \in \mathbb{T}$. In order to do that we use Theorems 2.1 and 2.2 to obtain some different sufficient oscillation criteria for equations (1.4) and (2.15), respectively.

Theorem 3.1. *Let $h_*^{\Delta}(t) > 0$ and $h_*(t) < t$ for $t \geq t_0 \in \mathbb{T}$. Assume that*

$$\int_T^{\infty} Q_1(u)\Delta u = \infty, \tag{3.1}$$

$$\int_T^{\infty} \left\{ \frac{1}{r_1(w)} \int_T^{w} \left[\frac{1}{\hat{r}_2(v)} \int_T^{v} \hat{Q}_2(u)\Delta u \right]^{1/\gamma_2} \Delta v \right\}^{1/\gamma_1} \Delta w = \infty, \tag{3.2}$$

$$\limsup_{t\to\infty} \int_{h_*(t)}^{t} \hat{Q}_3(u) \left(\int_{h_*(u)}^{h_*(t)} \frac{\Delta v}{\hat{r}_2^{1/\gamma_2}(v)} \right)^{\gamma_2} \Delta u > 1, \tag{3.3}$$

and

$$\int_T^{\infty} \left[\frac{1}{\hat{r}_2(w)} \int_T^{w} \bar{r}^{\gamma_2}(v)\hat{Q}_4(v)\Delta v \right]^{1/\gamma_2} \Delta w = \infty, \tag{3.4}$$

for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, where

$$\bar{r}(v) := \int_{h_*(v)}^{\infty} \frac{\Delta u}{\hat{r}_2^{1/\gamma_2}(u)} \quad \textit{with } \hat{r}_2(u) := r_2(u) e_{\frac{p}{r_2^{\sigma}}}(u, t_0),$$

and

$$\hat{Q}_i(u) := Q_i(u) e_{\frac{p}{r_2^{\sigma}}}(u, t_0) \qquad \textit{for } i = 2, 3, 4.$$

Then equation (1.4) is oscillatory.

Proof. Assume (1.4) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$, $x(h_i(t)) > 0$ on $[T, \infty)_{\mathbb{T}}$, $i = 1, 2$, and $x(h(t, s)) > 0$ on $[T, \infty)_{\mathbb{T}} \times [a, b]_{\hat{\mathbb{T}}}$. As shown in the proof of Theorem 2.1 we have $x^{[2]}$ is eventually of one sign. Therefore $\left[x^{[0]}\right]^{\Delta}$ and $\left[x^{[1]}\right]^{\Delta}$ are eventually of one sign. Therefore, we consider the following cases:

(I) $\left[x^{[0]}\right]^{\Delta} > 0$ and $\left[x^{[1]}\right]^{\Delta} > 0$ eventually. As seen in the proof of Theorem 2.1 we obtain that the second order dynamic inequality (2.2) has a positive solution $y(t) = x^{[1]}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$ for sufficiently large $T_1 \in [T, \infty)_{\mathbb{T}}$. Pick $T_2 \geq T_1$ such that $h_*(t) \geq T_1$ for $t \geq T_2$. Then by using the fact $y^{\Delta}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$, then $y(h_*(t)) > y(T_1)$ for $t \geq T_2$ and then

$$\phi_{\gamma_2}(y(h_*(t))) > \phi_{\gamma_2}(y(T_1)) =: L > 0.$$

Inequality (2.2) becomes

$$\begin{aligned} -\left\{r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta} &\geq p(t)\phi_{\gamma_2}\left(y^{\Delta^{\sigma}}(t)\right)+Q_1(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right) \\ &\geq LQ_1(t). \end{aligned} \tag{3.5}$$

Replacing t by u in (3.5), and integrating (3.5) from T_2 to $t \in [T_2,\infty)_{\mathbb{T}}$ we obtain

$$-r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)+r_2(T_2)\,\phi_{\gamma_2}\left(y^{\Delta}(T_2)\right) \geq L\int_{T_2}^{t} Q_1(u)\Delta u.$$

Hence by (3.1) we have $\lim_{t\to\infty} r_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right) = -\infty$, which contradicts the fact that $y^{\Delta}(t) > 0$ eventually. This completes the proof of this case.

(II) $\left[x^{[0]}\right]^{\Delta} < 0$ and $\left[x^{[1]}\right]^{\Delta} < 0$ eventually. As seen in the proof of Theorem 2.1 we obtain that dynamic inequality (2.3) has a positive solution $y(t) = -x^{[1]}(t) > 0$ on $[T_1,\infty)_{\mathbb{T}}$ for sufficiently large $T_1 \in [T,\infty)_{\mathbb{T}}$. Therefore (2.3) can be written as

$$\left\{\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta} - \hat{Q}_2(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right) \geq 0. \tag{3.6}$$

Pick $T_2 \geq T_1$ such that $h_*(t) \geq T_1$ for $t \geq T_2$. Then by using the fact $y^{\Delta}(t) > 0$ on $[T_1,\infty)_{\mathbb{T}}$, then $y(h_*(t)) > y(T_1)$ for $t \geq T_2$ and then

$$\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right) > \phi_{\gamma_2}\left(y(T_1)\right) =: L > 0.$$

Inequality (3.6) becomes

$$\left\{\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta} \geq \hat{Q}_2(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right) \geq L\hat{Q}_2(t). \tag{3.7}$$

Replacing t by u in (3.7), and integrating (3.7) from T_2 to $t \in [T_2,\infty)_{\mathbb{T}}$ we see that

$$\begin{aligned} \hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right) &\geq \hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)-\hat{r}_2(T_2)\,\phi_{\gamma_2}\left(y^{\Delta}(T_2)\right) \\ &\geq L\int_{T_2}^{t}\hat{Q}_2(u)\Delta u, \end{aligned}$$

which implies that

$$y^{\Delta}(t) \geq L^{1/\gamma_2}\left[\frac{1}{\hat{r}_2(t)}\int_{T_2}^{t}\hat{Q}_2(u)\Delta u\right]^{1/\gamma_2}.$$

Again, integrating the above inequality from T_2 to t we obtain

$$y(t) \geq y(t)-y(T_2) \geq L^{1/\gamma_2}\int_{T_2}^{t}\left[\frac{1}{\hat{r}_2(v)}\int_{T_2}^{v}\hat{Q}_2(u)\Delta u\right]^{1/\gamma_2}\Delta v,$$

which yields

$$x(T_2)-x(t) \geq L^{1/\gamma_1\gamma_2}\int_{T_2}^{t}\left\{\frac{1}{r_1(w)}\int_{T_2}^{w}\left[\frac{1}{\hat{r}_2(v)}\int_{T_2}^{v}\hat{Q}_2(u)\Delta u\right]^{1/\gamma_2}\Delta v\right\}^{1/\gamma_1}\Delta w.$$

Hence by (3.2) we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ eventually. This completes the proof of this case.

(III) $\left[x^{[0]}\right]^{\Delta}<0$ and $\left[x^{[1]}\right]^{\Delta}>0$ eventually. As seen in the proof of Theorem 2.1 we obtain that dynamic inequality (2.4) has a positive solution $y(t)=-x^{[1]}(t)>0$ on $[T_1,\infty)_{\mathbb{T}}$ for sufficiently large $T_1\in[T,\infty)_{\mathbb{T}}$. Therefore (2.4) can be written as

$$\left\{\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}-\hat{Q}_3(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right)\geq 0. \tag{3.8}$$

For $t\geq u\geq T_1$, we have

$$\begin{aligned} y\left(h_*(u)\right) &\geq y\left(h_*(u)\right)-y\left(h_*(t)\right)=-\int_{h_*(u)}^{h_*(t)} y^{\Delta}(v)\Delta v \\ &= -\int_{h_*(u)}^{h_*(t)} \frac{\left\{\hat{r}_2(v)\,\phi_{\gamma_2}\left(y^{\Delta}(v)\right)\right\}^{1/\gamma_2}}{\hat{r}_2^{1/\gamma_2}(v)}\Delta v \\ &\geq -\left\{\hat{r}_2\left(h_*(t)\right)\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right)\right\}^{1/\gamma_2}\int_{h_*(u)}^{h_*(t)}\frac{\Delta v}{\hat{r}_2^{1/\gamma_2}(v)}. \end{aligned} \tag{3.9}$$

Integrating the inequality (3.8) from $h_*(t)\geq T_1$ to t, we obtain

$$\begin{aligned} -\hat{r}_2\left(h_*(t)\right)\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right) &\geq \hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)-\hat{r}_2\left(h_*(t)\right)\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right) \\ &\geq \int_{h_*(t)}^{t}\hat{Q}_3(u)\phi_{\gamma_2}\left(y\left(h_*(u)\right)\right)\ \Delta u. \end{aligned} \tag{3.10}$$

Using (3.9) in (3.10), one can easily see that

$$\begin{aligned} &-\hat{r}_2\left(h_*(t)\right)\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right) \\ \geq\ &-\left\{\hat{r}_2\left(h_*(t)\right)\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right)\right\}\int_{h_*(t)}^{t}\hat{Q}_3(u)\left(\int_{h_*(u)}^{h_*(t)}\frac{\Delta v}{\hat{r}_2^{1/\gamma_2}(v)}\right)^{\gamma_2}\ \Delta u, \end{aligned}$$

or

$$1\geq\int_{h_*(t)}^{t}\hat{Q}_3(u)\left(\int_{h_*(u)}^{h_*(t)}\frac{\Delta v}{\hat{r}_2^{1/\gamma_2}(v)}\right)^{\gamma_2}\ \Delta u.$$

Taking the lim sup as $t\to\infty$ gives a contradiction to the condition (3.3).
(IV) $\left[x^{[0]}\right]^{\Delta}>0$ and $\left[x^{[1]}\right]^{\Delta}<0$ eventually. Proceeding as in the proof of Theorem 2.1 we have dynamic inequality (2.5) has a positive solution $y(t)=x^{[1]}(t)>0$ on $[T_1,\infty)_{\mathbb{T}}$ for sufficiently large $T_1\in[T,\infty)_{\mathbb{T}}$. Therefore (2.5) can be written as

$$\left\{\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta}+\hat{Q}_4(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right)\leq 0. \tag{3.11}$$

Pick $T_2 \geq T_1$ such that $h_*(t) \geq T_1$ for $t \geq T_2$. Using the fact that $\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)$ is decreasing, we obtain

$$\begin{aligned} -y(h_*(t)) &< y(\infty) - y(h_*(t)) = \int_{h_*(t)}^{\infty} \frac{\left(\hat{r}_2(u)\,\phi_{\gamma_2}\left(y^{\Delta}(u)\right)\right)^{1/\gamma_2}}{\hat{r}_2^{1/\gamma_2}(u)}\Delta u \\ &\leq \left(\hat{r}_2(h_*(t))\,\phi_{\gamma_2}\left(y^{\Delta}(h_*(t))\right)\right)^{1/\gamma_2} \int_{h_*(t)}^{\infty} \frac{\Delta u}{\hat{r}_2^{1/\gamma_2}(u)} \\ &\leq \left(\hat{r}_2(T_1)\,\phi_{\gamma_2}\left(y^{\Delta}(T_1)\right)\right)^{1/\gamma_2} \int_{h_*(t)}^{\infty} \frac{\Delta u}{\hat{r}_2^{1/\gamma_2}(u)} = L\,\bar{r}(t), \end{aligned}$$

where $L := \left(\hat{r}_2(T_1)\,\phi_{\gamma_2}\left(y^{\Delta}(T_1)\right)\right)^{1/\gamma_2} < 0$. From (3.11), we get for $t \geq T_2$,

$$\left\{\hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right)\right\}^{\Delta} \leq -\hat{Q}_4(t)\phi_{\gamma_2}\left(y\left(h_*(t)\right)\right) \leq L^{\gamma_2}\bar{r}^{\gamma_2}(t)\hat{Q}_4(t).$$

Hence, for $t \geq T_2$, we have

$$\begin{aligned} \hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right) &\leq \hat{r}_2(t)\,\phi_{\gamma_2}\left(y^{\Delta}(t)\right) - \hat{r}_2(T_2)\,\phi_{\gamma_2}\left(y^{\Delta}(T_2)\right) \\ &\leq L^{\gamma_2}\int_{T_2}^{t} \bar{r}^{\gamma_2}(u)\hat{Q}_4(u)\Delta u \\ &\leq L^{\gamma_2}\int_{T_2}^{t} \bar{r}^{\gamma_2}(u)\hat{Q}_4(u)\Delta u. \end{aligned}$$

It follows from this last inequality that

$$y(t) - y(T_2) \leq L^{\gamma_2/\gamma_1}\int_{T_2}^{t}\left[\frac{1}{\hat{r}_2(v)}\int_{T_2}^{v} \bar{r}^{\gamma_2}(u)\hat{Q}_4(u)\Delta u\right]^{1/\gamma_2}\Delta v.$$

Hence by (3.4), we have $\lim_{t\to\infty} y(t) = -\infty$, which contradicts the fact that y is a positive solution of (2.5). This completes the proof. □

The next theorem we apply Theorem 2.2 and then apply the main results of [5, 33].

Theorem 3.2. *Let $h^*(t) < t$ and $\bar{h}(t) < t$ for $t \geq t_0 \in \mathbb{T}$. Assume* (3.2) *and the following conditions hold:*

$$\limsup_{t\to\infty}\sup_{\tilde{\lambda}\in E_1}\{\tilde{\lambda}e_{-\tilde{\lambda}\bar{Q}_1}(t, h^*(t))\} < 1, \tag{3.12}$$

$$\limsup_{t\to\infty}\sup_{\tilde{\lambda}\in E_2}\{\tilde{\lambda}e_{-\tilde{\lambda}\bar{Q}_3}(t, \bar{h}(t))\} < 1, \tag{3.13}$$

and

$$\int_{T}^{\infty}\left[\frac{1}{r_2(v)}\int_{T}^{v}\bar{Q}_4(u)\Delta u\right]^{1/\gamma_2}\Delta v = \infty, \tag{3.14}$$

for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, where

$$e_{\bar{Q}}(t, s) = \exp\int_{s}^{t}\xi_{\mu(u)}\left(\bar{Q}(u)\right)\Delta u,$$

$$E_i = \{\tilde{\lambda} : \tilde{\lambda} > 0,\ 1 - \tilde{\lambda}\bar{Q}_i(t)\mu(t) > 0,\ t \in \mathbb{T}\},$$

and

$$\xi_\mu\left(\bar{Q}\right) = \begin{cases} \dfrac{\log\left(1+\mu\bar{Q}\right)}{\mu} & \text{if } \mu \neq 0, \\ \bar{Q}, & \text{if } \mu = 0. \end{cases}$$

Then equation (2.15) *is oscillatory.*

Proof. Assume (2.15) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$, $x(h_i(t)) > 0$ on $[T,\infty)_{\mathbb{T}}$, $i = 1,2$, and $x\left(h\left(t,s\right)\right) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]_{\hat{\mathbb{T}}}$. As seen in the proof of Theorem 2.2, we have $x^{[2]}(t)$ is nonincreasing on $[T,\infty)_{\mathbb{T}}$ and also $\left[x^{[0]}\right]^\Delta$ and $\left[x^{[1]}\right]^\Delta$ are eventually of one sign. Therefore, we consider the following cases:

(I) $\left[x^{[0]}\right]^\Delta > 0$ and $\left[x^{[1]}\right]^\Delta > 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^\Delta > 0 \text{ and } \left[x^{[1]}\left(t\right)\right]^\Delta > 0 \qquad \text{for } t \geq T_1.$$

Proceeding as in the proof of Theorem 2.2 we have that dynamic inequality (2.16) has a positive solution $z(t) = x^{[2]}\left(t\right) > 0$ on $[T_2,\infty)_{\mathbb{T}}$ for sufficiently large $T_2 \in [T_1,\infty)_{\mathbb{T}}$. Then, by [33, Corollary 2] (or [5]), we get a contradiction to (3.12).

(II) $\left[x^{[0]}\right]^\Delta < 0$ and $\left[x^{[1]}\right]^\Delta < 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^\Delta < 0 \text{ and } \left[x^{[1]}\left(t\right)\right]^\Delta < 0 \qquad \text{for } t \geq T_1.$$

As seen in the proof of Theorem 2.2 we get that dynamic inequality (2.17) has a positive solution $z(t) = -x^{[2]}\left(t\right) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Pick $T_2 \geq T_1$ such that $h_*(t) \geq T_1$ for $t \geq T_2$. Then by using the fact that $z^\Delta(t) > 0$ on $[T_1,\infty)_{\mathbb{T}}$, we have $z(h_*(t)) > z(T_1)$ for $t \geq T_2$ and then

$$z\left(h_*\left(t\right)\right) > z(T_1) =: L > 0.$$

Inequality (2.17) becomes

$$z^\Delta(t) \geq \bar{Q}_2(t) z\left(h_*(t)\right) \geq L\,\bar{Q}_2(t).$$

Then the same argument as in the proof of (II) of Theorem 3.1 leads to a contradiction to the assumption (3.2).

(III) $\left[x^{[0]}\right]^\Delta < 0$ and $\left[x^{[1]}\right]^\Delta > 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^\Delta < 0 \text{ and } \left[x^{[1]}\left(t\right)\right]^\Delta > 0 \qquad \text{for } t \geq T_1.$$

As shown in the proof of Theorem 2.2 we get that dynamic inequality (2.18)

$$z^\Delta(t) + \bar{Q}_3(t) z\left(\bar{h}(t)\right) \leq 0$$

has a positive solution $z(t) = x^{[2]}\left(t\right) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Then, by [33, Corollary 2] (or [5]), we get a contradiction to (3.13).

(IV) $\left[x^{[0]}\right]^{\Delta} > 0$ and $\left[x^{[1]}\right]^{\Delta} < 0$ eventually. Then there exists $T_1 \geq T$ such that

$$\left[x^{[0]}(t)\right]^{\Delta} > 0 \text{ and } \left[x^{[1]}(t)\right]^{\Delta} < 0 \qquad \text{for } t \geq T_1.$$

Proceeding as in the proof of Theorem 2.2 we have that dynamic inequality (2.19)

$$z^{\Delta}(t) - \bar{Q}_4(t) z\left(h_*(t)\right) \geq 0,$$

has a positive solution $z(t) = -x^{[2]}(t) > 0$ for $t \in [T_2, \infty)_{\mathbb{T}}$ for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$. Pick $T_2 \geq T_1$ such that $h_*(t) \geq T_1$ for $t \geq T_2$. Then by using the fact that $z^{\Delta}(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$, we have $z(h_*(t)) > z(T_1)$ for $t \geq T_2$ and then

$$z\left(h_*(t)\right) > z(T_1) =: L > 0.$$

Inequality (2.19) becomes

$$z^{\Delta}(t) \geq \bar{Q}_4(t) z\left(h_*(t)\right) \geq L\, \bar{Q}_4(t). \tag{3.15}$$

Replacing t by u in (3.15), and integrating from T_2 to $t \in [T_2, \infty)_{\mathbb{T}}$ we see that

$$z(t) \geq z(t) - z(T_2) \geq L \int_{T_2}^{t} \bar{Q}_4(u) \Delta u,$$

which implies that

$$-\left[x^{[1]}(t)\right]^{\Delta} \geq L^{1/\gamma_2} \left[\frac{1}{r_2(t)} \int_{T_2}^{t} \bar{Q}_4(u) \Delta u\right]^{1/\gamma_2}.$$

Again, integrating the above inequality from T_2 to t we obtain

$$-x^{[1]}(t) + x^{[1]}(T_2) \geq L^{1/\gamma_2} \int_{T_2}^{t} \left[\frac{1}{r_2(v)} \int_{T_2}^{v} \bar{Q}_4(u) \Delta u\right]^{1/\gamma_2} \Delta v.$$

Hence by (3.14) we have $\lim_{t \to \infty} x^{[1]}(t) = -\infty$, which contradicts the fact that $x^{[1]}(t) > 0$ eventually. This completes the proof. □

4. General Remarks

(1) The results here are valid for various type of time scales, e.g., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$, etc. (see [7]).
(2) The results of this paper are presented in a form that is essentially new and of a high degree of generality.
(3) We note that there are many criteria in the literature of first and second order dynamic equations and so by applying these results to inequalities (2.2)–(2.5), (2.16)–(2.19), we can obtain many oscillation results, more that those known in the literature. Here we omit the details.

(4) We note that our results on the asymptotic behavior of solutions are applicable to equations (1.4) and (2.15) for all $h^*(t), h_*(t)$ and $\bar{h}(t)$ while the oscillation results are applicable to equations (1.4) and (2.15) if $h^*(t) < t, h_*(t) < t$ and $\bar{h}(t) < t$. Thus, as it is known, it is the delay in equations (1.4) and (2.15) that can generate oscillations.

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