OSCILLATION AND NONOSCILLATION CRITERIA FOR FOUR DIMENSIONAL ADVANCED AND DELAY TIME-SCALE SYSTEMS

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ABSTRACT. We obtain oscillation and nonoscillation criteria for solutions to four-dimensional advanced and delay systems of first order dynamic equations on time scales. To establish oscillation criteria, we eliminate nonoscillatory solutions of the systems based on the sign of components of the solutions. Furthermore, some of our results are new in the discrete case.

1. Introduction. In this study, we consider the following systems on a time scale $\mathbb{T}$, i.e., arbitrary nonempty closed subset of the real numbers, see **[7, 8]**,

$$\begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = c(t)w^{\gamma}(t) \\ w^{\Delta}(t) = -d(t)x^{\lambda}(t), \end{cases} \tag{1}$$

$$\begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = c(t)w^{\gamma}(t) \\ w^{\Delta}(t) = -d(t)x^{\lambda}(k(t)), \end{cases} \tag{2}$$

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and

$$(3) \qquad \begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = c(t)w^{\gamma}(t) \\ w^{\Delta}(t) = -d(t)x^{\lambda}(g(t)), \end{cases}$$

where $k, g \in C_{rd}([t_0,\infty)_{\mathbb{T}}, [t_0,\infty)_{\mathbb{T}})$, $t \in [t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$ such that $g(t) \le t \le k(t)$ and $\lim_{t\to\infty} g(t) = \infty$. Here, C_{rd} is the set of rd-continuous functions. Systems (2) and (3) are so called advanced and delay systems, respectively. We also assume that the coefficient functions $a, b, c, d \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\alpha, \beta, \gamma, \lambda$ are the ratios of odd positive integers, and $\mathbb{T}$ is unbounded. By a solution (x, y, z, w) of system (1)((2) or (3)), we mean that functions x, y, z, w are delta-differentiable, their first delta-derivatives are rd-continuous, and satisfy system (1) ((2) or (3)) for all $t \ge t_0$. We call (x, y, z, w) a *proper* solution if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and $sup\{|x(s)|, |y(s)|, |z(s)|, |w(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$ for $t \ge t_0$. A solution (x, y, z, w) of system (1) is said to be *oscillatory* if all of its components are oscillatory, i.e., neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. Obviously, if one component of a solution is eventually of one sign, then all its components are eventually of one sign and so nonoscillatory solutions have all components nonoscillatory.

In **[6]**, oscillation and nonoscillation criteria of system (2) where $k(t) = \sigma(t)$, $t \in \mathbb{T}$ are investigated under the following condition

$$(4) \qquad \int_{t_0}^{\infty} a(t)\Delta t = \int_{t_0}^{\infty} b(t)\Delta t = \int_{t_0}^{\infty} c(t)\Delta t = \infty.$$

Assuming (4) shows that system (2) has two types of nonoscillatory solutions, namely

Type (a): $x > 0$, $y > 0$, $z > 0$, $w > 0$ eventually
Type (b): $x > 0$, $y > 0$, $z < 0$, $w > 0$ eventually.

In other words, if (x, y, z, w) is any nonoscillatory solution of system (2) such that $x > 0$, then (4) eliminates the rest of other nonoscillatory solutions of system (2), namely

Type (c): $x > 0$, $y < 0$, $z > 0$, $w > 0$ eventually
Type (d): $x > 0$, $y < 0$, $z < 0$, $w > 0$ eventually

Type (e): $x > 0$, $y > 0$, $z > 0$, $w < 0$ eventually
Type (f): $x > 0$, $y < 0$, $z < 0$, $w < 0$ eventually
Type (g): $x > 0$, $y > 0$, $z < 0$, $w < 0$ eventually
Type (h): $x > 0$, $y < 0$, $z > 0$, $w < 0$ eventually.

In **[9]** and **[11]**, Došlá and Krejčová consider a class of fourth order difference equations

$$\Delta(a_n(\Delta b_n(\Delta c_n(\Delta x_n)^{\gamma})^{\beta})^{\alpha}) + d_n x_{n+\tau}^{\lambda} = 0 \tag{5}$$

with $\tau \in \mathbb{Z}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are positive real sequences and present the oscillatory properties of the solutions of equation (5). The continuous analogue of (5) can be found in **[13]**.

As a unification of the studies above with the special case of $\alpha = \beta = \gamma = \lambda = 1$, Zhang et al. **[17]** consider oscillatory behavior of the fourth order delay dynamic equation

$$(c(t)(b(t)(a(t)x^{\Delta}(t))^{\Delta})^{\Delta})^{\Delta} + p(t)x(\tau(t)) = 0,$$

where $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$, and the fourth order advanced dynamic equation

$$(p(t)x^{\Delta^3}(t))^{\Delta} + q(t)f(x(\sigma(t))) = 0$$

is considered in Zhang et al. **[18]**, where $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and there exists $L > 0$ constant such that $\frac{f(y)}{y} > L$ for all $y \neq 0$. Previous of this study, Agarwal et al. **[1]** consider oscillatory behavior of an advanced nonlinear dynamic equation

$$(p(t)(x^{\Delta^2})^{\alpha})^{\Delta^2}(t) + q(t)f(x(\sigma(t))) = 0,$$

where α is the ratio of two positive odd integers, $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, and $f \in C_{rd}(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ and $f'(x) \geq 0$ for all $x \neq 0$.

Motivated by these studies, we establish some oscillation and nonoscillation results for systems (1), (2) and (3) without assuming (4). For the entire paper, we investigate the integral conditions of the coefficient functions a, b, c and d in each subsection in order to eliminate the indicated types above.

The proof of the following auxiliary lemma which plays a key role to obtain nonoscillatory criteria for systems (1)-(3) in the sublinear case, that is, $\alpha\beta\gamma\lambda < 1$ follows from the chain rule on a time scale, see **[2]**.

Lemma 1. *Let $f \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$. If $0 < \eta < 1$ and $f^\Delta < 0$ on $\mathbb{T}$, then*

$$\int_T^\infty -\frac{f^\Delta(t)}{f^\eta(t)} \Delta t < \infty, \ T \in \mathbb{T}.$$

2. Elimination of Nonoscillatory Solutions. Note that the condition

$$\int_{t_0}^\infty d(t)\Delta t = \infty \tag{6}$$

eliminates Type (a) and Type (b) nonoscillatory solutions of systems (1)-(3), see Lemma 2.3 in **[6]**. On the other hand, if

$$\int_{t_0}^\infty d(t)\Delta t < \infty, \tag{7}$$

then one can find necessary conditions in Theorem 3.1 and Theorem 4.1 in **[6]** to eliminate Type (b) and Type (a) nonoscillatory solutions of systems (1) and (2), respectively. The elimination criteria of these types for system (3) are stated in the next section.

In the following each subsection, we obtain nonoscillatory criteria to eliminate all the types from Type (c) to Type (h) for systems (1)-(3).

2.1. Type (c) Solutions.

Theorem 1. *Systems* (1) *and* (3) *have no solutions of Type* (c) *if any of the following conditions holds:*

(i) $\int_{t_0}^\infty b(t)\Delta t = \infty$;

(ii) $\int_{t_0}^\infty a(t) \left(\int_t^\infty b(s)\Delta s \right)^\alpha \Delta t = \infty$;

(iii) $\alpha\beta\gamma\lambda < 1$, *and*

$$\int_{t_0}^\infty d(s) \left(\int_{t_0}^s c(r)\Delta r \right)^{\beta\alpha\lambda} \left(\int_s^\infty a(r) \left(\int_r^\infty b(\tau)\Delta\tau \right)^\alpha \Delta r \right)^\lambda \Delta s$$
$$= \infty.$$

Proof. Assume that (x, y, z, w) is a Type (c) solution of system (1). By the monotonicity of z, there exist $t_0 \in \mathbb{T}$ and $m > 0$ such that

$$z(t) \geq m, \quad t \geq t_0. \tag{8}$$

Assume (i) holds. Plugging (8) into the integration of the second equation from t_0 to t yields

$$y(t) - y(t_0) = \int_{t_0}^{t} b(s) z^{\beta}(s) \Delta s \geq m^{\beta} \int_{t_0}^{t} b(s) \Delta s, \quad t \geq t_0.$$

As $t \to \infty$, $y(t) \to \infty$ by (i). However, this contradicts the negativity of y for large t. Hence, we have shown that system (1) has no solution of Type (c). Assume (ii) holds. Integrating the second equation of system (1) from t to ∞, we get

$$-y(t) \geq z^{\beta}(t) \int_{t}^{\infty} b(s) \Delta s, \quad t \geq t_0, \tag{9}$$

where we use the monotonicity of z. Now integrating the first equation from t_0 to t, plugging (9) into the resulting inequality, and using (8) yield

$$\begin{aligned} x(t) - x(t_0) &\leq -\int_{t_0}^{t} a(s) \left(\int_{s}^{\infty} b(r) z^{\beta}(r) \Delta r \right)^{\alpha} \Delta s \\ &\leq -m^{\beta\alpha} \int_{t_0}^{t} a(s) \left(\int_{s}^{\infty} b(r) \Delta r \right)^{\alpha} \Delta s, \quad t \geq t_0. \end{aligned}$$

As $t \to \infty$, $x(t) \to -\infty$ by (ii). But this contradicts the positivity of x for large t. Therefore, system (1) has no solution of Type (c). Assume (iii) holds. By integrating the third equation of system (1) from t_0 to t, we have

$$z(t) \geq \int_{t_0}^{t} c(s) w^{\gamma}(s) \Delta s, \quad t \geq t_0. \tag{10}$$

Now integrating the first equation of system (1) from t to ∞ and plugging (9) into the resulting inequality give us

$$x(t) \geq \int_{t}^{\infty} a(s)(-y(s))^{\alpha} \Delta s \geq z^{\beta\alpha}(t) \int_{t}^{\infty} a(s) \left(\int_{s}^{\infty} b(r) \Delta r \right)^{\alpha} \Delta s$$

for $t \geq t_0$, where we use the monotonicity of z. By plugging (10) into the equality above, taking λ power of both sides of the resulting inequality, and using the monotonicity of w, we get

$$x^{\lambda}(t) \geq w^{\alpha\beta\gamma\lambda}(t)\left(\int_{t_0}^{t} c(s)\Delta s\right)^{\beta\alpha\lambda}\left(\int_{t}^{\infty} a(s)\left(\int_{s}^{\infty} b(r)\Delta r\right)^{\alpha}\Delta s\right)^{\lambda} \tag{11}$$

for $t \geq t_0$. Multiplying (11) by $-d$, dividing both sides of the resulting inequality by $w^{\alpha\beta\gamma\lambda}$, and integrating from t_0 to t yield

$$\int_{t_0}^{t} -\frac{w^{\Delta}(s)}{w^{\alpha\beta\gamma\lambda}(s)}\Delta s \geq \int_{t_0}^{t} d(s)\left(\int_{t_0}^{s} c(r)\Delta r\right)^{\beta\alpha\lambda}\left(\int_{s}^{\infty} a(r)\left(\int_{r}^{\infty} b(\tau)\Delta\tau\right)^{\alpha}\Delta r\right)^{\lambda}\Delta s.$$

As $t \to \infty$, $\displaystyle\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{\alpha\beta\gamma\lambda}(s)}\Delta s = \infty$ by (iii). However, $\displaystyle\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{\alpha\beta\gamma\lambda}(s)}\Delta s < \infty$ by Lemma 1 and so this gives a contradiction and completes the proof for system (1). Note that the proof for system (3) can be shown similarly. □

Remark 1. If (i) or (ii) holds in Theorem 1, then system (2) has no solution of Type (c) either.

Since $\displaystyle\int_{t_0}^{\infty} b(t)\Delta t < \infty$ in Theorem 1 (ii), from changing the order of integration, see **[3]**, we obtain the following nonoscillation criteria.

Remark 2. If $\displaystyle\int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{\sigma(t)} a(s)\Delta s\right)\Delta t = \infty$, then systems (1), (2) and (3) with $\alpha = 1$ have no solutions of Type (c).

2.2. Type (d) Solutions.

Theorem 2. *Systems* (1) *and* (3) *have no solutions of Type* (d) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} a(t)\Delta t = \infty$;

(ii) $\displaystyle\int_{t_0}^{\infty} d(t)\left(\int_t^{\infty} a(s)\Delta s\right)^{\lambda} \Delta t = \infty;$

(iii) $\displaystyle\int_{t_0}^{\infty} c(t)\left(\int_t^{\infty} d(s)\left(\int_s^{\infty} a(r)\Delta r\right)^{\lambda} \Delta s\right)^{\gamma} \Delta t = \infty.$

Proof. Assume that (x, y, z, w) is a Type (d) solution of system (1). By the monotonicity of y, there exist $t_0 \in \mathbb{T}$ and $l < 0$ such that $y(t) \le l$ for $t \ge t_0$. Assume (i) holds. Plugging this inequality into the integration of the first equation of system (1) from t_0 to t yields

$$x(t) - x(t_0) \le l^{\alpha} \int_{t_0}^{t} a(s)\Delta s, \quad t \ge t_0.$$

As $t \to \infty$, $x(t) \to -\infty$ by (i). But, this contradicts the positivity of x. Hence, system (1) has no solution of Type (d). Assume (ii) holds. Integrating the first equation from t to ∞ yields

$$-x(t) \le l^{\alpha} \int_{t}^{\infty} a(s)\Delta s, \quad t \ge t_0. \tag{12}$$

Now integrating the fourth equation from t_0 to t and using (12) in the resulting integration shows that

$$w(t) - w(t_0) \le l^{\alpha\lambda} \int_{t_0}^{t} d(s)\left(\int_s^{\infty} a(r)\Delta r\right)^{\lambda} \Delta s, \quad t \ge t_0.$$

As $t \to \infty$, $w(t) \to -\infty$ by (ii). But, this contradicts the positivity of w. Hence, system (1) has no solution of Type (d). Assume (iii) holds. Substituting (12) in the integration of the fourth equation from t to ∞ yields

$$w(t) \ge -l^{\alpha\lambda} \int_{t}^{\infty} d(s)\left(\int_s^{\infty} a(r)\Delta r\right)^{\lambda} \Delta s, \quad t \ge t_0. \tag{13}$$

By integrating the third equation of system (1) from t_0 to t and plugging (13) into the resulting integration, we get

$$z(t) - z(t_0) \ge -l^{\alpha\lambda} \int_{t_0}^{t} c(s)\left(\int_s^{\infty} d(r)\left(\int_r^{\infty} a(\tau)\Delta \tau\right)^{\lambda} \Delta r\right)^{\gamma} \Delta s$$

for $t \ge t_0$. As $t \to \infty$, $z(t) \to \infty$ by (iii). But, this contradicts the negativity of z. Therefore, system (1) has no solution of Type (d).

When (i) holds, the proof for system (3) can be done similarly. The monotonicity of x is used in the proof of system (3) for (ii) and (iii). $\square$

Since $\int_{t_0}^{\infty} a(t)\Delta t < \infty$ in Theorem 2 (ii), from changing the order of integration, see **[3]**, we get the following result.

Remark 3. If $\int_{t_0}^{\infty} a(t)\left(\int_{t_0}^{\sigma(t)} d(s)\Delta s\right)\Delta t = \infty$, then systems (1) and (3) with $\lambda = 1$ have no solutions of Type (d).

Remark 4. If (i) holds in Theorem 2, then system (2) has no solution of Type (d) either.

In the following theorem, we introduce double and triple integral conditions to eliminate Type (d) solutions for system (2).

Theorem 3. *System* (2) *has no solution of Type* (d) *if any of the following conditions holds:*

(i) $\int_{t_0}^{\infty} d(t)\left(\int_{k(t)}^{\infty} a(s)\Delta s\right)^{\lambda}\Delta t = \infty;$

(ii) $\int_{t_0}^{\infty} c(t)\left(\int_{t}^{\infty} d(s)\left(\int_{k(s)}^{\infty} a(r)\Delta r\right)^{\lambda}\Delta s\right)^{\gamma}\Delta t = \infty.$

Proof. Assume that (x, y, z, w) is a Type (d) solution of system (2). By the monotonicity of y, there exist $t_0 \in \mathbb{T}$ and $l < 0$ such that $y(t) \leq l$ for $t \geq t_0$. Plugging this inequality into the integration of the first equation from $k(t)$ to ∞ yields

$$-x(k(t)) \leq l^{\alpha}\int_{k(t)}^{\infty} a(s)\Delta s, \quad t \geq t_0.$$

The rest of the proofs of (i) and (ii) can be completed similarly as in the proof of Theorem 2 (ii) and (iii), respectively. $\square$

2.3. Type (e) Solutions.

Theorem 4. *Systems (1), (2) and (3) have no solutions of Type (e) if any of the following conditions holds:*

(i) $\int_{t_0}^{\infty} c(t)\Delta t = \infty$;

(ii) $\int_{t_0}^{\infty} c(t)\left(\int_{t_0}^{t} d(s)\Delta s\right)^{\lambda} \Delta t = \infty$.

Proof. Assume that (x, y, z, w) is a Type (e) solution of system (1). Assume (i) holds. By the monotonicity of w, there exist $t_0 \in \mathbb{T}$ and $l < 0$ such that

$$w(t) \leq l, \quad t \geq t_0. \tag{14}$$

Plugging (14) into the integration of the third equation from t_0 to t yields

$$z(t) - z(t_0) = \int_{t_0}^{t} c(s)w^{\gamma}(s)\Delta s \leq l^{\gamma} \int_{t_0}^{t} c(s)\Delta s, \quad t \geq t_0. \tag{15}$$

Then as $t \to \infty$, $z(t) \to -\infty$ by the assumption, but this contradicts the positivity of z. Hence, it is shown that system (1) has no solution of Type (d). Assume (ii) holds. By the monotonicity of x, there exist $t_0 \in \mathbb{T}$ and $m > 0$ such that $x(t) \geq m$ for $t \geq t_0$. Substituting this inequality in the integration of the fourth equation from t_0 to t and plugging the resulting inequality into the integration of the third equation from t_0 to t yields

$$z(t) - z(t_0) = \int_{t_0}^{t} c(s)w^{\gamma}(s)\Delta s \leq -m^{\lambda\gamma} \int_{t_0}^{t} c(s)\left(\int_{t_0}^{s} d(r)\Delta r\right)^{\gamma} \Delta s$$

for $t \geq t_0$. As $t \to \infty$, $z(t) \to -\infty$ by (ii). But, we get a contradiction with the fact that $z(t) > 0$ for large t. Therefore, system (1) has no solution of Type (e). Note that the proof for systems (2) and (3) can be done similarly. $\square$

2.4. Type (f) Solutions.

Theorem 5. *Systems (1) and (3) have no solutions of Type (f) if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} a(t)\Delta t = \infty$;

(ii) $\displaystyle\int_{t_0}^{\infty} a(t)\left(\int_{t_0}^{t} b(s)\Delta s\right)^{\alpha} \Delta t = \infty$;

(iii) $\alpha\beta\gamma\lambda < 1$, *and*

$$\int_{t_0}^{\infty} a(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\left(\int_{t_0}^{r} d(\tau)\Delta\tau\right)^{\gamma} \Delta r\right)^{\beta} \Delta s\right)^{\alpha} \Delta t$$
$$= \infty.$$

Proof. Suppose that (x, y, z, w) is a Type (f) solution of system (1). The proof of (i) follows from the proof of Theorem 2 (i). Assume (ii) holds. There exist $t_0 \in \mathbb{T}$ and $l < 0$ such that

$$z(t) \le l, \quad t \ge t_0. \tag{16}$$

Integrating the second equation of system (1) from t_0 to t, using (16), and plugging the resulting inequality into the integration of the first equation of system (1) from t_0 to t give us

$$x(t) - x(t_0) \le l^{\beta\alpha} \int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\Delta r\right)^{\alpha} \Delta s, \quad t \ge t_0.$$

As $t \to \infty$, $x(t) \to -\infty$ by (ii). However, we get a contradiction with the fact that $x(t) > 0$ for large t. Assume (iii) holds. Integrating the fourth equation of system (1) from t_0 to t and the monotonicity of x yield

$$w(t) \le -x^{\lambda}(t) \int_{t_0}^{t} d(s)\Delta s, \quad t \ge t_0. \tag{17}$$

Plugging (17) into the integration of the third equation of system (1) from t_0 to t, and then plugging the resulting inequality into the integration of the second equation of system (1) from t_0 to t yield

$$y^{\alpha}(t) \le -x^{\lambda\gamma\beta\alpha}(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\left(\int_{t_0}^{r} d(\tau)\Delta\tau\right)^{\gamma} \Delta r\right)^{\beta} \Delta s\right)^{\alpha},$$

$t \ge t_0$, where we again use the monotonicity of x. After multiplying the above inequality by a, dividing the resulting inequality by $-x^{\lambda\gamma\beta\alpha}$, and integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} -\frac{x^\Delta(s)}{x^{\lambda\gamma\beta\alpha}(s)}\Delta s \geq \int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\tau)\left(\int_{t_0}^{\tau} d(\eta)\Delta\eta\right)^{\gamma}\Delta\tau\right)^{\beta}\Delta r\right)^{\alpha}\Delta s$$

for $t \geq t_0$. As $t \to \infty$, $\displaystyle\int_{t_0}^{\infty} -\frac{x^\Delta(s)}{x^{\lambda\gamma\beta\alpha}(s)}\Delta s = \infty$ by (iii). However, $\displaystyle\int_{t_0}^{\infty} -\frac{x^\Delta(s)}{x^{\lambda\gamma\beta\alpha}(s)}\Delta s < \infty$ by Lemma 1. This gives a contradiction and completes the proof. Therefore, (x, y, z, w) is not of Type (f) solution of system (1). The proof for system (3) can be shown in the same way. □

By the fact that the fourth equation is not used in the proof of Theorem 5 (i) and (ii), we have the following result for system (2).

Remark 5. If (i) or (ii) holds in Theorem 5, then system (2) has no solution of Type (f) either.

2.5. Type (g) Solutions.

Theorem 6. *Systems* (1) *and* (2) *have no solutions of Type* (g) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} b(t)\Delta t = \infty$;

(ii) $\displaystyle\int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{t} c(s)\Delta s\right)^{\beta}\Delta t = \infty$;

(iii) $\displaystyle\int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{t} c(s)\left(\int_{t_0}^{s} d(r)\Delta r\right)^{\gamma}\Delta s\right)^{\beta}\Delta t = \infty$;

(iv) $\alpha\beta\gamma\lambda < 1$, *and*

$$\int_{t_0}^{\infty} b(t)\left(\int_{t_0}^{t} c(s)\left(\int_{t_0}^{s} d(r)\left(\int_{t_0}^{r} a(\tau)\Delta\tau\right)^{\lambda}\Delta r\right)^{\gamma}\Delta s\right)^{\beta}\Delta t = \infty.$$

Proof. Assume that (x, y, z, w) is a Type (g) solution of system (1) and (i) holds. Plugging (16) into the integration of the second equation of system (1) from t_0 to t yields

$$y(t) - y(t_0) \leq l^{\gamma} \int_{t_0}^{t} b(s)\Delta s, \quad t \geq t_0.$$

As $t \to \infty$, $y(t) \to -\infty$ by (i). But, this contradicts the positivity of y for large t. Hence, we have shown that system (1) has no solution of Type (g). Assume (ii) holds. Then, (15) holds and substituting (15) into the integration of the second equation of system (1) from t_0 to t yields

$$y(t) - y(t_0) \leq l^{\gamma\beta} \int_{t_0}^{t} b(s) \left(\int_{t_0}^{s} c(r)\Delta r \right)^{\beta} \Delta s, \quad t \geq t_0.$$

As $t \to \infty$, $y(t) \to -\infty$ by (ii). Assume (iii) holds. There exist $t_0 \in \mathbb{T}$ and $n < 0$ such that $x(t) \geq n$ for $t \geq t_0$. Plugging this inequality into the integration of the fourth equation of system (1) from t_0 to t yields

$$w(t) \leq -n^{\lambda} \int_{t_0}^{t} d(s)\Delta s, \quad t \geq t_0. \tag{18}$$

After substituting (18) into the integration of the third equation from t_0 to t and then substituting the resulting inequality into the integration of the second equation of system (1) from t_0 to t, we obtain

$$y(t) - y(t_0) \leq -n^{\lambda\gamma\beta} \int_{t_0}^{t} b(s) \left(\int_{t_0}^{s} c(r) \left(\int_{t_0}^{r} d(\tau)\Delta\tau \right)^{\gamma} \Delta r \right)^{\beta} \Delta s.$$

As $t \to \infty$, $y(t) \to -\infty$ by (iii). However, we get a contradiction with the positivity of y. Therefore, system (1) has no solution of Type (g). Assume (iv) holds. The proof can be treated similarly as the proof of Theorem 5 (iii). First, substituting the integration of the first equation of system (1) from t_0 to t into the integration of the fourth equation of system (1) from t_0 to t, then substituting the resulting inequality into the integration of the third equation of system (1) from t_0 to t yield

$$z^{\beta}(t) \leq -y^{\alpha\lambda\gamma\beta}(t) \left(\int_{t_0}^{t} c(s) \left(\int_{t_0}^{s} d(r) \left(\int_{t_0}^{r} a(\tau)\Delta\tau \right)^{\lambda} \Delta\tau \right)^{\gamma} \Delta s \right)^{\beta},$$

where we use the monotonicity of y. Multiplying the above inequality by b, dividing the resulting inequality by $-y^{\alpha\lambda\gamma\beta}$ and integrating from t_0 to t, we get

$$\int_{t_0}^{t} -\frac{y^{\Delta}(s)}{y^{\alpha\lambda\gamma\beta}(s)}\Delta(s) \geq \int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\left(\int_{t_0}^{r} d(\tau)\left(\int_{t_0}^{\tau} a(\eta)\Delta\eta\right)^{\lambda}\Delta\tau\right)^{\gamma}\Delta r\right)^{\beta}\Delta s.$$

As $t \to \infty$, $\int_{t_0}^{\infty} -\frac{y^{\Delta}(s)}{y^{\alpha\lambda\gamma\beta}(s)}\Delta s = \infty$ by (iv). However, $\int_{t_0}^{\infty} -\frac{y^{\Delta}(s)}{y^{\alpha\lambda\gamma\beta}(s)}\Delta s < \infty$ by Lemma 1. This gives a contradiction and completes the proof. Therefore, (x, y, z, w) is not of Type (g) solution of system (1). The proof for system (2) can be shown similarly. □

Remark 6. If (i) or (ii) holds in Theorem 6, then system (3) has no solution of Type (g) either.

Theorem 7. *Let $\alpha\beta\gamma\lambda < 1$. System* (3) *has no solution of Type* (g) *if*

(19)
$$\int_{t_0}^{\infty} a(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\left(\int_{t_0}^{g(r)} d(\tau)\Delta\tau\right)^{\gamma}\Delta r\right)^{\beta}\Delta s\right)^{\alpha}\Delta t = \infty.$$

Proof. Assume that (x, y, z, w) is a Type (g) solution of system (3) and (19) hold. Integrating the first equation from t_0 to $g(t)$ and using the monotonicity of y yield

$$x(g(t)) \geq y^{\alpha}(g(t))\int_{t_0}^{g(t)} a(s)\Delta s, \quad g(t) \geq t_0. \tag{20}$$

The rest of the proof can be completed similarly as in the proof of Theorem 6 (iv). Therefore, (x, y, z, w) is not of Type (g) solution of system (3). □

2.6. Type (h) Solutions.

Theorem 8. *Systems* (1), (2) *and* (3) *have no solutions of Type* (h) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} c(t)\Delta t = \infty$;

(ii) $\displaystyle\int_{t_0}^{\infty} b(t)\left(\int_{t}^{\infty} c(s)\Delta s\right)^{\beta} \Delta t = \infty.$

Proof. Assume that (x, y, z, w) is a Type (h) solution of system (1). The proof of (i) follows from Theorem 4 (i). Assume (ii) holds. Then substituting (14) in the integration of the third equation from t to ∞ yields

$$z(t) \geq -l^{\gamma} \int_{t}^{\infty} c(s)\Delta s. \tag{21}$$

Plugging (21) into the integration of the second equation from t_0 to t yields

$$y(t) - y(t_0) \geq -l^{\gamma\beta} \int_{t_0}^{t} b(s)\left(\int_{s}^{\infty} c(r)\Delta r\right)^{\beta} \Delta s, \quad t \geq t_0.$$

As $t \to \infty$, $y(t) \to \infty$ by (ii). But, this contradicts the boundedness of $y(t)$ for large t. Hence, (x, y, z, w) is not of Type (h) solution of system (1). Note that the proof for systems (2) and (3) can be done similarly. □

In Theorem 8 (ii), since $\displaystyle\int_{t_0}^{\infty} c(t)\Delta t < \infty$, we get the following nonoscillation criteria in the special case of $\beta = 1$ by changing the order of integration, see **[3]**.

Remark 7. If $\displaystyle\int_{t_0}^{\infty} c(t)\left(\int_{t_0}^{\sigma(t)} b(s)\Delta s\right)\Delta t = \infty$, then systems (1), (2) and (3) with $\beta = 1$ have no solutions of Type (h).

3. Oscillatory Systems. In this section, we first introduce oscillation criteria for systems (1), (2) and then for system (3). We also finish this section with some open problems.

System (2) where $k(t) = \sigma(t)$, $t \in \mathbb{T}$ is considered in **[6]**. By the monotonicity of the first component of nonoscillatory solutions, all the results in **[6]** are valid for systems (1) and (2).

If (4) holds, then there are two types of nonoscillatory solutions of systems (1) and (2), namely Type (a) and Type (b). Assuming (6) is sufficient to eliminate these types. Therefore, this implies that systems (1) and (2) are oscillatory, see Lemma 2.3 in **[6]**. If (7) holds, then Theorems 3.1 and 4.1 in **[6]** eliminate Type (b) and Type (a) solutions of systems (1) and (2), respectively. In this case, if one of the conditions of Theorems 3.1 and Theorem 4.1 hold, then systems (1) and (2) are oscillatory.

If (4) does not hold, the theorems related with systems (1) and (2) in the previous section of this paper are to eliminate all nonoscillatory solutions from Type (c) to Type (h). In order to eliminate Type (a) and Type (b) solutions of systems (1) and (2), we have to assume one of the conditions of Theorems 3.1 and Theorem 4.1 in **[6]** or (6). From the discussions above, one can investigate oscillation criteria for systems (1) and (2).

In order to show that system (3) is oscillatory, we will make similar arguments. If (4) holds, system (3) does not have nonoscillatory solutions from Type (c) to Type (h). In addition, if (6) holds, then system (3) is oscillatory, see Lemmas 2.2 and 2.3 in **[6]**. If (7) holds, we need the following theorems to eliminate Type (a) and Type (b) solutions in order to show that system (3) is oscillatory. Note that these theorems follow from the proofs of Theorems 4.1 and 3.1 in **[6]**, respectively. However, we must assume that g is nondecreasing in (ii) and (iii) of Theorem 10.

Theorem 9. *System* (3) *has no solutions of Type* (a) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} a(s)\Delta s\right)^{\lambda}\Delta t=\infty;$

(ii) $\displaystyle\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{g(t)} a(s)\left(\int_{t_0}^{s} b(r)\Delta r\right)^{\alpha}\Delta s\right)^{\lambda}\Delta t=\infty;$

(iii) $\alpha\beta\gamma\lambda<1$ *and*
$$\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{g(t)} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\tau)\Delta\tau\right)^{\beta}\Delta r\right)^{\alpha}\Delta s\right)^{\lambda}\Delta t=\infty;$$

(iv) $\alpha\beta\gamma\lambda=1$ *and* $0<\varepsilon<1$

$$\int_{t_0}^{\infty} d(t) \left(\int_{t_0}^{g(t)} a(s) \left(\int_{t_0}^{s} b(r) \left(\int_{t_0}^{r} c(\tau)\Delta\tau \right)^{\beta} \Delta r \right)^{\alpha} \Delta s \right)^{\lambda(1-\varepsilon)} \Delta t = \infty.$$

Theorem 10. *System* (3) *has no solutions of Type* (b) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^{\infty} c(t) \left(\int_{t}^{\infty} d(s)\Delta s \right)^{\gamma} \Delta t = \infty;$

(ii) $\displaystyle\int_{t_0}^{\infty} b(t) \left(\int_{t}^{\infty} c(r) \left(\int_{r}^{\infty} d(\tau)\Delta\tau \right)^{\gamma} \Delta r \right)^{\beta} \Delta s = \infty;$

(iii) $\alpha\beta\gamma\lambda < 1$, *and*

$$\int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{g(t)} a(s)\Delta s \right)^{\lambda\gamma\beta} \left(\int_{t}^{\infty} c(r) \left(\int_{r}^{\infty} d(\tau)\Delta\tau \right)^{\gamma} \Delta r \right)^{\beta} \Delta t = \infty.$$

If (4) does not hold, similarly theorems related with system (3) are to eliminate all types of nonoscillatory solutions except for Type (a) and Type (b). In order to eliminate Type (a) and Type (b) nonoscillatory solutions, we have to assume one of the conditions of Theorems 9 and 10 or (6). Hence, under these assumptions, one can show that system (3) is oscillatory.

As a continuation of this study, first we would like to consider the oscillation and nonoscillation criteria of time-scale systems (1)-(3) in which the fourth dynamic equation does not have a negative sign, see **[10]** for the discrete case.

We also would like to consider nonoscillatory solutions of four-dimensional nonlinear neutral time-scale systems, see **[12]** for the discrete case.

There have been studies for the existence of nonoscillatory solutions of two and three dimensional dynamic systems, see **[4, 5, 14, 15, 16]**.

As a future work, one can consider the following system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = b(t)g(z(t)) \\ z^{\Delta}(t) = c(t)h(w(t)) \\ w^{\Delta}(t) = \lambda d(t)l(x(t)), \end{cases}$$

where $\lambda = \pm 1$ in order to show the existence and nonexistence of nonoscillatory solutions.

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