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Oscillation Properties of an Emden-Fowler Type Equation on Discrete Time Scales

ELVAN AKIN-BOHNER[a,*] and JOAN HOFFACKER[b,†]

[a]*Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409-0020, USA;* [b]*Department of Mathematics, University of Georgia, Athens, GA 30602-7403, USA*

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In this paper, we explore the oscillation properties of

$$u^{\Delta^2}(t) + p(t)u^{\gamma}(\sigma(t)) = 0$$

on a time scale $\mathbb{T}$ with only isolated points, where $p(t)$ is defined on $\mathbb{T}$ and γ is a quotient of odd positive integers. We define oscillation in this setting, and generate conditions on the integral of $p(t)$ which guarantee oscillation and find conditions which give the existence of a nonoscillatory solution. In addition, we consider the case when solutions of this equation has asymptotically positively bounded differences.

Keywords: Time scale; Oscillation; Nonlinear

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*Corresponding author. E-mail: akine@umr.edu

[†]E-mail: johoff@math.uga.edu

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INTRODUCTION AND PRELIMINARIES

Equations of the type

$$y'' + p(t)y = 0$$

have been studied for many years in the continuous setting. Classic oscillation and nonoscillation results can be found in numerous sources, including Refs. [3,7].

In this paper, we explore oscillation properties of

$$u^{\Delta^2}(t) + p(t)u^{\gamma}(\sigma(t)) = 0 \tag{1}$$

on a time scale $\mathbb{T}$ which contains only isolated points and is unbounded above. A point is *isolated* if it is left and right scattered. The function $p(t)$ is defined on $\mathbb{T}$ and γ is a quotient of odd positive integers. In Ref. [2], we defined oscillation in this setting, and found solution properties such as existence and uniqueness. In this paper, we will first give basic definitions and theorems on time scales so this paper is self contained. In Stefan Hilger's dissertation [8], the concept of a time scale was introduced to help unify the theory of differential and difference equations.

DEFINITION 1 *A time scale (measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.*

To reference points in the set $\mathbb{T}$ the forward and backward jump operators are defined.

DEFINITION 2 *For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\mathbb{T}$ has a maximum t^*, then we put $\sigma(t^*) = t^*$, and $\rho(t^\dagger) = t^\dagger$ if $\mathbb{T}$ has a minimum $t^\dagger$. When $\sigma(t) \neq t$ then t is called right scattered. When $\rho(t) \neq t$ then t is called left scattered.

It is convenient to have a graininess operator $\mu : \mathbb{T} \rightarrow [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$. Differentiation on time scales is defined in the following manner.

DEFINITION 3 *(Hilger [9]). Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}$. *Then we define* $f^{\Delta}(t)$ *to be the number (provided it exists) with the property that given any* $\epsilon > 0$, *there is a neighborhood* U *of* t *such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^{\Delta}(t)$ *the* <u>*delta derivative*</u> *of* f *at* t.

The next theorem contains two useful properties of derivatives on time scales.

THEOREM 1 *(Hilger [9], Bohner and Peterson [1]). Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function. Then we have the following:*

i) If f *is continuous at* t *and* $\sigma(t) \neq t$, *then* f *is differentiable at* t *with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

ii) If f *is differentiable at* t, *then*

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

One can also define integration on an appropriate class of functions.

DEFINITION 4 *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function, and* $a, b \in \mathbb{T}$. *If there exists a function* $F : \mathbb{T} \to \mathbb{R}$ *such that* $F^{\Delta}(t) = f(t)$ *for all* $t \in \mathbb{T}$, *then* F *is said to be an* <u>*antiderivative*</u> *of* f. *In this case the integral is given by the formula*

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

DEFINITION 5 *A function* f *is called* <u>*right-dense continuous*</u> *on* $\mathbb{T}$ *provided it is continuous at all right-dense points of* $\mathbb{T}$ *and has a left-sided limit (which is finite) at all left-dense points of* $\mathbb{T}$.

Notice that on our time scale, all functions are right-dense continuous by default.

THEOREM 2 *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is right dense continuous on* $\mathbb{T}$. *Then for* $t \in \mathbb{T}$,

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

In addition there are two basic integration by parts formulas which are given in the following theorem.

THEOREM 3 *(Bohner and Peterson [6]). Assume* $c, d \in \mathbb{T}$, *then*

i) $\int_c^d f(\sigma(t))g^{\Delta}(t)\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t)g(t)\Delta t.$

ii) $\int_c^d f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t)g(\sigma(t))\Delta t.$

THEOREM 4 *(Bohner and Peterson [6]). Let* $a \in \mathbb{T}$, $b \in \mathbb{T}$ *and assume* $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$. *Suppose further that for each fixed* $t \in \mathbb{T}$, $f(t, \tau)$ *and* $f^{\Delta}(t, \tau)$ *are right-dense continuous and*

$$f^{\Delta}(t, \tau) = \lim_{s \to t} \frac{f(t, \tau) - f(s, \tau)}{t - s}$$

uniformly with respect to τ *on compact subsets of* $\mathbb{T}$. *Also assume that* $k : \mathbb{T} \to \mathbb{R}$ *is right-dense continuous. Then*

$$h(t) := \int_t^b f(t, \tau)k(\tau)\Delta\tau \text{ implies}$$

$$h^{\Delta}(t) = \int_t^b f^{\Delta}(t, \tau)k(\tau)\Delta\tau - f(\sigma(t), t)k(t).$$

OSCILLATION

By an interval $[a, b]$ we mean $[a, b] \cap \mathbb{T}$ where $a, b \in \mathbb{T}$. Other intervals are defined similarly. A solution $u(t)$ of the given dynamic equation is a nontrivial solution which exists on $[a, \infty)$ for some $a \in \mathbb{T}$. We now proceed to defining oscillation in this context.

DEFINITION 6 *A solution* $u(t)$ *is called* *oscillatory* *if for any* $t_1 \in [a, \infty)$, *there exists a* $t_2 \in [t_1, \infty)$ *such that* $u(t_2)u(\sigma(t_2)) \leq 0$.

The given dynamic equation itself is called *oscillatory* if all its solutions are oscillatory. If the solution $u(t)$ is not oscillatory, then it is said to be *nonoscillatory*. Equivalently the following definition can be made.

DEFINITION 7 *The solution $u(t)$ is nonoscillatory if it is eventually positive or negative, i.e. there exists a $t_1 \in [a, \infty)$ such that $u(t)u(\sigma(t)) > 0$ for all $t \in [t_1, \infty)$.*

The given dynamic equation is called *nonoscillatory* if all of its solutions are nonoscillatory. The next result is very useful to prove our main results in this section.

LEMMA 1 *(Akin-Bohner and Hoffacker [2]). Assume that $p(t) \geq 0$ for all $t \in \mathbb{T}$, and for every $a \in \mathbb{T}$, $p(t) > 0$ for some $t \in [\sigma(a), \infty)$. If $u(t)$ is a nonoscillatory solution of Eq. (1) such that*

$$u(t) > 0$$

for all $t \in [a, \infty)$, then

$$u(\sigma(t)) > u(t) \tag{2}$$

and

$$0 < u^{\Delta}(\sigma(t)) \leq u^{\Delta}(t) \tag{3}$$

for all $t \in [a, \infty)$.

In the case $\mathbb{T} = \mathbb{R}$, the following theorem is due to Atkinson [3], which has been the impetus for much work on nonlinear oscillations over the years.

THEOREM 5 *Let $p(t)$ be as in Lemma 1, $a \in \mathbb{T}$, $a \geq 0$, and $\gamma > 1$. Then the dynamic equation (1) is oscillatory if and only if*

$$\int_a^{\infty} \sigma(l)p(l)\Delta l = \infty.$$

Proof If $\int_a^{\infty} \sigma(l)p(l)\Delta l = \infty$, then it was shown that the dynamic equation (1) is oscillatory in Akin-Bohner and Hoffacker [2]. The other direction follows from the proof of Theorem 6. □

Remark 1 Since all of the points in the time scale are isolated, one can rewrite $\int_a^{\infty} \sigma(l)p(l)\Delta l$ as $\sum_{l \in [a,\infty)} \mu(l)\sigma(l)p(l)$ (see Bohner and Peterson [6]).

Some of the proof techniques for the following theorems are similar to those found in Ref. [1].

THEOREM 6 *Let $p(t)$ be as in Lemma 1. Then Eq. (1) has a bounded nonoscillatory solution if and only if*

$$\int_a^\infty \sigma(l)p(l)\Delta l < \infty$$

where $a \in \mathbb{T}$, $a \geq 0$.

Proof *First suppose that Eq. (1) has a bounded nonoscillatory solution $u(t)$. Then there exists* $a \in \mathbb{T}$, $a \geq 0$, *such that* $u(t) > 0$ *for all* $t \in [a, \infty)$. *By Lemma 1, $u(t)$ is increasing on $[a,\infty)$. Therefore, $u(t)$ is bounded above and below by positive constants for all* $t \in [a, \infty)$. *Using the integration by parts formula in Theorem 3, we see that any solution $u(t)$ of Eq. (1) also satisfies*

$$tu^\Delta(t) = au^\Delta(a) + u(t) - u(a) - \int_a^t \sigma(l)p(l)u^\gamma(\sigma(l))\Delta l \tag{4}$$

for all $t \in [a, \infty)$. *If* $\int_a^t \sigma(l)p(l)\Delta l \to \infty$ *as* $t \to \infty$, *then the right side of Eq. (4) must approach* $-\infty$. *This implies that the left side of Eq. (4) is eventually negative. But this contradicts the fact that $u(t)$ is increasing.* *To prove the converse, suppose that* $\int_a^\infty \sigma(l)p(l)\Delta l < \infty$. *Using Theorem 4 it is easy to verify that any solution $u(t)$ of*

$$u(t) = 1 - \int_t^\infty (\sigma(l) - t)p(l)u^\gamma(\sigma(l))\Delta l \tag{5}$$

is also a solution of Eq. (1). Choose $a \in \mathbb{T}$, $a \geq 0$, *sufficiently large so that*

$$\max_{t\in[a,\infty)} \left\{ \int_t^\infty (\sigma(l) - t)p(l)\Delta l, \quad 2\gamma\int_t^\infty (\sigma(l) - t)p(l)\Delta l \right\} < \frac{1}{2}.$$

Consider the Banach space L_a of all bounded real functions $x(t)$, $t \in [a, \infty)$, *with the norm defined by*

$$\|x\| = \sup_{t\in[a,\infty)} |x(t)|.$$

We define a closed bounded subset S of L_a as

$$S := \left\{ x \in L_a : \frac{1}{2} \leq x(t) \leq 1 \right\}.$$

Let $T : S \to S$ be an operator such that

$$(Tx)(t) = 1 - \int_t^\infty (\sigma(l) - t)p(l)x^\gamma(\sigma(l))\Delta l$$

for $t \in [a, \infty)$. To see that the range of T is in S, note that if $x \in S$, then

$$(Tx)(t) \geq 1 - \int_t^\infty (\sigma(l) - t)p(l)\Delta l \geq \frac{1}{2}.$$

Clearly $(Tx)(t) \leq 1$. We will show that T is a contraction mapping on S. To see this, define $r(k) = k^\gamma$. By the continuous version of the Mean Value Theorem,

$$|k^\gamma - l^\gamma| \leq \left(\max_{k \leq \xi \leq l} (\xi^\gamma)' \right) |k - l|.$$

Thus for any $x, y \in S$,

$$\begin{aligned} |x^\gamma(t) - y^\gamma(t)| &\leq \left(\max_{x \leq \xi \leq y} (\xi^\gamma)' \right) |x(t) - y(t)| \\ &= \left(\max_{\frac{1}{2} \leq \xi \leq 1} (\xi^\gamma)' \right) |x(t) - y(t)| \\ &\leq \quad 2\gamma |x(t) - y(t)|. \end{aligned}$$

Therefore,

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \int_t^\infty (\sigma(l) - t)p(l)|x^\gamma(\sigma(l)) - y^\gamma(\sigma(l))|\Delta l \\ &\leq 2\gamma \int_t^\infty (\sigma(l) - t)p(l)|x(\sigma(l)) - y(\sigma(l))|\Delta l \\ &\leq 2\gamma \|x - y\| \int_t^\infty (\sigma(l) - t)p(l)\Delta l \\ &\leq \frac{1}{2} \|x - y\|. \end{aligned}$$

It follows that $\|Tx - Ty\| \leq (1/2)\,\|x - y\|$, *and hence* T *is a contraction mapping on* S. *Thus* T *has a unique fixed point in* S, *which is our desired bounded nonoscillatory solution of Eq. (5).* ☐

Example 1 Consider

$$u^{\Delta^2}(t) = -\frac{1}{t(\sigma(t))^2}\, u^{\gamma}(\sigma(t))$$

on $[a,\infty)$ where $a > 0$. Note that $p(t) = 1/t(\sigma(t))^2$ satisfies the conditions of Lemma 1 on $[a,\infty)$. Then

$$\int_a^\infty \sigma(s)p(s)\Delta s = \int_a^\infty \sigma(s)\frac{1}{s(\sigma(s))^2}\Delta s = \int_a^\infty \frac{1}{s\sigma(s)}\,\Delta s$$

$$= -\int_a^\infty \left(\frac{1}{s}\right)^{\Delta} \Delta s = \frac{1}{a} < \infty.$$

Therefore, by Theorem 6 this dynamic equation has a bounded nonoscillatory solution on $[a, \infty)$, regardless of the time scale chosen.

Example 2 Note that the oscillation of a dynamic equation depends on the time scale chosen. First consider

$$u^{\Delta^2}(t) = -\frac{1}{t\sigma(t)}u^{\gamma}(\sigma(t))$$

on $\mathbb{T} = \mathbb{Z}$ with $[a,\infty)$ where $\gamma > 1$, $a \in \mathbb{T}$, $a > 0$. Note that $p(t) = 1/t\sigma(t)$ satisfies the conditions of Lemma 1 on $[a,\infty)$. Then

$$\int_a^\infty \sigma(s)p(s)\Delta s = \int_a^\infty \sigma(s)\frac{1}{s\sigma(s)}\Delta s = \int_a^\infty \frac{1}{s}\Delta s = \sum_{s=a}^{\infty} \frac{1}{s} = \infty,$$

so by Theorem 5, this dynamic equation is oscillatory on $[a,\infty)$. If instead $\mathbb{T}$ is such that

$$\frac{\mu(t)}{t} \leq \frac{1}{2^t},$$

then

$$\int_a^\infty \sigma(s)p(s)\Delta s = \sum_{s\in[a,\infty)} \frac{\mu(s)}{s} \leq \sum_{s\in[a,\infty)} \frac{1}{2^s} < \infty,$$

so by Theorem 6, this dynamic equation has a bounded nonoscillatory solution on $[a,\infty)$.

For the case $\mathbb{T}=\mathbb{R}$, the following theorem, and others of its type, can be found in Belohorec [7,8].

THEOREM 7 *Assume* $p(t)$ *is as in Lemma 1 and* $0<\gamma<1$. *Then Eq. (1) is oscillatory if and only if*

$$\int_a^\infty (\sigma(l))^\gamma p(l)\Delta l=\infty$$

where $a\in\mathbb{T}$, $a\geq 0$.

Proof *Let* $u(t)$ *be a nonoscillatory solution of Eq. (1) such that* $u(t)>0$ *for all* $t\in[a,\infty)$ *where* $a\in\mathbb{T}$, $a\geq 0$. *By Lemma 1,* $u(t)$ *is increasing and* $u^\Delta(t)$ *is positive and nonincreasing for all* $t\in[a,\infty)$. *Fix* $j\in\mathbb{T}$ *such that* $j>2a$. *Then for all* $t\in[j,\infty)$, *we have*

$$u(t)=u(a)+\int_a^t u^\Delta(l)\Delta l>\int_a^t u^\Delta(t)\Delta l=(t-a)u^\Delta(t)>\frac{t}{2}u^\Delta(t),$$

i.e. $u(\sigma(t))/u^\Delta(\sigma(t))>\sigma(t)/2$. *Dividing Eq. (1) by* $(u^\Delta(\sigma(t)))^\gamma$, *using this inequality, and integrating from* j *to* t, *we obtain*

$$\int_j^t \frac{u^{\Delta^2}(l)}{(u^\Delta(\sigma(l)))^\gamma}\Delta l+\frac{1}{2^\gamma}\int_j^t p(l)(\sigma(l))^\gamma\Delta l<0 \tag{6}$$

for $t\in[j,\infty)$. *By hypothesis, the second integral in Eq. (6) approaches* ∞ *at* $t\to\infty$, *so the first term approaches* $-\infty$. *But we will show that this is impossible. To see this, let*

$$r(k)=u(l)+(k-l)u^\Delta(l);$$

$l\leq k\leq\sigma(l)$, $l\geq a$ *so that* r *is positive, continuous and increasing. Further, let*

$$s(k)=\frac{r(k+\mu(l))-r(k)}{\mu(l)},$$

$k \geq a$ so that s is positive and continuous. Since $l \leq k \leq \sigma(l)$, we have $\sigma(l) \leq k + \mu(l) \leq 2\sigma(l) - l$. Therefore,

$$\begin{aligned} r(k + \mu(l)) &= u(\sigma(l)) + (k + \mu(l) - \sigma(l))u^{\Delta}(\sigma(l)) \\ &= u(\sigma(l)) + (k - l)u^{\Delta}(\sigma(l)). \end{aligned}$$

This implies that

$$\begin{aligned} s(k) &= \frac{u(\sigma(l)) + (k-l)u^{\Delta}(\sigma(l)) - u(l) - (k-l)u^{\Delta}(l)}{\mu(l)} \\ &= \frac{u(\sigma(l)) - u(l)}{\mu(l)} + (k - l)\left[\frac{u^{\Delta}(\sigma(l)) - u^{\Delta}(l)}{\mu(l)}\right] \\ &= u^{\Delta}(l) + (k-l)u^{\Delta^2}(l). \end{aligned}$$

Therefore, $s'(k) = u^{\Delta^2}(l) \leq 0$ for $l < k < \sigma(l)$ which implies that s is nonincreasing and $0 < s(k) \leq s(l) = u^{\Delta}(l)$. Then for $l < k < \sigma(l)$ we have

$$\begin{aligned} \frac{u^{\Delta^2}(l)}{(u^{\Delta}(\sigma(l)))^{\gamma}} &= \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{u^{\Delta^2}(l)}{(u^{\Delta}(\sigma(l)))^{\gamma}}\,\mathrm{d}k \geq \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{s'(k)}{s^{\gamma}(\sigma(k))}\,\mathrm{d}k \\ &= \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{s'(k)}{s^{\gamma}(k)}\,\mathrm{d}k = \frac{1}{\mu(l)}\,\frac{1}{1-\gamma}[s^{1-\gamma}(\sigma(l)) - s^{1-\gamma}(l)] \\ &= \frac{1}{1-\gamma}(s^{1-\gamma})^{\Delta}(l). \end{aligned}$$

It follows that

$$\int_j^t \frac{u^{\Delta^2}(l)}{(u^{\Delta}(\sigma(l)))^{\gamma}}\Delta l \geq \frac{1}{1-\gamma}\int_j^t (s^{1-\gamma})^{\Delta}(l)\Delta l = \frac{1}{1-\gamma}[s^{1-\gamma}(t) - s^{1-\gamma}(j)].$$

But $s^{1-\gamma(t)} > 0$ and $0 < \gamma < 1$ for all $t \geq a$, so $\int_j^t (u^{\Delta^2}(l)/(u^{\Delta}(\sigma(l)))^{\gamma})\Delta l$ is bounded below which gives a contradiction and completes the proof. The necessary part is contained in the sufficiency part of the next theorem. □

DEFINITION 8 *A solution of Eq. (1) is said to have asymptotically positively bounded differences if there are positive constants a_1 and a_2*

such that

$$a_1 \leq u^{\Delta}(t) \leq a_2$$

for all $t \in [a, \infty)$ *for some* $a \in \mathbb{T}$.

THEOREM 8 *Let* $p(t)$ *be as in Lemma 1. Eq. (1) has a solution with asymptotically positively bounded differences if and only if*

$$\int_a^{\infty} p(l)(\sigma(l))^{\gamma} \Delta l < \infty$$

where $a \in \mathbb{T}$, $a \geq 0$.

Proof *Assume that* $\int_a^{\infty} p(l)(\sigma(l))^{\gamma} \Delta l < \infty$, *and fix* $a \in \mathbb{T}$, $a \geq 0$ *sufficiently large so that* $\int_a^{\infty} p(l)(\sigma(l))^{\gamma} \Delta l < (1/2)$. *Let* $u(t)$ *be the solution of Eq. (1) satisfying* $u(a) = 0$ *and* $u(\sigma(a)) = \mu(a)$ *so that* $u^{\Delta}(a) = 1$. *We want to show that* $1/2 \leq u^{\Delta}(t) \leq 1$ *for all* $t \in [a, \infty)$. *For this purpose, suppose that* $1/2 \leq u^{\Delta}(t) \leq 1$ *for all* $t \in [a, m]$ *where* $m \in \mathbb{T}$, $m \geq a$. *Then* $u(t) > 0$ *for all* $t \in (a, \sigma(m)]$. *However, from Eq. (1),* $u^{\Delta^2}(t) = -p(t)u^{\gamma}(\sigma(t)) \leq 0$ *for all* $t \in [a, m]$. *Therefore, for all* $t \in [a, \sigma(m)]$ *it follows that*

$$u(t) \leq u(a) + (t - a)u^{\Delta}(a) = t - a \leq t.$$

From Eq. (1) and the above inequalities we obtain

$$u^{\Delta}(m) = u^{\Delta}(a) - \int_a^m p(l)u^{\gamma}(\sigma(l))\Delta l \geq 1 - \int_a^m p(l)(\sigma(l))^{\gamma} \Delta l \geq \frac{1}{2}.$$

Also $u^{\Delta}(m) \leq u^{\Delta}(a) = 1$. *Therefore,* $1/2 \leq u^{\Delta}(m) \leq 1$ *and now by induction*

$$\frac{1}{2} \leq u^{\Delta}(t) \leq 1$$

holds for all $t \in [a, \infty)$.

Conversely let $u(t)$ *be a solution of Eq. (1) which has asymptotically positively bounded differences. Thus there exists an* $a \in \mathbb{T}$, $a \geq 0$, *such that* $u(t) > 0$ *for all* $t \in [a, \infty)$. *Then as in Theorem 7, we find that*

$$u(t) > \frac{t}{2} u^{\Delta}(t)$$

for all $t \in [j, \infty)$ *where* $j > 2a$. *Therefore, for all* $t \in [j, \infty)$ *it follows that*

$$u^{\Delta}(j) - u^{\Delta}(t) = \int_j^t p(l)u^{\gamma}(\sigma(l))\Delta l > \int_j^t p(l)\left(\frac{\sigma(l)}{2}u^{\Delta}(\sigma(l))\right)^{\gamma}\Delta l$$

$$\geq \left(\frac{a_1}{2}\right)^{\gamma}\int_j^t p(l)(\sigma(l))^{\gamma}\Delta l \geq 0.$$

If $\int_a^t p(l)(\sigma(l))^{\gamma}\Delta l \to \infty$ *as* $t \to \infty$, *then it must be the case that* $u^{\Delta}(t) \to -\infty$ *as* $t \to \infty$. *However, by Lemma 1,* $u^{\Delta}(t) > 0$ *for all* $t \in [a, \infty)$. *This implies that* $\int_a^{\infty} p(l)(\sigma(l))^{\gamma}\Delta l < \infty$. ☐

EXAMPLE 3 Consider

$$u^{\Delta^2}(t) = -\frac{1}{t(\sigma(t))^{\gamma}}\, u^{\gamma}(\sigma(t))$$

on $[a,\infty)$, where $a \in \mathbb{T}$, $a > 0$. The oscillation of this equation depends on the time scale chosen. On any time scale where

$$\sum_{t \in [a,\infty)} \frac{\mu(t)}{t} < \infty,$$

then this equation has a nonoscillatory solution by Theorem 8. However, if the time scale is such that

$$\sum_{t \in [a,\infty)} \frac{\mu(t)}{t} = \infty,$$

as in $\mathbb{T} = \mathbb{Z}$, then the equation is oscillatory when $0 < \gamma < 1$ by Theorem 7.

It is interesting to note that the proof technique impacts whether or not the theorem holds for $\mathbb{T} = \mathbb{R}$ as well as for the time scales we are interested in. For example, Theorem 6 uses the contraction mapping principle, and it is well known that this result is true for $\mathbb{T} = \mathbb{R}$. This suggests a modification of the proof techniques would generate the results on a wider selection of time scales, however, that is a topic for further research.

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