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Positive decreasing solutions of quasilinear dynamic equations

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Abstract

We consider a quasilinear dynamic equation reducing to a half-linear equation, an Emden–Fowler equation or a Sturm–Liouville equation under some conditions. Any nontrivial solution of the quasilinear dynamic equation is eventually monotone. In other words, it can be either positive decreasing (negative increasing) or positive increasing (negative decreasing). In particular, we investigate the asymptotic behavior of all positive decreasing solutions which are classified according to certain integral conditions. The approach is based on the Tychonov fixed point theorem.

Keywords: Measure chains; Time scales; Quasilinear equations; Half-linear equations; Sturm–Liouville equations

1. Introduction

In this paper, we consider a quasilinear dynamic equation

$$[a(t)\Phi_p(x^\Delta)]^\Delta = b(t)f(x^\sigma), \tag{1}$$

where a and b are real positive rd-continuous functions on a *time scale* $\mathbb{T}$ (an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$), $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ and $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$. Eq. (1) reduces to the half-linear dynamic equation, see Řehak [1],

$$[a(t)\Phi_p(x^\Delta)]^\Delta = b(t)\Phi_p(x^\sigma) \tag{2}$$

if $f = \Phi_p$ in (1) and to the Emden–Fowler dynamic equation, see Akın-Bohner and Hoffacker [2,3],

$$x^{\Delta^2} = b(t)\Phi_q(x^\sigma) \tag{3}$$

if $a(t) = 1$, $p = 2$ and $f = \Phi_q$, $q > 1$ in (1). Eq. (1) reduces to the quasilinear differential equation, see Cecchi, Došlá and Marini [4],

$$[a(t)\Phi_p(x')]' = b(t)f(x) \tag{4}$$

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when $\mathbb{T} = \mathbb{R}$ as well as to the quasilinear difference equation, see Cecchi, Došlá and Marini [5],

$$\Delta[a_k \Phi_p(\Delta x_k)] = b_k f(x_{k+1}) \tag{5}$$

when $\mathbb{T} = \mathbb{Z}$. Eq. (2) reduces to the half-linear differential equation, see Došly [6],

$$[a(t)\Phi_p(x')]' = b(t)\Phi_p(x) \tag{6}$$

when $\mathbb{T} = \mathbb{R}$ as well as to the half-linear difference equation, see Řehak [7],

$$\Delta[a_k \Phi_p(\Delta x_k)] = b_k \Phi_p(x_{k+1}) \tag{7}$$

when $\mathbb{T} = \mathbb{Z}$. Moreover, if $p = 2$, then a special case of Eq. (2) is the Sturm–Liouville dynamic equation, see Bohner and Peterson [8],

$$(a(t)x^{\Delta})^{\Delta} = b(t)x^{\sigma} \tag{8}$$

which covers the Sturm–Liouville differential equation, see Hartman [9],

$$(a(t)x')' = b(t)x \tag{9}$$

when $\mathbb{T} = \mathbb{R}$ and the Sturm–Liouville difference equation, see Agarwal [10],

$$\Delta(a_k \Delta x_k) = b_k x_{k+1} \tag{10}$$

when $\mathbb{T} = \mathbb{Z}$.

Such studies are essentially motivated by the dynamics of positive radial solutions of reaction–diffusion (flow through porous media, nonlinear elasticity) problems modelled by the nonlinear elliptic equation

$$-\mathrm{div}(\alpha(|\nabla u|)\nabla u) + \lambda f(u) = 0, \tag{11}$$

where $\alpha : (0, \infty) \mapsto (0, \infty)$ is continuous and such that $\delta(v) := \alpha(|v|)v$ is an odd increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, λ is a positive constant (the Thiele modulus) and f represents the ratio of the reaction rate at concentration u to the reaction rate at concentration unity, see Diaz [11] and Grossinho and Omari [12]. If $\alpha(|v|) = |v|^{p-2}$, then the differential operator in Eq. (11) is the one-dimensional analogue of the p-Laplacian $\Delta_p(u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and Eq. (11) leads to Eq. (4) when $\mathbb{T} = \mathbb{R}$.

Our main goal is to consider the asymptotic behavior of all positive decreasing solutions of Eq. (1) on time scales. More precisely, all solutions of Eq. (1) can be divided into several disjoint subsets by means of necessary and sufficient integral conditions which involve only the functions a and b, see Cecchi, Došlá and Marini [5,13] for the discrete case and see Cecchi, Došlá and Marini [4] for the continuous case. The approach used is based on the Tychonov fixed point theorem.

The set-up of this paper is as follows. In Section 2, we briefly introduce the concept of calculus of time scales including preliminary results. Then in Section 3, we first classify classes of solutions of Eq. (1), and then describe certain integral conditions and relations among them. In Section 4, we consider asymptotic behavior of all positive decreasing solutions of Eq. (1).

2. Calculus on time scales

There are two main purposes of the study of time scales: *Unification* and *Extension*. Choosing the time scale to be the set $\mathbb{R}$ corresponds to an ordinary differential equation, see Hartman [9], while choosing the time scale to be the integers $\mathbb{Z}$ corresponds to a difference equation, see Kelley and Peterson [14]. Therefore this subject helps proving results only once. On the other hand, there are time scales other than $\mathbb{R}$ and $\mathbb{Z}$ such as $h\mathbb{Z}$ for $h > 0$, the Cantor set, the set of harmonic numbers $\{\sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\}$, $q^{\mathbb{N}_0}$ for $q > 1$, etc., so that one can have much more general results. An introduction with applications and advances in dynamic equations are given in [8,15].

We define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}$$

for all $t \in \mathbb{T}$. In this definition we put $\inf(\emptyset) = \sup \mathbb{T}$. The *backward jump operator* ρ on $\mathbb{T}$ is defined by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$$

for all $t \in \mathbb{T}$. In this definition we put $\sup(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. Finally, the *graininess* function $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We define the interval $[t_0, \infty)$ in $\mathbb{T}$ by

$$[t_0, \infty) := \{t \in \mathbb{T} : t \geq t_0\}.$$

The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Assume $f : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of $f(t)$ at t, and it turns out that $^\Delta$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the usual forward difference operator Δ if $\mathbb{T} = \mathbb{Z}$.

It can be shown that if $f : \mathbb{T} \mapsto \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^\Delta(t) = \frac{f(x(\sigma(t))) - f(t)}{\mu(t)},$$

while if t is right-dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if the limit exists. If f is differentiable at t, then

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \text{where } f^\sigma = f \circ \sigma. \tag{12}$$

If $f, g : \mathbb{T} \mapsto \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then the product and quotient rules are as follows:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t)$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)} \quad \text{if } g(t)g^\sigma(t) \neq 0.$$

We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense $\lim_{s \to t^-} f(s)$ exists as a finite number. The set of rd-continuous functions $f : \mathbb{T} \mapsto \mathbb{R}$ will be denoted in this paper by C_{rd} and the set of functions $f : \mathbb{T} \mapsto \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C}^1_{\mathrm{rd}}$.

A function $F : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \mapsto \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a) \quad \text{for } t \in \mathbb{T}. \tag{13}$$

If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and $f \in \mathrm{C}_{\mathrm{rd}}$ on $[a, \infty)$, then we define the *improper integral* by

$$\int_a^\infty f(t)\Delta t := \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges, see Bohner and Guseinov [16]. Now we state some important results. One can find the proof of the following result in Bohner and Peterson [8, Theorem 1.75].

Theorem 2.1. *If* $f \in \mathrm{C_{rd}}$ *and* $t \in \mathbb{T}^\kappa$*, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

The following induction principle on $\mathbb{T}$ is a useful tool, see Bohner and Peterson [8, Theorem 1.7].

Theorem 2.2. *Let* $t_0 \in \mathbb{T}$ *and assume that*

$$\{I(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying:

(i) *The statement* $I(t_0)$ *is true.*
(ii) *If* $t \in [t_0, \infty)$ *is right-scattered and* $I(t)$ *is true, then* $I(\sigma(t))$ *is also true.*
(iii) *If* $t \in [t_0, \infty)$ *is right-dense and* $I(t)$ *is true, then there is a neighborhood* U *of* t *such that* $I(s)$ *is true for all* $s \in U \cap (t, \infty)$.
(iv) *If* $t \in (t_0, \infty)$ *is left-dense and* $I(s)$ *is true for all* $s \in [t_0, t)$*, then* $I(t)$ *is true.*

Then $I(t)$ *is true for all* $t \in [t_0, \infty)$.

The following L'Hospital's Rule on timescales can be found in Agarwal and Bohner [17, Theorem 3]. Let $\bar{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$. If $\infty \in \bar{\mathbb{T}}$, we call ∞ left-dense. For any left-dense $t_0 \in \mathbb{T}$ and any $\epsilon > 0$ the set

$$L_\epsilon(t_0) = \{t \in \mathbb{T} : 0 < t_0 - t < \epsilon\}$$

is nonempty, and so is $L_\epsilon(\infty) = \{t \in \mathbb{T} : t > \frac{1}{\epsilon}\}$ if $\infty \in \bar{\mathbb{T}}$.

Theorem 2.3. *Assume* f *and* g *are differentiable on* $\mathbb{T}$ *with*

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^-} g(t) = 0 \quad \text{for some left-dense } t_0 \in \bar{\mathbb{T}}.$$

Suppose there exists $\epsilon > 0$ *with*

$$g(t) > 0, \quad g^\Delta(t) < 0 \quad \text{for all } t \in L_\epsilon(t_0).$$

Then we have

$$\liminf_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} \le \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)}.$$

The following result is the chain rule on $\mathbb{T}$, see Bohner and Peterson [8, Theorem 1.90].

Theorem 2.4. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \mapsto \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))\mathrm{d}h \right\} g^\Delta(t)$$

holds.

3. Classes of solutions in terms of certain integrals

Throughout this paper we assume that $\mathbb{T}$ is unbounded above. By a *solution* we mean a delta-differentiable function x satisfying Eq. (1) such that $a\Phi_p(x^\Delta) \in \mathrm{C}^1_{\mathrm{rd}}$.

The following lemma is a crucial result that gives us that any solution of Eq. (1) is eventually monotone.

Lemma 3.1. *Let S be the set of nontrivial solutions of Eq. (1) on $[t_0, \infty)$. Any $x \in S$ is eventually monotone and belongs to one of the two classes:*

$$M^+ := \{x \in S : \text{ there exists } T \ge t_0 \text{ such that } x(t)x^\Delta(t) > 0 \text{ for } t \ge T\}$$
$$M^- := \{x \in S : x(t)x^\Delta(t) < 0 \text{ on } [t_0, \infty)\}.$$

Proof. Let $x \in S$ and define

$$F(t) := a(t)x(t)\Phi_p(x^\Delta(t)).$$

Then

$$\begin{aligned} F^\Delta(t) &= [a(t)\Phi_p(x^\Delta(t))]^\Delta x^\sigma(t) + a(t)\Phi_p(x^\Delta(t))x^\Delta(t) \\ &= b(t)f(x^\sigma(t))x^\sigma(t) + a(t)\Phi_p(x^\Delta(t))x^\Delta(t) \ge 0. \end{aligned}$$

Thus F is a nondecreasing function. If x was eventually constant, then the left-hand side of Eq. (1) would be eventually zero but not the right-hand side of Eq. (1). Therefore, x is not eventually constant and so there are only two possibilities:

(i) There exists $T \in [t_0, \infty)$ such that $F(t) > 0$ on $[T, \infty)$.
(ii) $F(t) < 0$ on $[t_0, \infty)$.

In case (i), we obtain that

$$x(t)x^\Delta(t) > 0 \quad \text{on } [T, \infty)$$

and so $x \in M^+$ is eventually strongly monotone. In case (ii), we obtain that

$$x(t)x^\Delta(t) < 0 \quad \text{on } [t_0, \infty)$$

so $x \in M^-$. Without loss of generality we assume $x(t_0) > 0$ and $x^\Delta(t_0) < 0$ and show that x is positive and decreasing by Theorem 2.2. Consider

$$\{I(t) : x \text{ is positive and decreasing on } [t_0, \infty)\}.$$

(i) Since $x(t_0) > 0$ and $x^\Delta(t_0) < 0$, $I(t_0)$ is true.
(ii) Assume that $t \in [t_0, \infty)$ is right-scattered and $I(t)$ is true. Then show that $I(\sigma(t))$ is also true. Assume not, i.e., $I(\sigma(t))$ is not true. If $x^\sigma(t_0) < 0$, then integrating Eq. (1) from t_0 to $\sigma(t_0)$ and using Theorem 2.1 give us

$$\begin{aligned} a(\sigma(t_0))\Phi_p(x^\Delta(t_0)) &< a(\sigma(t_0))\Phi_p(x^\Delta(\sigma(t_0))) - a(t_0)\Phi_p(x^\Delta(t_0)) \\ &= \int_{t_0}^{\sigma(t_0)} b(\tau)f(x^\sigma(\tau))\Delta\tau \\ &= \mu(t_0)b(t_0)f(x^\sigma(t_0)) < 0. \end{aligned}$$

Since a is positive, $\Phi_p(x^\Delta(\sigma(t_0)))$ is negative. Therefore $x^\Delta(\sigma(t_0)) < 0$. But this is a contradiction with $x \in M^-$.
(iii) Assume $t \in [t_0, \infty)$ is right-dense and $I(t)$ is true. Show that there exists a neighborhood U of t such that $I(s)$ is true for all $s \in U \cap (t, \infty)$. Then assume not. Since $I(t)$ is true when $t = \sigma(t)$, then $x(t) > 0$ and $x^\Delta(t) < 0$. Let U be a neighborhood of t and assume that there exists $s \in U \cap (t, \infty)$ such that $x(s) < 0$. Then integrating Eq. (1) from t to s yields that

$$\begin{aligned} a(s)\Phi_p(x^\Delta(s)) &< a(s)\Phi_p(x^\Delta(s)) - a(t)\Phi_p(x^\Delta(t)) \\ &= \int_t^s b(\tau)f(x^\sigma(\tau))\Delta\tau \\ &\le M_f \int_t^s b(\tau)\Delta\tau < 0, \end{aligned}$$

where $M_f = \max_{u \in [x(t), x(s)]} f(u)$. Since a is positive, $\Phi_p(x^\Delta(s)) < 0$ and so $x^\Delta(s) < 0$. But this is a contradiction with $x \in M^-$.

(iv) Assume $t \in (t_0, \infty)$ is left-dense and $I(s)$ is true for all $s \in [t_0, t)$, then $x(s) > 0$ and $x^{\Delta}(s) < 0$ for all $s \in [t_0, t)$. Assume that $x(t) < 0$. Integrating Eq. (1) from t_0 to t yields that

$$\begin{aligned} a(t)\Phi_p(x^{\Delta}(t)) &< a(t)\Phi_p(x^{\Delta}(t)) - a(t_0)\Phi_p(x^{\Delta}(t_0)) \\ &= \int_{t_0}^{t} b(\tau) f(x^{\sigma}(\tau))\Delta\tau \\ &\le M_f \int_{t_0}^{t} b(\tau)\Delta\tau < 0, \end{aligned}$$

where $M_f = \max_{u \in [x(\sigma(t_0)), x(t)]} f(u)$. Since a is positive, $\Phi_p(x^{\Delta}(t)) < 0$ and so $x^{\Delta}(t) < 0$. But then this is a contradiction.

Hence by Theorem 2.2 we obtain that x is positive and decreasing in case (ii). □

Remark 3.1. In general, the condition $xx^{\Delta} < 0$ does not ensure that x is eventually of one sign, i.e., is nonoscillatory. But this is true when $x \in S$ as follows from the proof of Lemma 3.1.

In this paper we study the class M^-. The qualitative behavior of solutions of the class M^+ is investigated in the next project. In the view of Lemma 3.1 M^- can be divided into the two subclasses:

$$M_B^- = \{x \in M^- : \lim_{t\to\infty} x(t) = l \neq 0\},$$
$$M_0^- = \{x \in M^- : \lim_{t\to\infty} x(t) = 0\}.$$

Solutions of M_B^- and M_0^- are called *asymptotically constant solutions* and *decaying solutions*, respectively. For the further results we describe this class in terms of certain integral conditions and define

$$\begin{aligned} Y_1 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t_0}^{t} b(s)\Delta s\right)\Delta t, \\ Y_2 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t}^{T} b(s)\Delta s\right)\Delta t, \\ Y_3 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right)\Delta t, \\ Y_4 &= \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Delta t, \end{aligned}$$

where Φ_{p^*} is the inverse of the map Φ_p, i.e., $\Phi_p(\Phi_{p^*}(u)) = \Phi_{p^*}(\Phi_p(u)) = u$. Then $\Phi_{p^*}(u) = |u|^{p^*-2}u$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. We conclude this section with some relationships between the convergence or divergence of Y_1, Y_2, Y_3 and Y_4.

Lemma 3.2. *We have*

(i) *If $Y_1 < \infty$, then $Y_3 < \infty$.*
(ii) *If $Y_2 < \infty$, then $Y_4 < \infty$.*
(iii) *If $Y_1 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.*
(iv) *If $Y_2 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.*
(v) *$Y_1 < \infty$ and $Y_2 < \infty$ if and only if $Y_3 < \infty$ and $Y_4 < \infty$.*

Proof. (i) Let $t_1 \in (t_0, T)$. Since

$$\begin{aligned} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Phi_{p^*}\left(\int_{t_0}^{s} b(t)\Delta t\right)\Delta s &= \int_{t_0}^{t_1} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Phi_{p^*}\left(\int_{t_0}^{s} b(t)\Delta t\right)\Delta s \\ &\quad + \int_{t_1}^{T} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Phi_{p^*}\left(\int_{t_0}^{s} b(t)\Delta t\right)\Delta s \end{aligned}$$

$$> \int_{t_0}^{t_1} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Phi_{p^*}\left(\int_{t_0}^{s} b(t)\Delta t\right) \Delta s + \Phi_{p^*}\left(\int_{t_0}^{t_1} b(s)\Delta s\right) \int_{t_1}^{T} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Delta s,$$

the assertion follows.

(ii) Let $t_1 \in (t_0, T)$. Since

$$\begin{aligned}\int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(\tau)}\right) \Phi_{p^*}\left(\int_{\tau}^{T} b(s)\Delta s\right) \Delta\tau &= \int_{t_0}^{T} \Phi_{p^*}\left(\int_{\tau}^{T} \frac{b(s)}{a(\tau)}\Delta s\right) \Delta\tau \\ &> \int_{t_0}^{t_1} \Phi_{p^*}\left(\int_{t_1}^{T} \frac{b(s)}{a(\tau)}\Delta s\right) \Delta\tau \\ &= \int_{t_0}^{t_1} \Phi_{p^*}\left(\frac{1}{a(\tau)}\right) \Phi_{p^*}\left(\int_{t_1}^{T} b(s)\Delta s\right) \Delta\tau \\ &> \left(\int_{t_1}^{T} b(s)\Delta s\right) \int_{t_0}^{t_1} \Phi_{p^*}\left(\frac{1}{a(\tau)}\right) \Delta\tau,\end{aligned}$$

the assertion holds.

(iii) Since

$$\int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t_0}^{t} b(s)\Delta s\right) \Delta t \le \Phi_{p^*}\left(\int_{t_0}^{T} b(s)\Delta s\right) \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Delta t,$$

the assertion holds.

(iv) Since

$$\int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t}^{T} b(s)\Delta s\right) \Delta t \le \Phi_{p^*}\left(\int_{t_0}^{T} b(s)\Delta s\right) \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Delta t,$$

the assertion holds.

(v) It follows immediately from (i)–(iv). □

Concerning the class M^- for Eq. (1), such a class can be empty when $\mathbb{T} = \mathbb{R}$, see Kiguradze and Chanturia [18, Corollary 17.3]. This fact has no discrete analogy, see Cecchi, Došlá and Marini [5, Theorem 1].

4. Limit behavior

In this section we study the positive decreasing solutions of Eq. (1) in the class M^- in terms of integrals Y_1 and Y_2. We start with necessary and sufficient conditions ensuring that $M_B^- \neq \emptyset$.

Theorem 4.1. *Eq. (1) has a solution in the class M_B^- if and only if $Y_2 < \infty$.*

Proof. Assume that $x \in M_B^-$. Without loss of generality suppose $x(t) > 0$ and $x^\Delta(t) < 0$ for $t \ge t_0$ and define $\lim_{t\to\infty} x(t) = l_x \neq 0$. Assume $Y_2 = \infty$ to get a contradiction. Integrating Eq. (1) from t to T, we obtain

$$-a(t)\Phi_p(x^\Delta(t)) > \int_t^T b(\tau) f(x^\sigma(\tau))\Delta\tau$$

and hence

$$\Phi_p(x^\Delta(t)) < -\frac{1}{a(t)} \int_t^T b(\tau) f(x^\sigma(\tau))\Delta\tau \le -\frac{1}{a(t)} m_f \int_t^T b(\tau)\Delta\tau,$$

where $m_f = \min_{u\in[x(\sigma(t)),x(T)]} f(u)$. Therefore

$$\begin{aligned}x^\Delta(t) &< \Phi_{p^*}\left(-\frac{1}{a(t)} m_f \int_t^T b(\tau)\Delta\tau\right) \\ &= -\Phi_{p^*}(m_f)\Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_t^T b(\tau)\Delta\tau\right).\end{aligned}$$

This implies that

$$x(t) < x(t_0) - \Phi_{p^*}(m_f) \int_{t_0}^{t} \Phi_{p^*}\left(\frac{1}{a(\tau)}\right) \Phi_{p^*}\left(\int_{\tau}^{t} b(s)\Delta s\right) \Delta\tau.$$

Note that $l_x > 0$. Since $Y_2 = \infty$, we have a contradiction. Therefore $Y_2 < \infty$.

Conversely, let $M_f = \max_{u\in[1/2,1]} f(u)$ and choose $t_1 \in [t_0, \infty)$ such that

$$\Phi_{p^*}(M_f) \int_{t_1}^{\infty} \Phi_{p^*}\left(\frac{1}{a(t)} \int_{\tau}^{\infty} b(s)\Delta s\right) \Delta\tau \le \frac{1}{2}. \tag{14}$$

Define X to be the Frěchet space of all continuous functions on $[t_1, \infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$. Let Ω be the nonempty subset of X given by

$$\Omega := \left\{ x \in X : \frac{1}{2} \le x(t) \le 1 \text{ for all } t \ge t_1 \right\}.$$

Clearly, Ω is closed, convex and bounded. Now we consider the operator $T : \Omega \mapsto X$ which assigns to any $x \in \Omega$ the continuous functions $T(x) = y_x$ given by

$$y_x(t) = T(x)(t) = \frac{1}{2} + \int_{t}^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)} \int_{s}^{\infty} b(\tau) f(x^{\sigma}(\tau))\Delta\tau\right) \Delta s. \tag{15}$$

We show that T satisfies the hypotheses of the Tychonov fixed point theorem.

Claim: $T : \Omega \mapsto \Omega$.

$$\begin{aligned} \frac{1}{2} \le T(x)(t) &= \frac{1}{2} + \int_{t}^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)} \int_{s}^{\infty} b(\tau) f(x^{\sigma}(\tau))\Delta\tau\right) \Delta s \\ &\le \frac{1}{2} + \Phi_{p^*}(M_f) \int_{t}^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)} \int_{s}^{\infty} b(\tau)\Delta\tau\right) \Delta s \\ &\le 1, \end{aligned}$$

by inequality (14).

Claim: T is continuous in $\Omega \subseteq X$.

Let $\{x_n\}$, $n \in \mathbb{N}$ be a sequence in Ω which is convergent to $\bar{x} \in X$, $\bar{x} \in \bar{\Omega} = \Omega$. Because for $s \ge t_1$

$$\Phi_{p^*}\left(\frac{1}{a(s)} \int_{s}^{\infty} b(\tau) f(x_n^{\sigma}(\tau))\Delta\tau\right) \le \Phi_{p^*}\left(\frac{M_f}{a(s)} \int_{s}^{\infty} b(\tau)\Delta\tau\right) < \infty,$$

Lebesgue's dominated convergence theorem gives the continuity of T in Ω.

Claim: T is relatively compact (i.e., equibounded and equicontinuous).

Since Ω is a bounded subset of X, T is equibounded. For any $x \in \Omega$,

$$0 \le -[T(x)(t)]^{\Delta} = \Phi_{p^*}\left(\frac{1}{a(t)} \int_{t}^{\infty} b(\tau) f(x^{\sigma}(\tau))\Delta\tau\right) \le \Phi_{p^*}\left(\frac{M_f}{a(t)} \int_{t}^{\infty} b(\tau)\Delta\tau\right) < \infty \tag{16}$$

which implies that functions in $T(\Omega)$ are equicontinuous on every compact subinterval of $[t_1, \infty)$. From the Tychonov fixed point theorem there exists an $\bar{x} \in \Omega$ such that $\bar{x} = T(\bar{x})$, i.e.,

$$\bar{x}(t) = \frac{1}{2} + \int_{t}^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)} \int_{s}^{\infty} b(\tau) f(\bar{x}^{\sigma}(\tau))\Delta\tau\right) \Delta s.$$

By (16) we have $\bar{x}^{\Delta}(t) = [T(\bar{x})(t)]^{\Delta} < 0$. Therefore by Lemma 3.1, $\left(\bar{x}\bar{x}^{\Delta}\right)(t) < 0$ on $[t_1, \infty)$. Hence $\bar{x} \in M_B^- \ne \emptyset$. □

Remark 4.1. As regards the class M_0^- it is not always true that there are solutions of (1) in the class M_0^- when $Y_2 < \infty$. Cecchi, Došlá, and Marini in [5, Example 1] show that $M_B^- \ne \emptyset$ by Theorem 4.1 but $M_0^- = \emptyset$ for the equation

$$\Delta^2 x_n = \frac{2}{n(n+2)^2} x_{n+1},$$

when $\mathbb{T} = \mathbb{Z}$.

Theorem 4.2. *If $Y_1 < \infty$ and $Y_2 < \infty$, then Eq. (1) has a solution in class M_0^-.*

Proof. We prove the statement $M_0^- \neq \emptyset$ by using a method similar to that given in the proof of Theorem 4.1. By Lemma 3.2 (v) we have $Y_3 < \infty$ and $Y_4 < \infty$. Choose $t_1 \geq t_0$ such that

$$\max_{u\in[0,Y_3]} f(u) \int_{t_1}^{\infty} b(\tau)\Delta\tau < \frac{1}{2}. \tag{17}$$

Let Ω be the nonempty subset of $C[t_1,\infty)$ given by

$$\Omega = \left\{ u \in C[t_1,\infty) : 0 \leq u(t) \leq \int_t^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s, \text{ for all } t \geq t_1 \right\}.$$

Clearly, Ω is bounded, closed and convex. Now we consider the operator $T : \Omega \mapsto C[t_1,\infty)$ which assigns to any $u \in \Omega$ the continuous function $T(u) = y_u$ given by

$$y_u(t) = T(u)(t) = \int_t^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)}\right) \Phi_{p^*}\left(1 - \int_{t_1}^{s} b(\tau) f(u^{\sigma}(\tau))\Delta\tau\right)\Delta s.$$

In view of inequality (17) we get

$$0 \leq \int_{t_1}^{t} b(\tau) f(u^{\sigma}(\tau))\Delta\tau \leq \max_{u\in[0,Y_3]} f(u) \int_{t_1}^{\infty} b(\tau)\Delta\tau < \frac{1}{2}.$$

Moreover, $\Phi_{p^*}(\frac{1}{2}) \leq 1$ and the operator T is well defined and

$$0 < T(u)(t) \leq \int_t^{\infty} \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s,$$

i.e., $T(\Omega) \subseteq \Omega$. In order to complete the proof, it is sufficient to use an argument similar to that given in the final part of the proof of Theorem 4.1 and to apply the Tychonov fixed point theorem. □

The following lemma is crucial to proving that every solution of Eq. (1) in the class M^- tends to a nonzero limit as $t \to \infty$, i.e., $M_0^- = \emptyset$, under certain conditions.

Lemma 4.1. *If $Y_3 = \infty$, then for any $x \in M^-$*

$$\lim_{t\to\infty} a(t)\Phi_p(x^{\Delta}(t)) = 0.$$

Proof. Since $a(t)\Phi_p(x^{\Delta}(t))$ is either negative and increasing or positive and decreasing for any solution $x \in M^-$ of Eq. (1), $\lim_{t\to\infty} a(t)\Phi_p(x^{\Delta}(t))$ exists. Assume that $x \in M^-$ such that

$$\lim_{t\to\infty} a(t)\Phi_p(x^{\Delta}(t)) = \lambda_x < 0.$$

Hence x is positive and the function $a\Phi_p(x^{\Delta})$ is negative and increasing. Consequently, since

$$a\Phi_p(x^{\Delta}) = \Phi_p\left(\frac{x^{\Delta}}{\Phi_{p^*}\left(\frac{1}{a}\right)}\right),$$

$\frac{x^{\Delta}}{\Phi_{p^*}(\frac{1}{a})}$ is negative increasing and

$$x^{\Delta}(t) < \Phi_{p^*}(\lambda_x)\Phi_{p^*}\left(\frac{1}{a(t)}\right).$$

Integrating both sides from t_0 to t, we have

$$x(t) < x(t_0) + \Phi_{p^*}(\lambda_x) \int_{t_0}^{t} \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s$$

which contradicts the fact that x is positive as $t \to \infty$ since $Y_3 = \infty$. The case $\lambda_x > 0$ is handled in a similar way. □

Theorem 4.3. *Assume $Y_1 = \infty$, $Y_2 < \infty$ and*

$$\limsup_{u \to 0} \frac{f(u)}{\Phi_p(u)} < \infty. \tag{18}$$

Then any $x \in M^-$ tends to a nonzero limit as $t \to \infty$, i.e., $M^- = M_B^- \neq \emptyset$, $M_0^- = \emptyset$.

Proof. Let $x \in M_0^-$. Without loss of generality assume $0 < x < 1$ and $x^\Delta < 0$ for all $t \geq t_0$. By Lemma 3.2, $Y_3 = \infty$ and thus, by Lemma 4.1, $\lim_{t\to\infty} a(t)\Phi_p(x^\Delta(t)) = 0$. Since Eq. (18) holds, there exists $M > 0$ such that

$$f(x^\sigma(t)) \leq M\Phi_p(x^\sigma(t)) \quad \text{for all } t \geq t_0.$$

By integrating Eq. (1) from t to ∞, we obtain

$$\begin{aligned} -a(t)\Phi_p(x^\Delta(t)) &= \int_t^\infty b(\tau) f(x^\sigma(\tau))\Delta\tau \\ &< M \int_t^\infty b(\tau)\Phi_p(x^\sigma(\tau))\Delta\tau \\ &< M\Phi_p(x^\sigma(t)) \int_t^\infty b(\tau)\Delta\tau. \end{aligned}$$

This implies that

$$\frac{x^\Delta(t)}{x^\sigma(t)} > -\Phi_{p^*}(M)\Phi_{p^*}\left(\int_t^\infty \frac{b(\tau)}{a(t)}\Delta\tau\right).$$

On the other hand,

$$[\ln(x(t))]^\Delta = x^\Delta(t) \int_0^1 \frac{1}{x(t) + \mu(t)hx^\Delta(t)} \mathrm{d}h \geq \frac{x^\Delta(t)}{x(t) + \mu(t)x^\Delta(t)} = \frac{x^\Delta(t)}{x^\sigma(t)},$$

where we use Theorem 2.4 and Eq. (12). By integrating the above inequality from t_0 to t we get

$$\ln(x(t)) - \ln(x(t_0)) > -\Phi_{p^*}(M) \int_{t_0}^t \Phi_{p^*}\left(\int_\tau^\infty \frac{b(s)}{a(\tau)}\Delta s\right)\Delta\tau,$$

which is a contradiction as $t \to \infty$. □

Remark 4.2. In general Theorem 4.3 does not hold without assuming (18). Cecchi, Došlá, and Marini in [5, Example 2] show that $M_0^- \neq \emptyset$ for the equation

$$\Delta^2 x_n = \frac{6\sqrt{n^{(2)}}}{(n+1)^{(4)}}\sqrt{|x_{n+1}|},$$

when $\mathbb{T} = \mathbb{Z}$. In this case, $Y_3 = \infty$ and $Y_2 < \infty$.

From the above results, we can summarize the situation as follows:

Theorem 4.4. *We have*

(i) *Assume $Y_1 = \infty$ and $Y_2 < \infty$. Then for Eq. (1) it holds that $M_B^- \neq \emptyset$. If (18) is satisfied, then $M^- = M_B^- \neq \emptyset$, $M_0^- = \emptyset$.*
(ii) *Assume $Y_1 < \infty$ and $Y_2 < \infty$. Then for (1) both solutions in M_0^- and M_B^- exist.*
(iii) *Assume (18) holds. Then Eq. (1) has solutions in the classes M_0^- and M_B^- if and only if $Y_1 < \infty$ and $Y_2 < \infty$.*

We finish this section with the following proposition, where we use the L'Hospital Rule on $\mathbb{T}$.

Proposition 4.1. *Assume $Y_1 < \infty$ and $Y_2 < \infty$, then there exists a solution x of Eq. (1) such that*

$$\lim_{t\to\infty} \frac{x(t)}{\int_t^\infty \Phi_{p^*}\left(\frac{1}{a(s)}\right)\Delta s}$$

exists finitely and it is different from zero.

Proof. Let x be the fixed point of operator T considered in the proof of Theorem 4.2. Then $x \in M_0^-$. From Eq. (1) the function $a\Phi_p(x^\Delta)$ is negative increasing. Since

$$a(t)\Phi_p(x^\Delta(t)) = \Phi_p\left(\frac{x^\Delta(t)}{\Phi_{p^*}\left(\frac{1}{a(t)}\right)}\right),$$

the function $\frac{x^\Delta}{\Phi_{p^*}\left(\frac{1}{a}\right)}$ is also negative and increasing. Then

$$\lim_{t\to\infty} \frac{x^\Delta(t)}{\Phi_{p^*}\left(\frac{1}{a(t)}\right)}$$

exists finitely and it is not zero because

$$-x^\Delta(t) = \Phi_{p^*}\left(\frac{1}{a(t)}\right)\Phi_{p^*}\left(1 - \int_{t_0}^t b(\tau)f(x^\sigma(\tau))\Delta\tau\right) \geq \Phi_{p^*}\left(\frac{1}{2}\right)\Phi_{p^*}\left(\frac{1}{a(t)}\right).$$

The assertion follows from Theorem 2.3. □

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