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OSCILLATORY THEOREMS FOR CERTAIN SECOND ORDER DAMPED DYNAMIC INCLUSIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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ABSTRACT. We shall establish some new criteria for the oscillation of second order nonlinear damped dynamic inclusions with distributed deviating arguments on time scales.

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1. INTRODUCTION

This paper deals with oscillatory behavior of all solutions of the nonlinear second order damped dynamic inclusions with distributed deviating arguments

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)\in\int_a^b q(t,\tau)F(t,x(g(t,\tau)))\Delta\tau \text{ for a.e. } t\geq t_0\in\mathbb{T}, \quad (1.1)$$

on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$ and $0<a<b$, $a,b\in\mathbb{T}$. Whenever we write $t\geq t_1$, we mean $t\in[t_1,\infty)\cap\mathbb{T}:=[t_1,\infty)_{\mathbb{T}}$.

We assume that

1. α is the ratio of positive odd integers;
2. $p,r:\mathbb{T}\mapsto\mathbb{R}^+$ are single-valued rd-continuous functions such that $r^{\Delta}(t)\geq 0$ for $t\in\mathbb{T}$ and
$$\int_{t_0}^{\infty}\left(\frac{1}{r(s)}e_{\frac{-p}{r}}(s,t_0)\right)^{\frac{1}{\alpha}}\Delta s=\infty; \quad (1.2)$$
where $e_p(t,t_0)$ is the exponential function satisfying the semigroup property $e_p(a,b)e_p(b,c)=e_p(a,c)$.
3. $q:\mathbb{T}\times[a,b]_{\mathbb{T}}\to\mathbb{R}^+$ is a rd-continuous function;

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4. $g : \mathbb{T} \times [a, b]_{\mathbb{T}} \mapsto \mathbb{T}$ is decreasing with respect to its second variable, $g(t, \tau) \leq t$ and $g(t, \tau) \to \infty$ as $t \to \infty$, $\tau \in [a, b]_{\mathbb{T}}$;
5. $F : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto 2^{\mathbb{R}}$ is a multi function (here $2^{\mathbb{R}}$ denotes the family of nonempty subsets of $\mathbb{R}$).

We note that the usual standard notation in inclusion theory is used here, e.g.,

$$|F(t, u)| = \sup\{|v| : v \in F(t, u)\}$$

and

$$F(t, u) > 0 \text{ means } w > 0 \text{ for each } w \in F(t, u).$$

By a solution to inclusion (1.1), we mean a function $x \in \mathrm{C_{rd}}$ with $r(x^{\Delta})^{\alpha}$ and $p(x^{\Delta})^{\alpha} \in \mathrm{C_{rd}}$ and $(r(x^{\Delta})^{\alpha})^{\Delta} + p(x^{\Delta})^{\alpha} \in L^1_{loc}[t_0, \infty)_{\mathbb{T}}$. We assume throught that inclusion (1.1) possesses such solutions. We recall that a solution of (1.1) is said to be nonoscillatory if there exists a $t_1 \in \mathbb{T}$ such that $x(t)x^{\sigma}(t) > 0$ for all $t \in [t_0, \infty)$. Otherwise, it is said to be oscillatory. Inclusion (1.1) is said to be oscillatory if all its solutions are oscillatory.

The theory of time scales which has recently received a lot of attention, was introduced in [17], in order to unify continuous and discrete analysis.

In recent years, there has been much research acitivity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales, and we refer the reader to the papers [8,18]. However, there are few results dealing with the oscillation of second order dynamic inclusions on time scales [2-7] and for second order dynamic equations with distributed deviating arguments [10].

In this article, we study oscillatory behaviour of all solutions of inclusion (1.1). We also establish an oscillation result for (1.1) via comparison with second order dynamic equations whose oscillatory character are known. We also present similar oscillation results for (1.1) when condition (1.2) fails. Finally, we investigate some possible extensions to the results obtained. The results of this paper are new for the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

2. MAIN RESULTS

We assume that

$$\begin{cases} F(t, x) < 0 & \text{for } (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^+ \\ F(t, x) > 0 & \text{for } (t, x) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^-. \end{cases} \tag{2.1}$$

We now present the following result.

Lemma 2.1. *Let conditions* (1)–(5), (1.2) *and* (2.1) *hold. Assume that* $x(t)$ *is an eventually positive solution of* (1.1). *Then there exists a* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$x(t) > 0, \quad x^{\Delta}(t) > 0 \quad \textit{and} \quad x^{\Delta\Delta}(t) \leq 0 \textit{ eventually.} \tag{2.2}$$

Proof. Let x be an eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(g(t,\tau))>0$ for $t\geq t_1$ and $\tau\in[a,b]_{\mathbb{T}}$. Let

$$\begin{cases} y(t) := (r(x^{\Delta})^{\alpha})^{\Delta}(t) + p(x^{\Delta})^{\alpha}(t) \text{ with } y(t) \in \int_a^b q(t,\tau)F(t,x(g(t,\tau)))\Delta\tau \\ \text{and} \\ y\in L^1_{loc}[t_0,\infty)_{\mathbb{T}}. \end{cases} \tag{2.3}$$

From (2.1), we obtain

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t) + p(x^{\Delta})^{\alpha}(t) \leq 0 \text{ for a.e. } t\geq t_1$$

or

$$w^{\Delta}(t) + \frac{p(t)}{r(t)}w(t) \leq 0 \text{ for a.e. } t\geq t_1,$$

where

$$w(t) = r(x^{\Delta})^{\alpha}(t), \quad t\geq t_1.$$

We assert that $\frac{w(t)}{e_{-\frac{p}{r}}(t,t_0)}$ is decreasing for $t\geq t_1$. Clearly, we see that

$$\begin{aligned}\left(\frac{w(t)}{e_{-\frac{p}{r}}(t,t_0)}\right)^{\Delta} &= \frac{w^{\Delta}(t)e_{-\frac{p}{r}}(t,t_0) - \left(-\frac{p(t)}{r(t)}\right)e_{-\frac{p}{r}}(t,t_0)w(t)}{e_{-\frac{p}{r}}(t,t_0)e^{\sigma}_{-\frac{p}{r}}(t,t_0)} \\ &= \frac{w^{\Delta}(t)+\frac{p(t)}{r(t)}w(t)}{e^{\sigma}_{-\frac{p}{r}}(t,t_0)}\end{aligned}$$

and the assertion is proved. Next, we claim that $x^{\Delta}(t)>0$ for $t\geq t_1$. To this end, assume that $x^{\Delta}(t)<0$ eventually. Then there exists a $t_2\in[t_1,\infty)$ such that $x^{\Delta}(t)<0$ for $t\geq t_2$. Using the fact that $\frac{w(t)}{e_{-\frac{p}{r}}(t,t_0)}$ is decreasing we have

$$\frac{r(t)(x^{\Delta}(t))^{\alpha}}{e_{-\frac{p}{r}}(t,t_0)} \leq \frac{r(t_2)(x^{\Delta}(t_2))^{\alpha}}{e_{-\frac{p}{r}}(t_2,t_0)} := b<0 \text{ for } t\in[t_2,\infty)_{\mathbb{T}}.$$

Thus,

$$x^{\Delta}(t) \leq -b^{\frac{1}{\alpha}}\left[\frac{e_{-\frac{p}{r}}(t,t_0)}{r(t)}\right]^{\frac{1}{\alpha}} \text{ for } t\in[t_2,\infty)_{\mathbb{T}}.$$

Integrating this inequality from t_2 to $t\geq t_0$ we find

$$x(t) \leq x(t_2) - b^{\frac{1}{\alpha}}\int_{t_2}^{t}\left(\frac{e_{-\frac{p}{r}}(s,t_0)}{r(s)}\right)^{\frac{1}{\alpha}}\Delta s.$$

As $t\to\infty$, $x(t)$ approaches to $-\infty$, which contradicts the fact that $x(t)$ is eventually positive, and thus $x^{\Delta}(t)>0$ eventually. Next, we prove that $x^{\Delta\Delta}(t)\leq 0$ eventually. From [2, Theorem 1.90] we find

$$\begin{aligned}(x^{\alpha}(t))^{\Delta} &\geq \alpha\int_0^1[hx+(1-h)x]^{\alpha-1}x^{\Delta}(t)dh \\ &= \alpha x^{\alpha-1}(t)x^{\Delta}(t),\end{aligned}$$

and then we calculate

$$\left((x^{\Delta}(t))^{\alpha}\right)^{\Delta} \geq \alpha \left(x^{\Delta}(t)\right)^{\alpha-1} x^{\Delta\Delta}(t). \tag{2.4}$$

Hence

$$\begin{aligned}(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} &= r^{\Delta}(t)(x^{\Delta}(t))^{\alpha} + r^{\sigma}(t)\left(x^{\Delta}(t))^{\alpha}\right)^{\Delta} \\ &\geq qr^{\Delta}(t)(x^{\Delta}(t))^{\alpha} + \alpha r^{\sigma}(t)(x^{\Delta}(t))^{\alpha-1}x^{\Delta\Delta}(t),\end{aligned}$$

in view of the fact that $x^{\Delta}(t) > 0$ and $r^{\Delta}(t) \geq 0$, we obtain $x^{\Delta\Delta}(t) \leq 0$ eventually. This completes the proof. □

Lemma 2.2 ([15]). *Let the hypotheses of Lemma* 2.1 *hold. Then there exist a constant* $\bar{c} \in (0,1)$ *and a* $\bar{t} \in \mathbb{T}$, $\bar{t} > t_0$ *such that*

$$\frac{x(g(t))}{g(t)} \geq \bar{c}\frac{x^{\sigma}(t)}{\sigma(t)} \text{for all } t \geq \bar{t}, \tag{2.5}$$

where g *is as in* (4).

We let

$$Q(t) = \int_a^b q(t,\tau)\Delta\tau$$

and

$$g(t) = g(t,b).$$

We now present the following oscillation results for (1.1).

Theorem 2.3. *Let conditions* (1)–(5), (1.2) *and* (2.1) *hold and assume that*

$$\begin{cases} \text{there exists } f : [t_0,\infty)\times\mathbb{R} \mapsto \mathbb{R} \text{ with } xf(t,x) > 0 \text{ for a.e. } t \geq t_0 \\ \qquad \text{and } x \neq 0 \text{ with } |\frac{f(t,x)}{x^{\alpha}}| \geq c > 0 \text{ for a.e. } t \geq t_0, \\ \text{and} \\ \qquad |F(t,x)| \geq f(t,x) \text{ for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{+} \\ \qquad |F(t,x)| \leq -f(t,x) \text{ for } (t,x) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^{-}. \end{cases} \tag{2.6}$$

If there exists a function $\xi(t) \in C^1_{rd}(\mathbb{T},\mathbb{R}^{+})$ *such that*

$$p(t) \geq \frac{r^{\sigma}(t)\xi^{\sigma}(t)}{\xi^2(t)}\xi^{\Delta}(t) \text{ for } t \in [t_0,\infty)_{\mathbb{T}} \tag{2.7}$$

and

$$\int^{\infty} \xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s = \infty, \tag{2.8}$$

then inclusion (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Suppose $x(t)>0$ and $x(g(t,\tau))>0$ for $t\geq t_0$ and $a\leq\tau\leq b$. Let

$$\begin{cases} y(t) := (r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t) \text{ with } y(t)\in\int_a^b q(t,\tau)F(t,x(g(t,\tau)))\Delta\tau \\ \text{and} \\ y\in L^1_{loc}[t_0,\infty)_{\mathbb{T}}. \end{cases}$$

From (2.1) and (2.6) we have

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)\leq 0$$

and

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)+\int_a^b q(t,\tau)f(t,x(g(t,\tau)))\Delta\tau\leq 0$$

or

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)+\int_a^b cq(t,\tau)x^{\alpha}(g(t,\tau))\Delta\tau\leq 0 \quad \text{a.e. } t\geq t_0. \tag{2.9}$$

By Lemma 2.1, x is an increasing function on $[t_0,\infty)_{\mathbb{T}}$ and by (4), (2.9) becomes

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)+cQ(t)x^{\alpha}(g(t))\leq 0 \quad \text{a.e. } t\geq t_0. \tag{2.10}$$

By Lemma 2.2, then there exist a $t_1\geq t_0$ and a constant $\bar{c}\in(0,1)$ such that (2.5) holds. Using (2.5) in (2.10) we get

$$(r(x^{\Delta})^{\alpha})^{\Delta}(t)+p(x^{\Delta})^{\alpha}(t)+c(\bar{c})^{\alpha}Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha}x^{\alpha}(\sigma(t))\leq 0 \quad \text{a.e. } t\geq t_1. \tag{2.11}$$

Define the function $w(t)$ by

$$w(t)=\xi(t)\frac{r(t)(x^{\Delta}(t))^{\alpha}}{x^{\alpha}(t)} \text{ on } [t_1,\infty)_{\mathbb{T}}. \tag{2.12}$$

Then we have $w(t)>0$ and

$$w^{\Delta}(t)=\xi^{\Delta}(t)\frac{r^{\sigma}(t)(x^{\Delta}(\sigma(t)))^{\alpha}}{x^{\alpha}(\sigma(t))}+\xi(t)\frac{(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta}}{x^{\alpha}(\sigma(t))}-\xi(t)\frac{r(t)(x^{\Delta}(t))^{\alpha}(x^{\alpha}(t))^{\Delta}}{x^{\alpha}(t)x^{\alpha}(\sigma(t))}. \tag{2.13}$$

Using (2.4) and (2.11) in (2.13) we obtain

$$\begin{aligned} w^{\Delta}(t)\leq{}& \frac{\xi^{\Delta}(t)}{\xi(t)}w(\sigma(t))-\frac{\xi(t)}{x^{\alpha}(\sigma(t))}\left[p(t)(x^{\Delta}(t))^{\alpha}+c(\bar{c})^{\alpha}Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha}x^{\alpha}(\sigma(t))\right] \\ &-\alpha\xi(t)\frac{r(t)(x^{\Delta}(t))^{\alpha+1}}{x(t)x^{\alpha}(\sigma(t))} \end{aligned}$$

or

$$\begin{aligned} w^{\Delta}(t)\leq{}& -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha}+\frac{\xi^{\Delta}(t)}{\xi^{\sigma}(t)}w(\sigma(t))-\frac{\xi(t)(x^{\Delta}(t))^{\alpha}}{x^{\alpha}(\sigma(t))}p(t) \\ &-\alpha\xi(t)\frac{r(t)(x^{\Delta}(t))^{\alpha+1}}{x(t)x^{\alpha}(\sigma(t))} \text{ for a.e. } t\geq t_1. \end{aligned} \tag{2.14}$$

By Lemma 2.1, $x^{\Delta\Delta}(t) \le 0$ and $x^{\Delta}(t) > 0$, we get

$$x^{\Delta}(t) \ge x^{\Delta}(\sigma(t)) \quad \text{and} \quad x(t) \le x(\sigma(t)), \quad t \ge t_1. \tag{2.15}$$

Using (2.15) in (2.14), one can easily find

$$w^{\Delta}(t) \le -c(\bar{c})^{\alpha}\xi(t)Q(t)(\frac{g(t)}{\sigma(t)})^{\alpha} + \left[\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} - \frac{\xi(t)p(t)}{r(\sigma(t))\xi(\sigma(t))}\right] w(\sigma(t))$$
$$- \alpha\frac{\xi(t)r(t)}{(\xi(\sigma(t))r(\sigma(t)))^{\frac{\alpha+1}{\alpha}}}(w(\sigma(t)))^{\frac{\alpha+1}{\alpha}} \text{ for a.e. } t \ge t_1. \tag{2.16}$$

Using conditions (2.7) in (2.16) we get

$$w^{\Delta}(t) \le -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} \text{ for a.e. } t \ge t_1.$$

Integrating this inequality from t_1 to t we have

$$0 < w(t) \le w(t_1) - c(\bar{c})^{\alpha}\int_{t_1}^{t}\xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s \to -\infty \text{ as } t \to \infty,$$

a contradiction. This completes the proof. □

The following result is concerned with the oscillation of (1.1) when condition (2.7) fails.

Theorem 2.4. *Let conditions* (1)–(5), (1.2), (2.1) *and* (2.6) *hold. If there exists a function* $\xi(t) \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$ *such that*

$$p(t) \le \frac{r(\sigma(t))\xi(\sigma(t))}{\xi^2(t)}\xi^{\Delta}(t) \textit{ for } t \in [t_1, \infty)_{\mathbb{T}} \tag{2.17}$$

and for $t \ge t_1$ *and for every constant* $\bar{c} \in (0, 1)$

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\right.$$
$$\left. - \frac{1}{c\bar{c}^{2\alpha}}\left(\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} - \frac{p(s)\xi(s)}{r(\sigma(s))\xi(\sigma(s))}\right)\left(\frac{r(\sigma(s)\xi(\sigma(s))}{\sigma^{\alpha}(s)}\right)\right]\Delta s = \infty, \tag{2.18}$$

then inclusion (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t, \tau)) > 0$ for $t \ge t_0$ and $a \le \tau \le b$. Proceeding as in the proof of Theorem 2.3, we obtain (2.16) which takes the form

$$w^{\Delta}(t) \le -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t))}\right)^{\alpha}$$
$$+ \left[\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} - \frac{\xi(t)p(t)}{r(\sigma(t))\xi(\sigma(t))}\right] w(\sigma(t)) \text{ for a.e. } t \ge t_1. \tag{2.19}$$

From (2.5) we have

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} \le \left(\frac{1}{\bar{c}t}\right)^{\alpha} \text{ for } t \ge t_1 \tag{2.20}$$

and so,

$$w(t) = \xi(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} \leq (\bar{c})^{-\alpha}\frac{\xi(t)r(t)}{t^{\alpha}} \text{ for } t \geq t_1. \tag{2.21}$$

Using (2.21) in (2.19) we get

$$\frac{1}{c(\bar{c})^{\alpha}}w^{\Delta}(t) \leq -\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} + \frac{1}{c(\bar{c})^{2\alpha}}\Big(\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} - \frac{\xi(t)p(t)}{r(\sigma(t))\xi(\sigma(t))}\Big)\frac{\xi(\sigma(t))r(\sigma(t))}{\sigma^{\alpha}(t)} \text{ for } t \geq t_1. \tag{2.22}$$

Integrating this inequality from t_1 to t and taking lim sup of both sides of the resulting inequality as $t \to \infty$, we arrive at the desired contradiction. This completes the proof. □

Following corollary is immediate.

Corollary 2.5. *Let the hypotheses of Theorem* 2.4 *hold and condition* (2.18) *be replaced by condition* (2.8) *and*

$$\int^{\infty}\left[\frac{\xi^{\Delta}(s)}{\xi(\sigma(s))} - \frac{\xi(s)p(s)}{r(\sigma(s))\xi(\sigma(s))}\right]\left(\frac{r(\sigma(s))\xi(\sigma(s))}{\sigma^{\alpha}(s))}\right)\Delta s < \infty,$$

then conclusion of Theorem 2.4 *holds.*

Next, we establish the following oscillation result for (1.1).

Theorem 2.6. *Let conditions* (1)–(5), (1.2), (2.1) *and* (2.6) *hold. If*

$$\limsup_{t\to\infty}\frac{t^{\alpha}}{r(t)}\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s = \infty, \tag{2.23}$$

then inclusion (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t,\tau)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $a \leq \tau \leq b$. Proceeding as in the proof of Lemma 2.1 we obtain

$$\left(\frac{r(t)(x^{\Delta}(t))^{\alpha}}{e_{-\frac{p}{r}}(t,t_0)}\right)^{\Delta} \leq \frac{(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + p(t)(x^{\Delta}(t))^{\alpha}}{e^{\sigma}_{-\frac{p}{r}}(s,t_0)} \leq 0$$

for all $t \in [t_1,\infty)_{\mathbb{T}}, t_1 \geq t_0$. Now, for all $u \geq t$, $u,t \in [t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\frac{r(u)(x^{\Delta}(u))^{\alpha}}{e_{-\frac{p}{r}}(u,t_0)} &= \frac{r(t)(x^{\Delta}(t))^{\alpha}}{e_{-\frac{p}{r}}(t,t_0)} + \int_t^u\left(\frac{r(s)(x^{\Delta}(s))^{\alpha}}{e_{-\frac{p}{r}}(s,t_0)}\right)^{\Delta}\Delta s \\ &\leq \frac{r(t)(x^{\Delta}(t))^{\alpha}}{e_{-\frac{p}{r}}(t,t_0)} - c(\bar{c})^{\alpha}\int_t^u\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t_0)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}x^{\alpha}(s)\Delta s.\end{aligned}$$

Letting $u \to \infty$, we have

$$\frac{r(t)(x^{\Delta}(t))^{\alpha}}{e_{-\frac{p}{r}}(t,t_0)} \geq c(\bar{c})^{\alpha}\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t_0)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}x^{\alpha}(s)\Delta s$$
$$\geq qc(\bar{c})^{\alpha}x^{\alpha}(t)\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t_0)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s;$$

or

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} \geq \frac{c(\bar{c})^{\alpha}}{r(t)}\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s \text{ for a.e. } t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.24}$$

Using (2.20) in (2.24) we have

$$\left(\frac{1}{\bar{c}t}\right)^{\alpha} \geq \frac{c(\bar{c})^{\alpha}}{r(t)}\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s$$

$$\frac{1}{c(\bar{c})^{2\alpha}} \geq \frac{t^{\alpha}}{r(t)}\int_t^{\infty}\frac{Q(s)}{e_{-\frac{p}{r}}(\sigma(s),t)}\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s.$$

Taking lim sup of both sides of this inequality as $t \to \infty$ we obtain a contradiction to condition (2.23). This completes the proof. □

Theorem 2.7. *Let conditions* (1)–(5), (1.2), (2.1) *and* (2.6) *hold and* $\alpha \geq 1$. *If there exists a function* $\xi(t) \in C^1_{rd}(\mathbb{T},\mathbb{R}^+)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\Bigg\{c(\bar{c})^{\alpha}\xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha} - \frac{(r(\sigma(s)))^2}{\alpha(\bar{c})^{\alpha-1}\sigma^{\alpha-1}(s)r(t)\xi(t)}\left[\xi^{\Delta}(s)-\frac{p(s)\xi(s)}{r(\sigma(s))}\right]^2\Bigg\}\Delta s = \infty, \tag{2.25}$$

where $\bar{c} \in (0,1)$, *then inclusion* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t,\tau)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $a \leq \tau \leq b$. Proceeding as in the proofs of Theorems 2.3 and 2.4 to obtain (2.16) and (2.21). Now, from (2.16) we obtain

$$w^{\Delta}(t) \leq -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} + \left[\frac{\xi^{\Delta}(t)}{\xi^{\sigma}(t)} - \frac{p(t)\xi(t)}{r^{\sigma}(t)\xi^{\sigma}(t)}\right]w^{\sigma}(t) - \alpha\frac{\xi(t)r(t)}{(\xi^{\sigma}(t)r^{\sigma}(t))^{\frac{\alpha+1}{\alpha}}}w^2(\sigma(t))w^{\frac{1}{\alpha}-1}(\sigma(t)) \tag{2.26}$$

for $t \geq t_1$. Using (2.21) in (2.26) and noting that $\alpha \geq 1$ we have

$$w^{\Delta}(t) \leq -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} + \left[\frac{\xi^{\Delta}(t)}{\xi^{\sigma}(t)} - \frac{p(t)\xi(t)}{r^{\sigma}(t)\xi^{\sigma}(t)}\right]w^{\sigma}(t) - \alpha(\bar{c})^{\alpha-1}\sigma^{\alpha-1}(t)\frac{\xi(t)r(t)}{(\xi^{\sigma}(t)r^{\sigma}(t))^2}w^2(\sigma(t)) \tag{2.27}$$

for $t \geq t_1$ and so

$$w^{\Delta}(t) \leq -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} - \frac{(r(\sigma(t))^2}{\alpha(\bar{c})^{\alpha-1}\sigma^{\alpha-1}(t)r(t)\xi(t)}\left[\xi^{\Delta}(t) - \frac{p(t)\xi(t)}{r(\sigma(t))}\right]^2$$
$$= \left(\sqrt{R(t)}w(\sigma(t)) - \frac{P(t)}{2\sqrt{R(t)}}\right)^2,$$

where

$$P(t) = \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} - \frac{p(t)\xi(t)}{r(\sigma(t))\xi(\sigma(t))}$$

and

$$R(t) = \alpha(\bar{c})^{\alpha-1}\sigma^{\alpha-1}(t)\frac{\xi(t)r(t)}{(\xi^{\sigma}(t)r^{\sigma}(t))^2}.$$

Thus,

$$w^{\Delta}(t) \leq -c(\bar{c})^{\alpha}\xi(t)Q(t)\left(\frac{g(t)}{\sigma(t)}\right)^{\alpha} - \frac{(r(\sigma(t))^2}{\alpha(\bar{c})^{\alpha-1}\sigma^{\alpha-1}(t)r(t)\xi(t)}\left[\xi^{\Delta}(t) - \frac{p(t)\xi(t)}{r(\sigma(t))}\right]^2$$

for $t \geq t_1$. Integrating this inequality from t_0 to t and taking lim sup of both sides of the resulting inequality as $t \to \infty$, we arrive at the desired conclusion. This completes the proof. □

Next and for convenience, we consider the set

$$D = \{(t, s) : t \geq s \geq t_0, t, s \in [t_0, \infty)_{\mathbb{T}}\}.$$

We say that a function $H(t, s) \in \mathrm{C_{rd}}(D, \mathbb{R})$ satisfies condition [H] if

$$[H] \qquad H(t, t) = 0 \text{ for } t \geq t_0, \quad H(t, s) > 0 \text{ for } t \geq s \geq t_0,\ t, s \in [t_0, \infty)_{\mathbb{T}}$$

and has nonpositive Δ- partial derivative $H_s^{\Delta}(t, s)$ with respect to the second variable, i.e.,

$$H_s^{\Delta}(t, s) \in \mathrm{C_{rd}} \quad \text{and} \quad H_s^{\Delta}(t, s) \leq 0.$$

Theorem 2.8. *Let conditions* (1)–(5), (1.2), (2.1) *and* (2.6) *hold and H satisfies condition* [H]. *If there exist Δ-differentiable functions $\xi, \eta : \mathbb{T} \to \mathbb{R}^+$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[c(\bar{c})^{\alpha} H(t, s)\xi(s)\eta(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha} \Delta s - \frac{(r(\sigma(s))\eta(\sigma(s))h(t, s))^2}{4\alpha(\bar{c})^{\alpha-1}H(t, s)\eta(s)\sigma^{\alpha-1}(s)r(s)\xi(s)}\right]\Delta s = \infty$$

for some $t_1 \geq t_0$, where constants c, $\bar{c}$ are as in Theorem 2.3 *and*

$$h(t, s) = (H(t, s)\eta(s))_s^{\Delta} + H(t, s)\eta(s)\left[\frac{\xi^{\Delta}(s)}{\xi(\sigma(s))} - \frac{p(s)\xi(s)}{\xi(\sigma(s))r(\sigma(s))}\right], \qquad (2.28)$$

then inclusion (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t,\tau)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $a \leq \tau \leq b$. Proceeding as in the proof of Theorem 2.7, we obtain (2.27). Now, multiplying both sides of (2.27) by $\eta(s)H(t,s)$ and integrating from $t_1 \geq t_0$ to t, we obtain

$$
\begin{aligned}
c(\bar{c})^{\alpha}\int_{t_1}^{t} H(t,s)\eta(s)\xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s \leq -\int_{t_1}^{t} H(t,s)\eta(s)w^{\Delta}(s)\Delta s \\
+\int_{t_1}^{t} H(t,s)\eta(s)\left[\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} - \frac{p(s)\xi(s)}{r^{\sigma}(s)\xi^{\sigma}(s)}\right] w^{\sigma}(s)\Delta s \\
-\alpha(\bar{c})^{\alpha-1}\int_{t_1}^{t} H(t,s)\eta(s)\sigma^{\alpha-1}(s)\frac{\xi(s)r(s)}{(\xi^{\sigma}(s)r^{\sigma}(s))^2}w^2(\sigma(s))\Delta s \\
= -\left[H(t,s)\eta(s)w(s)\right]_{t_1}^{t} + \int_{t_1}^{t}(H(t,s)\eta(s))_s^{\Delta}w^{\sigma}(s))\Delta s \\
+\int_{t_1}^{t} H(t,s)\eta(s)\left[\frac{\xi^{\Delta}(s)}{\xi(s)} - \frac{p(s)\xi(s)}{r^{\sigma}(s)\xi^{\sigma}(s)}\right] w^{\sigma}(s)\Delta s \\
-\alpha(\bar{c})^{\alpha-1}\int_{t_1}^{t} H(t,s)\eta(s)\sigma^{\alpha-1}(s)\frac{\xi(s)r(s)}{(r^{\sigma}(s)\xi^{\sigma}(s))^2}w^2(\sigma(s))\Delta s \\
\leq H(t,t_1)\eta(t_1)w(t_1) \\
+\int_{t_1}^{t}\left\{(H(t,s)\eta(s))_s^{\Delta} + H(t,s)\eta(s)\left[\frac{\xi^{\Delta}(s)}{\xi^{\sigma}(s)} - \frac{p(s)\xi(s)}{r^{\sigma}(s)\xi^{\sigma}(s)}\right]\right\} w^{\sigma}(s)\Delta s \\
-\alpha(\bar{c})^{\alpha-1}\int_{t_1}^{t} H(t,s)\eta(s)\sigma^{\alpha-1}(s)\frac{\xi(s)r(s)}{(r^{\sigma}(s)\xi^{\sigma}(s))^2}w^2(\sigma(s))\Delta s \\
=: H(t,t_1)\eta(t_1)w(t_1) + \int_{t_1}^{t} h(t,s)w^{\sigma}(s)\Delta s \\
-\alpha(\bar{c})^{\alpha-1}\int_{t_1}^{t} H(t,s)\eta(s)\sigma^{\alpha-1}(s)\frac{\xi(s)r(s)}{(r^{\sigma}(s)\xi^{\sigma}(s))^2}w^2(\sigma(s))\Delta s \\
= H(t,t_1)\eta(t_1)w(t_1) - \int_{t_1}^{t}\left[\frac{h^2(t,s)\left(r^{\sigma}(s)\ \xi^{\sigma}(s)\right)^2}{4\alpha(\bar{c})^{\alpha-1}H(t,s)\eta(s)\sigma^{\alpha-1}(s)r(s)\xi(s)}\right]\Delta s \\
-\int_{t_1}^{t}\left(\sqrt{R(t,s)}w^{\sigma}(s) - \frac{h(t,s)}{2\sqrt{R(t,s)}}\right)^2\Delta s
\end{aligned}
$$

and so

$$
\begin{aligned}
c(\bar{c})^{\alpha}\int_{t_1}^{t} H(t,s)\eta(s)\xi(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}\Delta s \leq H(t,t_1)\eta(t_1)w(t_1) \\
-\int_{t_1}^{t}\left[\frac{\left(h(t,s)r^{\sigma}(s)\xi^{\sigma}(s)\right)^2}{4\alpha(\bar{c})^{\alpha-1}H(t,s)\eta(s)\sigma^{\alpha-1}(s)r(s)\xi(s)}\right]\Delta s,
\end{aligned}
$$

where

$$
R(t,s) = \alpha(\bar{c})^{\alpha-1}H(t,s)\eta(s)\sigma^{\alpha-1}(s)\frac{\xi(s)r(s)}{\left(\xi(\sigma(s))r(\sigma(s))\right)^2}.
$$

Now, we can easily see that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\Bigg[c(\bar{c})^{\alpha}H(t,s)\eta(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha} - \frac{(h(t,s)r^{\sigma}(s)\xi^{\sigma}(s))^2}{4\alpha(\bar{c})^{\alpha-1}H(t,s)\eta(s)\sigma^{\alpha-1}(s)r(s)\xi(s)}\Bigg]\Delta s \le \eta(t_1)w(t_1).$$

Taking lim sup of both sides of this inequality as $t\to\infty$, we obtain a contradiction. This completes the proof. □

The following result is immmediate.

Theorem 2.9. *Let conditions* (1)–(5)*,* (1.2)*,* (2.1) *and* (2.6) *hold and H satisfies condition* [H]*. If there exist Δ-differentiable functions $\xi,\eta:\mathbb{T}\to\mathbb{R}^+$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\xi(s)\eta(s)Q(s)\left(\frac{g(s)}{\sigma(s)}\right)^{\alpha}-\frac{h(t,s)}{c\bar{c}^{\alpha}}\frac{r(\sigma(s))\xi(\sigma(s))}{\sigma^{\alpha}(s)}\right]\Delta s=\infty, \tag{2.29}$$

for some $t_1\in[t_0,\infty)$, $\bar{c}\in(0,1)$ and h is defined as in (2.28)*, then inclusion* (1.1) *is oscillatory.*

Remark 2.10.
- In condition (4), we may select $g(t,\tau)$ to be nondecreasing with respect to the second variable τ. In this case, we assume $g(t)=g(t,a)$ and the obtained results are valid.
- In Theorems 2.8 and 2.9, we may assume H to be

$$H(t,s)=(t-s)^m,\quad t\ge s\ge t_0,\quad t,s\in[t_0,\infty)_{\mathbb{T}} \text{ and } m\ge 1,$$
$$\left(\ln\frac{t+1}{s+1}\right)^m,\quad t\ge s\ge t_0,\quad t,s\in[t_0,\infty)_{\mathbb{T}} \text{ and } m\ge 1$$

etc. The details are left to the readers.

REFERENCES

[1] E. Akın-Bohner, and S. Sun, Oscillation Criteria for a class of second order strongly superlinear and sublinear dynamic inclusions. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 19, 2012, no. 3, 285–294.

[2] R. P. Agarwal, S. R. Grace and D. O'Regan, Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations. Kluwer, Dordrecht, 2002.

[3] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, Discrete Oscillation Theory, *Hindawi Publishing Corporation*, 2005.

[4] R. P. Agarwal, S. R. Grace, D. O'Regan, On nonoscillatory solutions of differential inclusions, Prec. Amer. Math. Soc., 181(2003), 129–140.

[5] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation criteria for sublinear and superlinear second order differential inclusions, Mem. Diff. Eqns. Math. Physic. 28(2003), 1–12.

[6] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation theorems for second order differential inclusions. Ont. J. Dynamic System. Diff. Eqns., 1(2007), 85–88.

[7] R. P. Agarwal, S. R. Grace, D. O'Regan, Some nonoscillation criteria for inclusions, J. Aust. Math. Soc., 80(2006), 1–12.

[8] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation criteria for second order differential inclusions, Adv. Stud. Contem. Math 16(2008), 47–56.

[9] M. Bohner, Some oscillation criteria for first order delay dynamic equations. Far J. Appl. Math. 18(2005), 289–304.

[10] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston, 2001.

[11] T. Candan, Oscillation of second order nonlinear neutral dynamic equation on time scales with distributed deviating arguments. Compt. Math. Appl. 62(11)(2011), 471–491.

[12] L. Erbe, A. Peterson and P. Rehak, Comparison theorems for linear dynamic equations on time scales, J. Math. Anal. Appl., 275(2002), 418–438.

[13] L. Erbe, T. Hassan and A. Peterson, Oscillation criteria for nonlinear functional neutral dynamic equations on time scales, J. Diff. Eqn. Appl., 15(2009), 1097–116.

[14] S. R. Grace, R. P. Agarwal and D. O'Regan, A selection of oscillation criteria for second order differential inclusions, Appl. Math Letters. 22(2009), 153–158.

[15] S. R. Grace, R. P. Agarwal, M. Bohner and D. O'Regan, Oscillation of second order strongly superlinear and strongly sublinear dynamic inclusions, Comum. Nonlinear Sci. Numer. Stimul., 14(2009), 3463–3471.

[16] S. R. Grace, R. P. Agarwal, B. Kaymakalan and W. Sae-jie, Oscillation theorems for second order nonlinear dynamic equations, Appl. Math. Comput., 32(2010), 205–218.

[17] S. R. Grace, E. Akın and M. Dikmen, On the oscillation of second order nonlinear neutral dynamic equations with distributed deviating arguments on time scales, Dynamic Systems and Applications. To appear, 2014.

[18] S. R. Grace, M. Bohner and R. P. Agarwal,On the oscillation of second order half-linear dynamic equations, J. Difference Eqn. Appl., 15(2009), 451–460.

[19] S. Hilger, Analysis on measure chains- a unified approach to continuous and discrete calculus, Results Math., 19(1990), 18–56.

[20] S. H. Saker, Oscillation criteria for second order halflinear dynamic equations on time scales, J. Comput. Appl. Math., 177(2005), 375–387.