Boundary Value Problems For A Differential Equation On A Measure Chain

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Abstract

We will prove existence and uniqueness theorems for solution of the boundary value problem $x^{\Delta\Delta}(t) = f(t, x^\sigma(t)),\ x(a) = A, x(\sigma^2(b)) = B$ for t in a measure chain $\mathbb{T}$. In one of our results we use upper and lower solutions to prove the existence of a solution to this boundary value problem (BVP). We then use this result to show that if for each fixed t, $f(t, x)$ is strictly increasing in x, then this BVP has a unique solution. In our last result we get an existence-uniqueness theorem in the case where f satisfies a one sided Lipschitz condition.

Key words: *measure chains*, *lower and upper solutions*
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1 Introduction

We are concerned with the boundary value problem (BVP)

$$x^{\Delta\Delta} = f(t, x^\sigma(t)),$$

$$x(a) = A, \quad x(\sigma^2(b)) = B$$

on a measure chain $\mathbb{T}$, where we assume $f(t, x)$ is continuous on $[a, b] \times \mathbb{R}$. We need some preliminary definitions and theorems.

Definition A *measure chain* (time scale) is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition Let $\mathbb{T}$ be a measure chain and define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) := inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. In this definition we put $\sigma(\emptyset) = \sup \mathbb{T}$ and the *backward jump operator* ρ on $\mathbb{T}$ by

$$\rho(t) := sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. In this definition we put $\rho(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$ we say t is *left- dense*.

Throughout this paper we make the blanket assumption that $a \leq b$ are points in $\mathbb{T}$.

Definition Define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Other types of intervals are defined similarly. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

We are concerned with calculus on measure chains whose introduction is given in S. Hilger [7]. Some recent papers concerning differential equations on measure chains were written by Agarwal and Bohner [1, 2], Agarwal, Bohner, and Wong [3], Erbe and Hilger [5], Erbe and Peterson [6]. Some preliminary definitions and theorems on measure chains can also be found in Kaymakçalan, Lakshmikantham, and Sivasundaram [8].

Definition Assume $f : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$, then we define $f^\Delta(t)$ to be the number (provided it exists) with property that given any $\epsilon > 0$, there is a neighorhood U of t such that

$$\mid [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \mid \leq \epsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of $f(t)$ and it turns out that f^Δ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

Some elementary facts that we will use concerning the delta derivative are contained in the following theorem due to Hilger.

Theorem 1 *Assume $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

1. *If f is differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

3. *If f is differentiable and t is right dense, then*

$$f^\Delta(t) = lim_{s \to t} \frac{f(t) - f(t)}{t - s}.$$

4. *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + (\sigma(t) - t)f^\Delta(t)$$

Definition A function $F : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \mapsto \mathbb{R}$ provided

$$F^\Delta(t) = f(t)$$

holds for all $t \in \mathbb{T}^\kappa$. We define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for $t \in \mathbb{T}$.

Definition We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *right-dense continuous* provided at any right-dense point $t \in \mathbb{T}$

$$lim_{s \to t^+} f(s) = f(t)$$

and

if $t \in \mathbb{T}$ is left-dense we assume

$$lim_{s \to t^-} f(s)$$

exists and is finite.

Definition Let $a, b \in \mathbb{T}$ and assume that $\sigma^2(b) \in \mathbb{T}$. We want to consider $Lx(t) = 0$ on the interval $[a, \sigma^2(b)]$. We say a nontrivial solution of $Lx(t) = 0$ has a *generalized zero* at a iff $x(a) = 0$. We say a nontrivial solution x has a generalized zero at $t_0 \in (a, \sigma^2(b)]$ provided either $x(t_0) = 0$ or $x(\rho(t_0))x(t_0) < 0$. Finally we say that $Lx(t) = 0$ is *disconjugate* on $[a, \sigma^2(b)]$ provided there is no nontrivial solution of $Lx(t) = 0$ with two (or more) generalized zeros in $[a, \sigma^2(b)]$.

Definition Let X and Y be Banach spaces. We say $T : X \mapsto Y$ is *compact* provided it is continuous and T maps bounded sequences into sequentially compact sequences. In this paper we will make use of the following well known theorem whose proof is given in Deimling [4] and Zeidler [10].

Theorem 2 *(Schauder Fixed Point Theorem) Assume X is a Banach space and K is a closed, bounded and convex subset of X. If $T : K \mapsto K$ is compact, then T has a fixed point in K.*

An excellent explanation of nonlinear BVPs for difference equation can be found in Kelley and Peterson ([9],Chapter 9).

2 Main Results

Definition Let

$$D := max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G(t,s)| \Delta s$$

where $G(t, s)$ is the Green's function for the BVP

$$x^{\Delta\Delta}(t) = 0,$$

$$x(a) = 0, \quad x(\sigma^2(b)) = 0$$

on a measure chain $\mathbb{T}$. If $\mathbb{T}$=$\mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, then it is well known that $D = \frac{(\sigma^2(b)-a)^2}{8}$.

Theorem 3 *Assume $f(t,x)$ is continuous on $[a,b]\times\mathbb{R}$. If $M>0$ satisfies $M \geq max\{|A|,|B|\}$ and $D \leq \frac{M}{Q}$ where $Q>0$ satisfies*

$$Q \geq max\{|f(t,x)| : t \in [a,b], |x| \leq 2M\},$$

then the BVP

$$x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)), \quad t \in [a,b] \tag{1}$$

$$x(a) = A, \quad x(\sigma^2(b)) = B \tag{2}$$

has a solution.

Proof: Define X to be the Banach space $X = C[a, \sigma^2(b)]$ equipped with the norm $\|\cdot\|$ defined by

$$\|x\| := max_{t\in[a,\sigma^2(b)]}|x(t)|.$$

Let

$$K := \{x \in X : ||x|| \leq 2M\}.$$

It can be shown that K is a closed, bounded and convex subset of X. Define $A : K \mapsto X$ by

$$Ax(t) := z(t) + \int_a^{\sigma(b)} G(t,s) f(s, x^{\sigma}(s))\Delta s$$

for $t \in [a, \sigma^2(b)]$, where $z(t)$ is the solution of the BVP

$$z^{\Delta\Delta}(t) = 0,$$

$$z(a) = A, \quad z(\sigma^2(b)) = B.$$

It can be shown that $A : K \mapsto X$ is continuous.
Claim $A : K \mapsto K$:
Let $x \in K$. Consider

$$\begin{aligned} |Ax(t)| &= |z(t) + \int_a^{\sigma(b)} G(t,s) f(s, x^{\sigma}(s))\Delta s| \\ &\leq |z(t)| + \int_a^{\sigma(b)} |G(t,s)||f(s, x^{\sigma}(s))|\Delta s \\ &\leq M + Q\int_a^{\sigma(b)} |G(t,s)|\Delta s \\ &\leq M + QD \\ &\leq M + Q\frac{M}{Q} \\ &= 2M \end{aligned}$$

for all $t \in [a, \sigma^2(b)]$. But this implies that $||Ax|| \leq 2M$. Hence $A : K \mapsto K$.
It can be shown that $A : K \mapsto K$ is a compact operator by the Ascoli-Arzela Theorem. Hence A has a fixed point in K by Theorem 2. □

Corollary 4 *If $f(t,x)$ is continuous and bounded on $[a,b]\times\mathbb{R}$, then the BVP (1), (2) has a solution.*

Proof: Choose $P > \sup\{|f(t,x)| : a \le t \le b, x \in \mathbb{R}\}$. Then, pick M large enough so that

$$D < \frac{M}{P}$$

and

$$|A| \le M, \ \ |B| \le M.$$

Then there is a number $Q > 0$ such that

$$P \ge Q \ \ where \ \ Q \ge max\{|f(t,x)| : t \in [a,b], \ \ |x| \le 2M\}.$$

Hence

$$D < \frac{M}{P} \le \frac{M}{Q}$$

and so, the given BVP has a solution by Theorem 3. □

Define

$$\mathbb{D} := \{x : x^{\Delta}(t) \ is\ continuous\ on\ [a,\sigma(b)]\ and\ x^{\Delta\Delta}(t)\ is\ right-dense\ continuous\ on\ [a,b]\}.$$

Definition We say $\alpha \in \mathbb{D}$ is a *lower solution* of (1) on $[a,\sigma^2(b)]$ provided

$$\alpha^{\Delta\Delta}(t) \ge f(t,\alpha^{\sigma}(t))$$

on $[a,b]$. We say $\beta \in \mathbb{D}$ is an *upper solution* of (1) on $[a,\sigma^2(b)]$ provided

$$\beta^{\Delta\Delta}(t) \le f(t,\beta^{\sigma}(t))$$

on $[a,b]$.

Theorem 5 *Assume $f(t,x)$ is continuous on $[a,b]\times\mathbb{R}$ and there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1) and*

$$\alpha(a) \le A \le \beta(a), \ \ \alpha(\sigma^2(b)) \le B \le \beta(\sigma^2(b))$$

such that

$$\alpha(t) \le \beta(t)$$

on $[a,\sigma^2(b)]$. Then the BVP (1), (2) has a solution $x(t)$ with

$$\alpha(t) \le x(t) \le \beta(t)$$

on $[a,\sigma^2(b)]$.

Proof: Define the modification of f with respect to α and β by for each fixed $t \in [a,b]$

$$F(t,x) = \begin{cases} f(t,\beta^\sigma(t)) + \frac{x-\beta^\sigma(t)}{1+|x|} & \text{if } x \geq \beta^\sigma(t) \\ f(t,x) & \text{if } \alpha^\sigma(t) \leq x \leq \beta^\sigma(t) \\ f(t,\alpha^\sigma(t)) + \frac{x-\alpha^\sigma(t)}{1+|x|} & \text{if } x \leq \alpha^\sigma(t). \end{cases}$$

Note that $F(t,x)$ is continuous and bounded on $[a,b] \times \mathbb{R}$ and $F(t,x) = f(t,x)$ if $\alpha^\sigma(t) \leq x \leq \beta^\sigma(t)$ for $t \in [a,b]$.
By Corollary 4, the BVP

$$x^{\Delta\Delta} = F(t, x^\sigma(t)),$$

$$x(a) = A, \quad x(\sigma^2(b)) = B$$

has a solution $x(t)$. To complete the proof it suffices to show that

$$\alpha(t) \leq x(t) \leq \beta(t)$$

on $[a, \sigma^2(b)]$.
Claim $x(t) \leq \beta(t)$ for $t \in [a, \sigma^2(b)]$:
Assume not, then if $z(t) := x(t) - \beta(t)$, then $z(t)$ has a positive maximum in $(a, \sigma^2(b))$.
Choose $c \in (a, \sigma^2(b))$ so that $z(c) = max\{z(t) : t \in [a, \sigma^2(b)]\} > 0$ and $z(t) < z(c)$ for $t \in (c, \sigma^2(b)]$.
There are four cases to consider:

1. $\rho(c) = c < \sigma(c)$
2. $\rho(c) < c < \sigma(c)$
3. $\rho(c) < c = \sigma(c)$
4. $\rho(c) = c = \sigma(c)$.

We will show that the first case is impossible and in the other cases we will show that

$$z^\Delta(c) \leq 0 \;\; and \;\; z^{\Delta\Delta}(\rho(c)) \leq 0.$$

Case 1: $\rho(c) = c < \sigma(c)$.
Claim this case is impossible:
Assume $z^\Delta(c) \geq 0$. If $z^\Delta(c) > 0$, then $z(\sigma(c)) > z(c)$. But this contradicts the way c was chosen. If $z^\Delta(c) = 0$, then $z(\sigma(c)) = z(c)$. But this also contradicts the way c was chosen.
Assume $z^\Delta(c) < 0$, then $lim_{t \to c^-} z^\Delta(t) = z^\Delta(c) < 0$. This implies that there exits a $\delta > 0$ such that $z^\Delta(t) < 0$ on $(c-\delta, c]$. Hence $z(t)$ is strictly decreasing on $(c-\delta, c]$. But this contradicts the way c was chosen. Therefore this case is impossible.
Case 2: $\rho(c) < c < \sigma(c)$.
It is easy to check that $z^\Delta(c) < 0$ and $z^{\Delta\Delta}(\rho(c)) < 0$.

Case 3: $\rho(c) < c = \sigma(c)$.

Claim $z^{\Delta}(c) \leq 0$ and $z^{\Delta\Delta}(\rho(c)) \leq 0$:

Assume $z^{\Delta}(c) > 0$, then $lim_{t \to c^+} z^{\Delta}(t) = z^{\Delta}(c) > 0$. This implies that there exists a $\delta > 0$ such that $z^{\Delta}(t) > 0$ on $[c, c+\delta)$. Hence $z(t)$ is strictly increasing on $[c, c+\delta)$. But this contradicts the way c was chosen. Therefore $z^{\Delta}(c) \leq 0$. Since $\rho(c)$ is right-scattered,

$$z^{\Delta\Delta}(\rho(c)) = \frac{z^{\Delta}(c) - z^{\Delta}(\rho(c))}{c - \rho(c)} \leq 0.$$

Case 4: $\rho(c) = c = \sigma(c)$.

Claim $z^{\Delta}(c) = 0$ and $z^{\Delta\Delta}(\rho(c)) \leq 0$:

Using the same proof as in Case 3 we have that $z^{\Delta}(c) \leq 0$. Assume $z^{\Delta}(c) < 0$, then $lim_{t \to c} z^{\Delta}(t) = z^{\Delta}(c) < 0$. This implies that there exists a $\delta > 0$ such that $z^{\Delta}(t) < 0$ on $(c-\delta, c]$. Hence $z(t)$ is strictly decreasing on $(c-\delta, c]$. But this contradicts the way c was chosen.

Assume $z^{\Delta\Delta}(\rho(c)) > 0$, then $lim_{t \to \rho(c)} z^{\Delta\Delta}(t) = z^{\Delta\Delta}(\rho(c)) = z^{\Delta\Delta}(c) > 0$. This implies that there exists a $\delta > 0$ such that $z^{\Delta\Delta}(t) > 0$ on $(c-\delta, c+\delta)$. Hence $z^{\Delta}(t)$ is strictly increasing on $(c-\delta, c+\delta)$. But $z^{\Delta}(c) = 0$ hence $z^{\Delta}(t) > 0$ on $(c, c+\delta)$. This implies that $z(t)$ is strictly increasing on $(c, c+\delta)$. But this contradicts the way c was chosen. Therefore $z^{\Delta\Delta}(\rho(c)) \leq 0$.

Hence

$$\begin{aligned} x(c) &> \beta(c) \\ x^{\Delta}(c) &\leq \beta^{\Delta}(c) \\ x^{\Delta\Delta}(\rho(c)) &\leq \beta^{\Delta\Delta}(\rho(c)). \end{aligned}$$

But

$$\begin{aligned} x^{\Delta\Delta}(\rho(c)) &= F(\rho(c), x^{\sigma}(\rho(c))) \\ &= f(\rho(c), \beta^{\sigma}(\rho(c)) + \frac{x^{\sigma}(\rho(c)) - \beta^{\sigma}(\rho(c))}{1 + |x^{\sigma}(\rho(c))|} \\ &= f(\rho(c), \beta^{\sigma}(\rho(c))) + \frac{x(c) - \beta(c)}{1 + |x(c)|} \\ &> f(\rho(c), \beta^{\sigma}(\rho(c))) \\ &\geq \beta^{\Delta\Delta}(\rho(c)) \end{aligned}$$

since $\sigma(\rho(c)) = c$, $x(c) > \beta(c)$ and β is an upper solution of (1) on $[a, \sigma^2(b)]$.

Hence $x^{\Delta\Delta}(\rho(c)) > \beta^{\Delta\Delta}(\rho(c))$. But this contradicts the fact that $x^{\Delta\Delta}(\rho(c)) \leq \beta^{\Delta\Delta}(\rho(c))$. Therefore $x(t) \leq \beta(t)$ for $t \in [a, \sigma^2(b)]$.

Similarly, one can show that $\alpha(t) \leq x(t)$ for $t \in [a, \sigma^2(b)]$. Therefore $x(t)$ solves the BVP (1), (2). □

Example 6 *Consider the BVP*

$$x^{\Delta\Delta}(t) = -\cos x^{\sigma}(t),$$

$$x(0) = 0, \quad x(\sigma^2(b)) = 0.$$

First, note that $\alpha(t) = 0$ is a lower solution on $[0, \sigma^2(b)]$ since

$$\alpha^{\Delta\Delta}(t) = 0 > -\cos 0 = -1.$$

Next, let $\beta(t) = \int_0^t (c-s)\Delta s$ where $c = \frac{1}{\sigma^2(b)} \int_0^{\sigma^2(b)} \tau \Delta\tau$. Then

$$\beta^{\Delta\Delta}(t) = -1 < -\cos \beta^\sigma(t),$$

so $\beta(t)$ is an upper solution on $[0, \sigma^2(b)]$.

Note that $\alpha(0) = 0 = \beta(0)$, $\alpha(\sigma^2(b)) = \beta(\sigma^2(b))$ and $\beta(t) = -\int_0^{\sigma(b)} G(t,s)\Delta s$ is a solution of BVP

$$\beta^{\Delta\Delta}(t) = -1,$$

$$\beta(0) = 0, \ \ \beta(\sigma^2(b)) = 0.$$

Since $G(t,s) \leq 0$ for $t \in [0, \sigma^2(b)]$ and $s \in [0,b]$, $\beta(t) \geq 0$ on $[0, \sigma^2(b)]$.

Therefore we can conclude that there is a solution $x(t)$ with

$$0 \leq x(t) \leq \int_0^t (c-s)\Delta s$$

on $[0, \sigma^2(b)]$ by Theorem 5.

In the following theorem we see that if for each fixed t, $f(t,x)$ is strictly increasing in x, then the BVP (1), (2) has a unique solution.

Theorem 7 *Assume $f(t,x)$ is continuous on $[a,b] \times \mathbb{R}$ and assume for each fixed $t \in [a,b]$, $f(t,x)$ is nondecreasing in x, $-\infty < x < \infty$. Then the BVP (1), (2) has a solution.*

If, for each $t \in [a,b]$, $f(t,x)$ is strictly increasing in x, then the BVP (1), (2) has a unique solution.

Proof: Choose $M \geq \max\{|f(t,0)| : t \in [a,b]\}$. Let $u(t)$ be the solution of the BVP

$$u^{\Delta\Delta}(t) = M, \ \ t \in [a,b]$$

$$u(a) = 0, \ \ u(\sigma^2(b)) = 0.$$

This implies that $u(t) \leq 0$ on $[a, \sigma^2(b)]$. Pick $K \geq \max\{|A|, |B|\}$.

Set

$$\alpha(t) = u(t) - K$$

on $[a, \sigma^2(b)]$. Then

$$\begin{aligned} \alpha^{\Delta\Delta}(t) = u^{\Delta\Delta}(t) = M &\geq f(t,0) \\ &\geq f(t, \alpha^\sigma(t)) \end{aligned}$$

on $[a,b]$ since $f(t,x)$ is nondecreasing in x. Therefore $\alpha(t)$ is a lower solution of (1) on $[a,\sigma^2(b)]$. Next, let $v(t)$ be the solution of the BVP

$$v^{\Delta\Delta}(t) = -M,$$

$$v(a) = 0, \quad v(\sigma^2(b)) = 0.$$

This implies that $v(t) \geq 0$ *on* $[a,\sigma^2(b)]$. Then set

$$\beta(t) = v(t) + K$$

on $[a,\sigma^2(b)]$. It follows that

$$\begin{aligned} \beta^{\Delta\Delta}(t) = v^{\Delta\Delta}(t) = -M &\leq f(t,0) \\ &\leq f(t,\beta^\sigma(t)) \end{aligned}$$

on $[a,b]$ since $f(t,x)$ is nondecreasing in x. Therefore $\beta(t)$ is an upper solution of (1) on $[a,\sigma^2(b)]$.

Note that $\alpha(a) \leq A \leq \beta(a), \quad \alpha(\sigma^2(b)) \leq B \leq \beta(\sigma^2(b))$ and $\alpha(t) \leq \beta(t)$ on $[a,\sigma^2(b)]$. Therefore there exits a solution $x(t)$ of the BVP (1), (2) with

$$\alpha(t) \leq x(t) \leq \beta(t)$$

on $[a,\sigma^2(b)]$ by Theorem 5.

Now assume for each $t \in [a,b]$, $f(t,x)$ is strictly increasing in x, $-\infty < x < \infty$. Assume $x(t)$, $y(t)$ are distinct solutions of the BVP (1), (2). Without loss of generality, assume $x(t) > y(t)$ at some points in $(a,\sigma^2(b))$. This implies that $x(t) - y(t)$ has a positive maximum in $(a,\sigma^2(b))$. Hence, there exists $c \in (a,\sigma^2(b))$ such that

$$x(c) > y(c)$$

$$x^\Delta(c) \leq y^\Delta(c)$$
$$x^{\Delta\Delta}(\rho(c)) \leq y^{\Delta\Delta}(\rho(c)).$$

But

$$\begin{aligned} x^{\Delta\Delta}(\rho(c)) &= f(\rho(c), x^\sigma(\rho(c))) \\ &> f(\rho(c), y^\sigma(\rho(c))) \\ &= y^{\Delta\Delta}(\rho(c)) \end{aligned}$$

since $\sigma(\rho(c)) = c$ but this contradicts the fact that $x^{\Delta\Delta}(\rho(c)) \leq y^{\Delta\Delta}(\rho(c))$. Therefore the BVP (1), (2) has a unique solution. □

The following example is a simple implication of Theorem 7.

Example 8 *If $c(t)$, $d(t)$ and $e(t)$ are right-dense continuous functions on $[a, \sigma^2(b)]$ with $c(t) \geq 0$, $d(t) \geq 0$ on $[a, b]$, then the BVP*

$$x^{\Delta\Delta}(t) = c(t)x^\sigma(t) + d(t)[x(\sigma(t))]^3 + e(t),$$

$$x(a) = A, \quad x(\sigma^2(b)) = B$$

has a solution. Further if $c(t) + d(t) > 0$ on $[a, b]$, then the above BVP has a unique solution.

The next theorem is a generalization of the uniqueness of solutions of initial value problems(IVP's) for (1).

Theorem 9 *Assume $f(t, x)$ is continuous on $[a, b] \times \mathbb{R}$ and solutions of IVPs for $x^{\Delta\Delta} = f(t, x^\sigma)$ are unique. Assume α and β are lower and upper solutions of (1) respectively on $[a, \sigma^2(b)]$ such that $\alpha(t) \leq \beta(t)$ on $[a, \sigma^2(b)]$. If there exists $t_0 \in [a, \sigma(b)]$ such that*

$$\alpha(t_0) = \beta(t_0)$$

$$\alpha^\Delta(t_0) = \beta^\Delta(t_0),$$

then $\alpha(t) \equiv \beta(t)$ on $[a, \sigma^2(b)]$.

Proof: Assume $\alpha(t) \not\equiv \beta(t)$ on $[a, \sigma^2(b)]$.
First consider the case where $t_0 < \sigma^2(b)$ and $\alpha(t) < \beta(t)$ for at least one point in $(t_0, \sigma^2(b)]$. Pick

$$t_1 = \max\{t : \alpha(s) = \beta(s), t_0 \leq s \leq t\} < \sigma^2(b).$$

We have two cases to consider:
Case 1: $\sigma(t_1) = t_1$.
There exists $t_2 \in \mathbb{T}$ with $t_1 < t_2$ such that $\alpha(t) < \beta(t)$ on $(t_1, t_2]$.
By Theorem 5, the BVP

$$x^{\Delta\Delta}(t) = f(t, x^\sigma(t)),$$

$$x(t_1) = \beta(t_1), \;\; x(t_2) = \beta(t_2)$$

has a solution $x_1(t)$ satisfying $\alpha(t) \leq x_1(t) \leq \beta(t)$ on $[t_1, t_2]$.
Similarly, the BVP

$$x^{\Delta\Delta}(t) = f(t, x^\sigma(t)),$$

$$x(t_1) = \alpha(t_1), \;\; x(t_2) = \alpha(t_2)$$

has a solution $x_2(t)$ satisfying $\alpha(t) \leq x_2(t) \leq \beta(t)$ on $[t_1, t_2]$.
Since $\alpha(t) \leq x_i(t) \leq \beta(t)$, $i = 1, 2$ on $[t_1, t_2]$, $x_1^\Delta(t_1) = x_2^\Delta(t_1)$. Since solution of IVPs are unique, $x_1(t) \equiv x_2(t)$. But this contradicts the fact that $x_1(t_2) \neq x_2(t_2)$.
Case 2: $\sigma(t_1) > t_1$.
First we need to show that $t_1 > t_0$.

Assume not, then $t_1 = t_0$. By assumption,

$$\alpha(t_0) = \beta(t_0)$$

$$\alpha^{\Delta}(t_0) = \beta^{\Delta}(t_0),$$

and hence $\alpha(\sigma(t_0) = \beta(\sigma(t_0))$. But this contradicts the way t_1 was chosen and hence $t_1 > t_0$.
There are two subcases:
Subcase 1: $\rho(t_1) < t_1$.
Since

$$\alpha(\rho(t_1)) = \beta(\rho(t_1)), \ \ \alpha(t_1) = \beta(t_1) \ \ and \ \ \alpha(\sigma(t_1)) < \beta(\sigma(t_1)),$$
$$\beta^{\Delta\Delta}(\rho(t_1)) > \alpha^{\Delta\Delta}(\rho(t_1)).$$

But

$$\begin{aligned} \alpha^{\Delta\Delta}(\rho(t_1)) &\geq f(\rho(t_1), \alpha^{\sigma}(\rho(t_1))) \\ &= f(\rho(t_1), \alpha(t_1)) \\ &= f(\rho(t_1), \beta(t_1)) \\ &= f(\rho(t_1), \beta^{\sigma}(\rho(t_1))) \\ &\geq \beta^{\Delta\Delta}(\rho(t_1)) \end{aligned}$$

since $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1) on $[a, \sigma^2(b)]$. This is a contradiction.
Subcase 2: $\rho(t_1) = t_1$.
By continuity

$$\begin{aligned} \beta^{\Delta}(t_1) &= lim_{t \to t_1^-} \beta^{\Delta}(t) \\ &= lim_{t \to t_1^-} \alpha^{\Delta}(t) \\ &= \alpha^{\Delta}(t_1). \end{aligned}$$

This implies that $\beta(\sigma(t_1)) = \alpha(\sigma(t_1))$ and we get a contradiction to the way t_1 was chosen.
Therefore $\alpha(t) \equiv \beta(t)$ on $[t_0, \sigma^2(b)]$.
Next consider the other case where $a < t_0$ and $\alpha(t) < \beta(t)$ for at least one point in $[a, t_0)$.
This time pick

$$t_1 = min\{t : \alpha(s) = \beta(s), t \leq s \leq t_0\} > a.$$

We have two cases:
Case 1: $\rho(t_1) = t_1$.
There exists $t_2 \in \mathbb{T}$ with $t_2 < t_1$ such that $\alpha(t) < \beta(t)$ on $[t_2, t_1)$.
By Theorem 5, the BVP

$$x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t))$$

$$x(t_1) = \beta(t_1), \ \ x(t_2) = \beta(t_2)$$

has a solution $x_1(t)$ satisfying $\alpha(t) \leq x_1(t) \leq \beta(t)$ on $[t_2, t_1]$.

Similarly, the BVP

$$x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t))$$

$$x(t_1) = \alpha(t_1), \;\; x(t_2) = \alpha(t_2)$$

has a solution $x_2(t)$ satisfying $\alpha(t) \le x_2(t) \le \beta(t)$ on $[t_2, t_1]$.
Since $\alpha(t) \le x_i(t) \le \beta(t)$ for $i = 1, 2$, $t \in [t_2, t_1]$, $x_1^{\Delta}(t_1) = x_2^{\Delta}(t_1)$. Since solutions of IVPs are unique, $x_1(t) \equiv x_2(t)$ on $[t_2, t_1]$. But this contradicts the fact that $x_1(t_2) \neq x_2(t_2)$.
Case 2: $\rho(t_1) < t_1$.
Note that $\alpha(t_1) = \beta(t_1)$, $\alpha^{\Delta}(t_1) = \beta^{\Delta}(t_1)$ and $\alpha(\rho(t_1)) < \beta(\rho(t_1))$. Hence

$$\beta^{\Delta\Delta}(\rho(t_1)) > \alpha^{\Delta\Delta}(\rho(t_1)).$$

But

$$\begin{aligned} \alpha^{\Delta\Delta}(\rho(t_1)) &\ge f(\rho(t_1), \alpha^{\sigma}(\rho(t_1))) \\ &= f(\rho(t_1), \beta^{\sigma}(\rho(t_1))) \\ &\ge \beta^{\Delta\Delta}(\rho(t_1)) \end{aligned}$$

and so this is a contradiction.
Therefore $\alpha(t) \equiv \beta(t)$ on $[a, t_0]$.
Hence $\alpha(t) \equiv \beta(t)$ on $[a, \sigma^2(b)]$. □

In the next theorem we prove an existence-uniqueness theorem for solutions of the BVP (1), (2) where we assume $f(t, x)$ satisfies a one sided Lipschitz condition.

Theorem 10 *Assume $f(t, x)$ is continuous on $[a, b] \times \mathbb{R}$, solutions of the IVPs are unique and exist on $[a, \sigma^2(b)]$ for (1), and there exists a right-dense continuous function $k(t)$ on $[a, b]$ such that*

$$f(t, x) - f(t, y) \ge k(t)\ (x - y)$$

for $x \ge y$, $t \in [a, b]$.
If $x^{\Delta\Delta} = k(t)x^{\sigma}$ is disconjugate on $[a, \sigma^2(b)]$, then the BVP (1), (2) has a unique solution.

Proof: Let $x(t, m)$ be the solution of the IVP

$$x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)),$$

$$x(a) = A, \quad x^{\Delta}(a) = m.$$

Define $S := \{x(\sigma^2(b), m) : m \in \mathbb{R}\}$. By continuity of solutions on initial conditions, S is a connected set. We want to show that S is neither bounded above nor below.
Fix $m_1 > m_2$ and let

$$w(t) := x(t, m_1) - x(t, m_2).$$

Note that $w(a) = 0$ and $w^{\Delta}(a) = m_1 - m_2 > 0$.
Claim: $w(t) > 0$ on $(a, \sigma^2(b)]$:

Pick

$$t_1 = max\{t \in [a, \sigma^2(b)] : w(s) \geq 0 \ \ for\ s\ \in [a, t]\}.$$

Then

$$\begin{aligned} w^{\Delta\Delta}(t) &= x^{\Delta\Delta}(t, m_1) - x^{\Delta\Delta}(t, m_2) \\ &= f(t, x^\sigma(t, m_1)) - f(t, x^\sigma(t, m_2)) \\ &\geq k(t)\,[x^\sigma(t, m_1) - x^\sigma(t, m_2)] \end{aligned}$$

on $[a, \rho(t_1)]$.

Hence $w^{\Delta\Delta}(t) - k(t)w^\sigma(t) \geq 0$ on $[a, \rho(t_1)]$.

Define

$$Lw(t) := w^{\Delta\Delta}(t) - k(t)w^\sigma(t) \ \ for\ \ t \in [a, b]$$

and let $v(t)$ be the solution of the IVP

$$Lu(t) := u^{\Delta\Delta}(t) - k(t)u^\sigma(t) = 0,$$

$$u(a) = 0, \quad u^\Delta(a) = 1.$$

Take

$$v(t) = (m_1 - m_2)\; u(t).$$

Note that

$$Lw(t) \geq Lv(t)$$

on $[a, \rho(t_1)]$, and

$$w(a) = v(a), w^\Delta(a) = v^\Delta(a).$$

Hence $w(t) \geq v(t)$ on $[a, \sigma(t_1)]$ by the Comparison Theorem given by Erbe and Peterson [6, Theorem 9]. Letting $t = \sigma(t_1)$ we get that

$$w(\sigma(t_1)) \geq v(\sigma(t_1)) > 0$$

using the fact that $Lw(t) = 0$ is disconjugacy on $[a, \sigma^2(b)]$.

We have two cases to consider:

Case 1: $t_1 < \sigma(t_1)$.

If t_1 is right-scattered, then $w^\sigma(t_1) < 0$. But this contradicts the fact that

$$w(\sigma(t_1)) \geq v(\sigma(t_1)) > 0.$$

Case 2: $t_1 = \sigma(t_1)$.

If t_1 is right-dense, then $w(t_1) = 0$. But this also contradicts the fact that

$$w(\sigma(t_1)) = w(t_1) \geq v(\sigma(t_1)) = v(t_1) > 0.$$

Hence $w(t) > 0$ on $(a, \sigma^2(b)]$. In particular

$$w(\sigma^2(b)) \geq (m_1 - m_2)\; u(\sigma^2(b)) > 0.$$

Fix m_2 and let $m_1 \to \infty$. This implies that

$$lim_{m_1 \to \infty} \;\; x(\sigma^2(b), m_1) = \infty.$$

Therefore S is not bounded above. Fix m_1 and let $m_2 \to -\infty$. This implies that

$$lim_{m_2 \to -\infty} \;\; x(\sigma^2(b), m_2) = -\infty.$$

Therefore S is not bounded below.
Hence $S = \mathbb{R}$ and so $B \in S$. This implies there is some $m_0 \in \mathbb{R}$ such that

$$x(\sigma^2(b), m_0) = B.$$

Hence the BVP (1), (2) has a solution. Uniqueness follows immediately from the fact that $m_1 > m_2$ implies $x(\sigma^2(b), m_1) > x(\sigma^2(b), m_2)$. $\square$

References

[1] R. P. Agarwal and M. Bohner. Quadratic functionals for second order matrix equations on time scales. *Nonlinear Anal.*, 33:675–692, 1998.

[2] R. P. Agarwal and M. Bohner. Basic calculus on time scales and some of its applications. *Results Math.*, 35(1-2):3–22, 1999.

[3] R. P. Agarwal, M. Bohner, and P. J. Y. Wong. Positive solutions and eigenvalues of conjugate boundary value problems. *Proc. Edinburgh Math. Soc.*, 42:349–374, 1999.

[4] K. Deimling. *Nonlinear Functional Analysis*. Springer, New York, 1985.

[5] L. Erbe and S. Hilger. Sturmian theory on measure chains. *Differential Equations Dynam. Systems*, 1(3):223–244, 1993.

[6] L. Erbe and A. Peterson. Green's functions and comparison theorems for differential equations on measure chains. *Dynam. Contin. Discrete Impuls. Systems*, 6(1):121–137, 1999.

[7] S. Hilger. Analysis on measure chains – a unified approach to continuous and discrete calculus. *Results Math.*, 18:18–56, 1990.

[8] B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram. *Dynamic Systems on Measure Chains*, volume 370 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1996.

[9] W. G. Kelley and A. C. Peterson. *Difference Equations: An Introduction with Applications*. Academic Press, San Diego, second edition, 2001.

[10] E. Zeidler. *Nonlinear Functional Analysis and its applications I*. Springer Verlag, New York, 1985.