Dynamic Systems and Applications 25 (2016) 219-236

CLASSIFICATION OF NONOSCILLATORY SOLUTIONS OF NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

ÖZKAN ÖZTÜRK AND ELVAN AKIN

Missouri University of Science Technology,
202 Rolla Building, MO, 65409-0020

ABSTRACT. We study the asymptotic behavior of nonoscillatory solutions of nonlinear dynamic equations on time scales. More precisely, all eventually monotone solutions of nonlinear dynamic equations can be divided into several disjoint subsets by means of necessary and sufficient integral conditions. Examples are given to illustrate some of our main results.

AMS (MOS) Subject Classification. 39A10.

1. **Introduction**

This paper deals with the asymptotic behavior of solutions of the nonlinear dynamic equation

$$\left[a(t)|x^{\Delta}(t)|^{\alpha}\operatorname{sgn}x^{\Delta}\right]^{\Delta}=b(t)|x^{\sigma}(t)|^{\beta}\operatorname{sgn}x^{\sigma}(t), \tag{1.1}$$

where $a,b\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$ and $\alpha,\beta>0$. A time scale, denoted by $\mathbb{T}$, is a closed subset of real numbers. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. By a solution we mean a delta differentiable function x satisfying equation (1.1) such that $\left[a(t)|x^{\Delta}(t)|^{\alpha}\operatorname{sgn}x^{\Delta}\right]\in \mathrm{C}^1_{\mathrm{rd}}$, where the set of rd-continuous functions and the set of functions that are differentiable and whose derivative is rd-continuous will be denoted by C_{rd} and $\mathrm{C}^1_{\mathrm{rd}}$, respectively. We also assume that $x(t)$ is a proper solution on $[t_0,T)_{\mathbb{T}}$, i.e., $x(t)$ exists and $x(t)\neq 0$ on $[t_0,T)_{\mathbb{T}}$. Whenever we write $t\geq t_1$, we mean that $t\in[t_1,\infty)_{\mathbb{T}}:=[t_1,\infty)\cap\mathbb{T}$.

Equation (1.1) reduces to the nonlinear differential equation, see Cecchi, Došlá, Marini and Vrkoč [8], and Tanigawa [15],

$$[a(t)|x'(t)|^{\alpha}\operatorname{sgn}x']=b(t)|x(t)|^{\beta}\operatorname{sgn}x \tag{1.2}$$

when $\mathbb{T}=\mathbb{R}$, and the nonlinear difference equation, see Cecchi, Došlá, Marini [9],

$$\Delta(a_n|\Delta x_n|^{\alpha}\operatorname{sgn}\Delta x_n)=b_n|x_{n+1}|^{\beta}\operatorname{sgn}x_{n+1} \tag{1.3}$$

when $\mathbb{T}=\mathbb{Z}$.

Received February 15, 2015

Such dynamic equations are studied by Akın-Bohner in [1, 2, 3], by Erbe, Baoguo and Peterson in [12] and Akın-Bohner, Bohner, and Saker in [4]. Such studies are motivated by the dynamics of positive radial solutions of reaction-diffusion (flow through porous media, nonlinear elasticity) problems, see Diaz [11] and Grossinho and Omari [13]. Our results and methods extend those stated and used in the continuous case in [1] and [8], and in the discrete case in [9, 10], see also references therein.

Our goal is to investigate the asymptotic behavior of nonoscillatory solutions of (1.1) by certain types of integrals depending on a, b, α and β. In Section 2, we classify eventually monotone solutions in two types, introduce the sub-classes that are obtained by using equation (1.1) and show the existence and non-existence of nonoscillatory solutions of (1.1). In Section 3, we investigate the convergence and divergence of more general integrals and use those results in Section 4 to show the co-existence of solutions of (1.1) in these sub-classes when $\alpha > \beta$, $\alpha < \beta$ and $\alpha = \beta$. Finally, we construct examples to highlight some of our results in the last section.

An excellent introduction of time scales calculus can be found in [6] and [7] by Bohner and Peterson. Therefore, we only give the preliminary results that we use in our proofs.

Theorem 1.1 ([6, Theorem 1.75])**.** *If $f \in \mathrm{C_{rd}}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 1.2 ([6, Theorem 1.77])**.** *If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}$, then*

$$\int_a^b f(\sigma(t))g^\Delta(t) = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t;$$

or

$$\int_a^b f(t)g^\Delta(t) = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$$

Theorem 1.3 ([6, Theorem 1.90])**.** *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh \right\} g^\Delta(t)$$

holds.

Theorem 1.4. [6, Theorem 1.98] *Assume $\nu : \mathbb{T} \longrightarrow \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$*

$$\int_a^b f(t)\nu^\Delta(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s.$$

Theorem 1.5 (Integral Minkowski Inequality) [5, Theorem 2.1])**.** *Let* $(X, \mathscr{M}, \mu_\Delta)$ *and* $(Y, \mathscr{L}, \nu_\Delta)$ *be time scale measure spaces and let* u, v *and* f *be nonnegative functions on* X, Y, *and* $X \times Y$, *respectively. If* $p \geq 1$, *then*

$$\left[\int_X \left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^p u(x)d\mu_\Delta(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)d\nu_\Delta(y) \tag{1.4}$$

holds provided all integrals in (1.4) *exist. If* $0 < p < 1$ *and*

$$\int_X \left(\int_Y fvd\nu_\Delta\right)^p ud\mu_\Delta > 0, \quad \int_Y fvd\nu_\Delta > 0 \tag{1.5}$$

then (1.4) *is reversed. If* $f < 0$ *and (1.5) and*

$$\int_X f^p ud\mu_\Delta > 0$$

hold, then (1.4) *is reversed, as well.*

Theorem 1.6 (Hölder's Inequality) [5, Theorem 1.3])**.** *For* $p \neq 1$, *define* $q = p/(p-1)$. *Let* $(E, \mathscr{F}, \mu_\Delta)$ *be a time scale measure space. Assume* w, f, g *are nonnegative functions such that* $wf^p, wg^p, w(f+g)^p$ *are* $\Delta-$*integrable on* E. *If* $p > 1$, *then*

$$\int_E w(t)f(t)g(t)d\mu_\Delta(t) \leq \left(\int_E w(t)f^p(t)d\mu_\Delta(t)\right)^{\frac{1}{p}} \left(\int_E w(t)g^q(t)d\mu_\Delta(t)\right)^{\frac{1}{q}}. \tag{1.6}$$

If $0 < p < 1$ *and* $\int_E wg^q d\mu_\Delta > 0$, *or if* $p < 0$ *and* $\int_E wf^p d\mu_\Delta > 0$, *then* (1.6) *is reversed.*

We also use the algebraic inequality

$$(a+b)^p \leq 2^p(a^p + b^p) \tag{1.7}$$

for $a \geq 0$, $b \geq 0$ and $p > 0$, see [14].

It is shown by Akın-Bohner in [1] that any nontrivial solutions of equation (1) on $[t_0, \infty)_{\mathbb{T}}$ is eventually monotone and belongs to one of the following classes:

$$M^+ := \{x \text{ is a solution of } (1) : \exists\ t_1 \geq t_0 \quad \text{such that} \quad x(t)x^\Delta(t) > 0 \text{ for } t \geq t_1\},$$
$$M^- := \{x \text{ is a solution of } (1) : x(t)x^\Delta(t) < 0 \text{ for } t \geq t_0\}.$$

For equation (1.1), M^+ can be empty when $\mathbb{T} = \mathbb{R}$, see [1]. However, it is not true when $\mathbb{T} = \mathbb{Z}$, see [9]. In addition, M^- can be empty when $\mathbb{T} = \mathbb{R}$, see [1], while this is an open problem in the case $\mathbb{T} = \mathbb{Z}$.

In this paper, we study the solutions of (1.1) in M^+ and M^- described by the following integrals:

$$J_1 = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_{t_0}^{t} b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t,$$

$$K_1 = \lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{t_0}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right)^{\beta} \Delta t,$$

$$J_2 = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_{\sigma(t)}^{T} b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t,$$

$$K_2 = \lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{\sigma(t)}^{T} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right)^{\beta} \Delta t,$$

$$J_3 = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \Delta t,$$

$$K_3 = \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Delta t.$$

We now present the convergence and divergence relationships between above integrals. One can prove the followings similar to [2, Lemma 2.1].

Lemma 1.7. *For the integrals J_1, K_1, J_2, K_2, J_3 and K_3, we have the following relationships:*
(a) *If $J_1 < \infty$, then $J_3 < \infty$.*
(b) *If $K_1 < \infty$, then $K_3 < \infty$.*
(c) *If $J_1 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*
(d) *If $K_1 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*
(e) *$J_1 < \infty$ and $K_1 < \infty$ if and only if $J_3 < \infty$ and $K_3 < \infty$,*
(f) *If $J_2 < \infty$, then $K_3 < \infty$.*
(g) *If $K_2 < \infty$, then $J_3 < \infty$.*
(h) *If $J_2 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*
(i) *If $K_2 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*
(j) *$J_2 < \infty$ and $K_2 < \infty$ if and only if $J_3 < \infty$ and $K_3 < \infty$.*

2. **Classification of Nonoscillatory Solutions of (1.1)**

In this section, we obtain the existence and non-existence of solutions of (1.1) in M^+ and M^- depending on J_1, K_1, and J_2, K_2, respectively.

For the convenience, we denote

$$x^{[1]} = a(t)|x^{\Delta}|^{\alpha} \operatorname{sgn} x^{\Delta}, \tag{2.1}$$

so-called the quasi-derivative of x. Let $x(t)$ be a proper solution of (1) in M^+ on $[t_0,\infty)_{\mathbb{T}}$, and without loss of generality assume that $x(t)>0$ for $[t_0,\infty)_{\mathbb{T}}$. By equation (1.1) we have that $x^{[1]}(t)$ is increasing for $t\geq t_0$. Then either there exists $t_1\geq t_0$ such that $x^{[1]}(t)>0,\ t\geq t_1$ or $x^{[1]}(t)<0,\ t\geq t_0$. If $x^{[1]}(t)>0,\ t\geq t_1$, then $x^{\Delta}(t)>0$ for $t\geq t_1$ and $x^{[1]}(t)$ tends to a positive constant or infinity as $t\to\infty$. Clearly, x has a positive limit or infinite limit. Similarly, if $x^{[1]}(t)<0,\ t\geq t_0$, then $x^{\Delta}(t)<0$ for $t\geq t_0$ and so $x^{[1]}(t)$ tends to a non-positive constant as $t\to\infty$ while $x(t)$ goes to a non-negative constant $t\to\infty$.

So in the light of this information, we can have the following lemmas:

Lemma 2.1. *For positive real numbers c and d, M^+ can be a divided into the following sub-classes according to the asymptotic behavior of solution x of* (1.1) *and $x^{[1]}$:*

$$\begin{aligned}
M^+_{B,B}&=\left\{x\in M^+:\lim_{t\to\infty}|x(t)|=c,\quad \lim_{t\to\infty}|x^{[1]}(t)|=d\right\},\\
M^+_{\infty,B}&=\left\{x\in M^+:\lim_{t\to\infty}|x(t)|=\infty,\quad \lim_{t\to\infty}|x^{[1]}(t)|=d\right\},\\
M^+_{B,\infty}&=\left\{x\in M^+:\lim_{t\to\infty}|x(t)|=c,\quad \lim_{t\to\infty}|x^{[1]}(t)|=\infty\right\},\\
M^+_{\infty,\infty}&=\left\{x\in M^+:\lim_{t\to\infty}|x(t)|=\infty,\quad \lim_{t\to\infty}|x^{[1]}(t)|=\infty\right\}.
\end{aligned}$$

Lemma 2.2. *For positive real numbers c and d, M^- can be divided into the following sub-classes according to the asymptotic behavior of solution x of* (1.1) *and $x^{[1]}$:*

$$\begin{aligned}
M^-_{B,B}&=\left\{x\in M^-:\lim_{t\to\infty}|x(t)|=c,\quad \lim_{t\to\infty}|x^{[1]}(t)|=d\right\},\\
M^-_{B,0}&=\left\{x\in M^-:\lim_{t\to\infty}|x(t)|=c,\quad \lim_{t\to\infty}|x^{[1]}(t)|=0\right\},\\
M^-_{0,B}&=\left\{x\in M^-:\lim_{t\to\infty}|x(t)|=0,\quad \lim_{t\to\infty}|x^{[1]}(t)|=d\right\},\\
M^-_{0,0}&=\left\{x\in M^+:\lim_{t\to\infty}|x(t)|=0,\quad \lim_{t\to\infty}|x^{[1]}(t)|=0\right\}.
\end{aligned}$$

In the literature, any eventually nontrivial solution $x\in M^+$ is called *regularly (weakly) increasing* if at least one of $\lim_{t\to\infty}|x(t)|$, $\lim_{t\to\infty}|x^{[1]}(t)|$ exists finitely. Otherwise, it is called a *strongly increasing* solution. Similarly, a solution in $M^-_{0,B}$ is called *regularly (weakly) decaying* while a solution in $M^-_{0,0}$ is called *strongly decaying.*

The following theorem gives us the existence of proper solutions of (1.1) in sub-classes of M^+ based on the integrals J_1 and K_1.

Theorem 2.3. *For solutions of* (1.1) *in M^+, we have the followings:*
(a) *$J_1<\infty$ and $K_1<\infty$ if and only if $M^+_{B,B}\neq\emptyset$.*
(b) *$J_1<\infty$ and $K_1=\infty$ if and only if $M^+_{B,\infty}\neq\emptyset$.*
(c) *If $M^+_{\infty,B}\neq\emptyset$, then $J_1=\infty$ and $K_1<\infty$.*
(d) *If $J_1=K_1=\infty$, then every solution in M^+ belongs to $M^+_{\infty,\infty}$.*

Proof. (a) Suppose that there exists a solution of (1.1) in $M_{B,B}^{+}$. Without loss of generality we assume that $x(t) > 0$ for $t \geq t_1$. Then $x^{[1]}(t)$ is increasing for $t \geq t_1$. By [2, Theorem 3.1], if x has a finite limit, then $J_1 < \infty$. So it is enough to prove that $K_1 < \infty$. Since $x^{[1]}(t)$ is increasing for $t \geq t_1$, $x^{[1]}(t) \geq M$, where $x^{[1]}(t_1) = M \in \mathbb{R}^+$. This implies that

$$x^{\Delta}(t) \geq M^{\frac{1}{\alpha}} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}}, \quad t \geq t_1.$$

Integrating the last inequality from t_1 to t yields

$$x(t) > M^{\frac{1}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s, \quad t \geq t_1$$

or

$$x^{\sigma}(t) > M^{\frac{1}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s, \quad t \geq t_1 \tag{2.2}$$

by the monotonicity of x. Taking the β^{th} power of both sides of (2.2) and multiplying the resulting by b yield

$$(x^{\sigma}(t))^{\beta}\, b(t) > M^{\frac{\beta}{\alpha}} b(t) \left[\int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^{\beta}, \quad t \geq t_1.$$

From (1.1) we get

$$\left[x^{[1]}(t)\right]^{\Delta} > M^{\frac{\beta}{\alpha}} b(t) \left[\int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^{\beta}, \quad t \geq t_1.$$

Finally, integrating the last inequality from t_1 to t yields

$$x^{[1]}(t) > M^{\frac{\beta}{\alpha}} \int_{t_1}^{t} b(s) \left[\int_{t_1}^{s} \left(\frac{1}{a(\tau)}\right)^{\frac{1}{\alpha}} \Delta \tau\right]^{\beta} \Delta s, \quad t \geq t_1. \tag{2.3}$$

Since $x^{[1]}$ has a finite limit, $K_1 < \infty$ from the above inequality.

Conversely, suppose that $J_1 < \infty$ and $K_1 < \infty$. Without loss of generality assume that $x(t) > 0$ for $t \geq t_1$. By [2, Theorem 3.1], there exists a solution x of (1.1) such that $\lim_{t\to\infty} x(t) = c$, where $0 < c < \infty$. So it is enough to show that $x^{[1]}(t)$ converges to a finite number as $t \to \infty$. Since $x(t)$ has a finite limit, there exists $t_2 \geq t_1$ such that $x^{\sigma}(t) < c$ for $t \geq t_2$. Integrating equation (1.1) from t_2 to t gives

$$x^{[1]}(t) = x^{[1]}(t_2) + \int_{t_2}^{t} b(s)\,(x^{\sigma}(s))^{\beta}\, \Delta s < x^{[1]}(t_2) + c^{\beta} \int_{t_2}^{t} b(s) \Delta s. \tag{2.4}$$

By Lemma 1.7(b), $K_3 < \infty$. Therefore, taking the limit of both sides of (2.4) as $t \to \infty$ proves the assertion.

(b) Suppose that there exists a solution x of (1.1) in $M_{B,\infty}^{+}$. It is enough to show that $K_1 = \infty$ since we show in Theorem 2.3(a) that if there exists a bounded solution of (1.1), then $J_1 < \infty$. By Lemma 1.1(b), it is enough to show that $K_3 = \infty$. Without loss of generality, we assume that $x(t) > 0$ for $t \geq t_1$. Integrating equation (1) from t_1 to t yields

$$x^{[1]}(t) = x^{[1]}(t_1) + \int_{t_1}^{t} b(s)\left(x^{\sigma}(s)\right)^{\beta} \Delta s \leq x^{[1]}(t_1) + \left(x^{\sigma}(t)\right)^{\beta} \int_{t_1}^{t} b(s)\Delta s, \quad t \geq t_1.$$

Taking the limit of both sides of the inequality above as $t \to \infty$ gives us that $K_3 = \infty$.

Conversely, suppose that $J_1 < \infty$ and $K_1 = \infty$. By Theorem 2.3(a), we have the existence of a bounded solution x of (1.1) in M^+. By the estimate (2.3) and the divergence of K_1, we obtain that $x^{[1]}$ has an infinite limit. So this completes the proof.

(c) Suppose that there exists a solution of (1.1) in $M^+_{\infty,B}$. By [2, Corollary 3.1], $J_1 = \infty$. So it suffices to prove that $K_1 < \infty$. The proof is very similar to the proof of Theorem 2.3(a). So from estimate (2.3) and since $x^{[1]}$ has a finite limit, we obtain that $K_1 < \infty$.

(d) It follows from Theorem 2.3 (a). □

In the following corollary, we obtain the necessary conditions for the non-existence of solutions in sub-classes of M^+ based on the integrals J_1 and K_1 and the proof follows from Theorem 2.3.

Corollary 2.4. *For solutions of* (1.1) *in* M^+, *we have the followings:*
(a) *If* $J_1 = \infty$ *or* $K_1 = \infty$, *then* $M^+_{B,B} = \emptyset$.
(b) *If* $J_1 = \infty$ *or* $K_1 < \infty$, *then* $M^+_{B,\infty} = \emptyset$.
(c) *If* $J_1 < \infty$ *or* $K_1 = \infty$, *then* $M^+_{\infty,B} = \emptyset$.

We finish this section by showing the existence and non-existence of solutions of equation (1.1) in sub-classes of M^-. In order to do that we define the following integral

$$I = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_{t}^{T} b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t.$$

The proofs of (b) and (d) below can be found in [3, Theorem 2.1, Theorem 2.3] and [3, Theorem 2.4], respectively. So we only prove parts (a) and (c). We use Schauder-Tychonoff fixed point theorem in order to show some of the existence of solutions in M^-.

Theorem 2.5. *For solutions of* (1.1) *in* M^-, *we have the followings:*
(a) $M^-_{B,B} \neq \emptyset$ *if and only if* $I < \infty$ *and* $K_2 < \infty$.
(b) $M^-_{0,B} \neq \emptyset$ *if and only if* $K_2 < \infty$.
(c) *If* $I < \infty$ *and* $K_2 = \infty$, *then* $M^-_{B,0} \neq \emptyset$
(d) *If* $J_2 = K_2 = \infty$, *then every solution in* M^- *belongs to* $M^-_{0,0}$.

Proof. (a) Suppose that $M^-_{B,B} \neq \emptyset$. Then for $c > 0$ and $d > 0$, there exists a solution $x \in M^-$ of (1.1) such that $|x(t)| \to c$ and $|x^{[1]}(t)| \to d$ as $t \to \infty$. By [1, Theorem 4.1], we have that $I < \infty$. So it is enough to show that $K_2 < \infty$. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0$. Then integrating (2.1) from $\sigma(t)$ to ∞ gives us

$$x^\sigma(t) > \int_{\sigma(t)}^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[-x^{[1]}(s)\right]^{\frac{1}{\alpha}} \Delta s > d^{\frac{1}{\alpha}} \int_{\sigma(t)}^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s. \tag{2.5}$$

Taking the β^{th} power and multiplying both sides of (2.5) by b yield us

$$\left[-x^{[1]}(t)\right]^\Delta > d^{\frac{\beta}{\alpha}} b(t) \left[\int_{\sigma(t)}^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^\beta. \tag{2.6}$$

Integrating (2.6) from t_0 to t gives us

$$0 < -x^{[1]}(t_0) + (d)^{\frac{\beta}{\alpha}} \int_{t_0}^t b(s) \left[\int_{\sigma(s)}^\infty \left(\frac{1}{a(\tau)}\right)^{\frac{1}{\alpha}} \Delta\tau\right]^\beta \Delta s < -x^{[1]}(t).$$

As $t \to \infty$ the assertion follows.

Conversely, assume that $I < \infty$ and $K_2 < \infty$. Since $J_3 < \infty$ by Lemma 1.7(g), for arbitrarily given $c > 0$ and $d > 0$, take $t_1 \geq t_0$ so large that

$$\int_{t_1}^\infty \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left[d + (2c)^\beta \int_t^\infty b(s)\Delta s\right]^{\frac{1}{\alpha}} \Delta t \leq c.$$

Define X to be the Frečhet space of all continuous functions on $[t_1, \infty)_{\mathbb{T}}$ endowed with the topology of uniform convergence on compact sub-intervals of $[t_1, \infty)_{\mathbb{T}}$. Let Ω be the nonempty subset of X given by

$$\Omega := \{x \in X : \ c \leq x(t) \leq 2c, \ t \geq t_1\}.$$

Define

$$(\mathscr{F}x)(t) = c + \int_t^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[d + \int_s^\infty b(\tau)(x^\sigma(\tau))^\beta \Delta t\right]^{\frac{1}{\alpha}} \Delta s.$$

Clearly Ω is closed, convex and bounded. One can also show that $\mathscr{F} : \Omega \to \Omega$ is a continuous mapping and relatively compact. Then by the Schauder-Tychonoff fixed point theorem, $\mathscr{F}$ has a fixed element $x \in \Omega$ such that $x = \mathscr{F}(x)$, i.e.,

$$x(t) = (\mathscr{F}x)(t) = c + \int_t^\infty \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[d + \int_s^\infty b(\tau)(x^\sigma(\tau))^\beta \Delta t\right]^{\frac{1}{\alpha}} \Delta s. \tag{2.7}$$

So by (2.7), we have $x^\Delta(t) < 0$ for $[t_1, \infty)_{\mathbb{T}}$, i.e., $x(t)x^\Delta(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Taking the limit as $t \to \infty$ proves the assertion.

(c) Suppose that $I < \infty$ and $K_2 = \infty$. By [1, Theorem 4.1], we have that there exists a solution x of (1.1) such that $|x(t)| \to c$ as $t \to \infty$. So we only show that $x^{[1]}$ has a zero limit. Since $K_2 = \infty$, by Lemma 1.7(i), $J_3 = \infty$ or $K_3 = \infty$. But since

$I < \infty$ implies that $J_2 < \infty$, we have that $K_3 < \infty$ by Lemma 1.7(f). Hence $J_3 = \infty$. Therefore by [3, Lemma 1.3], the proof is complete. $\square$

The following corollary gives us the non-existence of solutions of (1) in sub-classes of M^-.

Corollary 2.6. *For solutions of* (1.1) *in* M^-*, we have the following results:*
(a) $M^-_{B,B} = \emptyset$ *if and only if* $I = \infty$ *or* $K_2 = \infty$.
(b) $M^-_{0,B} = \emptyset$ *if and only if* $K_2 = \infty$.
(c) *Let* $\beta \geq \alpha$. $M^-_{0,0} = \emptyset$ *if* $I < \infty$ *or* $K_2 < \infty$.
(d) *Let* $\beta \geq \alpha$. *If* $J_2 = \infty$ *or* $K_2 < \infty$, *then* $M^-_{B,0} = \emptyset$.

Proof. (a) and (b) immediately follow from Theorem 2.5(a) and (b), respectively. The part (c) was proved in [3, Theorem 2.2]. For part (d), non-existence of such a solution of (1.1) can be found in [1, Theorem 4.1] and limit behavior of $x^{[1]}$ can be shown with the similar idea as in [3, Theorem 2.2(ii)]. $\square$

3. **Integral Relations**

In this section, we introduce more general integrals than J_i and K_i, $i = 1, 2$. The goal is to obtain not only integral relations between these integrals but also some preliminary results in order to investigate the co-existence of solutions in M^+ and M^-.

Let $r, q \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and $\lambda, \gamma > 0$. Define

$$L_\lambda(r, q) = \lim_{T\to\infty} \int_{t_0}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s\right)^{\lambda} \Delta t \tag{3.1}$$

and

$$M_\gamma(r, q) = \lim_{T\to\infty} \int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta t. \tag{3.2}$$

We can rewrite the integrals J_1, J_2, K_1 and K_2 by using (3.1) and (3.2) as follows:

$$J_1 = L_{\frac{1}{\alpha}}(b, A), \quad J_2 = M_\alpha(A, b), \quad K_1 = L_\beta(A, b), \quad K_2 = M_{\frac{1}{\beta}}(b, A),$$

where $A = \left(\frac{1}{a}\right)^{\frac{1}{\alpha}}$. It is clear that if

$$\lim_{T\to\infty} \int_{t_0}^{T} q(t)\Delta t = \infty, \tag{3.3}$$

then

$$L_\lambda(r, q) = M_\gamma(r, q) = \infty.$$

The following follows from Theorem 1.2.

Lemma 3.1. *If* $\lambda = \gamma = 1$, *then* $L_1(r, q) = M_1(r, q)$.

The following lemmas show the convergence and divergence of (3.1) and (3.2) by using λ and γ.

Lemma 3.2. *Let $\lambda = \gamma \leq 1$. If $M_\lambda(r,q) = \infty$, then $L_\lambda(r,q) = \infty$.*

Proof. Let $p = \frac{1}{\lambda}$. So $L_\lambda(r,q)$ and $M_\lambda(r,q)$ can be rewritten as

$$L_{\frac{1}{p}}(r,q) = \lim_{T \longrightarrow \infty} \int_{t_0}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s \right)^{\frac{1}{p}} \Delta t,$$
$$M_{\frac{1}{p}}(r,q) = \lim_{T \longrightarrow \infty} \int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s \right)^{p} \Delta t.$$

Set

$$r(t,s) = \begin{cases} 0; & s \leq \sigma(t) \\ r(t); & s > \sigma(t). \end{cases}$$

Then we have

$$\left[\int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s \right)^{p} \Delta t \right]^{\frac{1}{p}} = \left[\int_{t_0}^{T} \left(\int_{\sigma(t)}^{T} (r(t))^{\frac{1}{p}} q(s)\Delta s \right)^{p} \Delta t \right]^{\frac{1}{p}}$$
$$= \left[\int_{t_0}^{T} \left(\int_{t_0}^{T} (r(t,s))^{\frac{1}{p}} q(s)\Delta s \right)^{p} \Delta t \right]^{\frac{1}{p}} \leq \int_{t_0}^{T} q(s) \left(\int_{t_0}^{T} r(t,s)\Delta t \right)^{\frac{1}{p}} \Delta s$$
$$= \int_{t_0}^{T} q(s) \left(\int_{t_0}^{s} r(t)\Delta t \right)^{\frac{1}{p}} \Delta s,$$

where $u = 1$, $f = r^{\frac{1}{p}}$ and $v = q$ in Theorem 1.5. Taking limit as $T \to \infty$ completes the proof. $\square$

Lemma 3.3. *Let $\lambda = \gamma \geq 1$. If $L_\lambda(r,q) = \infty$, then $M_\lambda(r,q) = \infty$.*

Proof. Suppose that $L_\lambda(r,q) = \infty$ and $\lambda \geq 1$. Let

$$q(t,s) = \begin{cases} 0; & s \geq t \\ q(t); & s < t. \end{cases}$$

Then we have

$$\left[\int_{t_0}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s \right)^{\lambda} \Delta t \right]^{\frac{1}{\lambda}} = \left[\int_{t_0}^{T} \left(\int_{t_0}^{t} (q(t))^{\frac{1}{\lambda}} r(s)\Delta s \right)^{\lambda} \Delta t \right]^{\frac{1}{\lambda}}$$
$$= \left[\int_{t_0}^{T} \left(\int_{t_0}^{T} (q(t,s))^{\frac{1}{\lambda}} r(s)\Delta s \right)^{\lambda} \Delta t \right]^{\frac{1}{\lambda}} \leq \int_{t_0}^{T} r(s) \left(\int_{t_0}^{T} q(t,s)\Delta t \right)^{\frac{1}{\lambda}} \Delta s$$
$$= \int_{t_0}^{T} r(s) \left(\int_{\sigma(s)}^{T} q(t)\Delta t \right)^{\frac{1}{\lambda}} \Delta s,$$

where $f = q^{\frac{1}{\lambda}}$, $v = r$ and $u = 1$ in Theorem 1.5. As $T \to \infty$, the assertion follows. $\square$

Now we will obtain similar results for $\lambda \neq \gamma$. But in order to do that we need the following two lemmas.

Lemma 3.4. *Let*

$$Q_T(t) = \int_t^T q(s)\Delta s. \tag{3.4}$$

If $\eta < 1$ *and*

$$\lim_{T\to\infty} \int_{t_0}^T q(s)\Delta s < \infty,$$

then

$$\lim_{T\longrightarrow\infty} \int_{t_0}^T \frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t)]^{\eta}}\Delta t < \infty.$$

Proof. Set $\nu(t) = -Q_T(t)$ and $f(t) = \frac{1}{[Q_T(\sigma(t)]^{\eta}}$. Since $-Q_T(t)$ is increasing on $[t_0, T)_{\mathbb{T}}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ on $[t_0, T)_{\mathbb{T}}$, by Theorem 1.4, we have

$$\int_{t_0}^T \frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t))]^{\eta}}\Delta t = \int_{-\int_{t_0}^T q(s)\Delta s}^0 \frac{dt}{[Q_T((-Q_T)^{-1}(t))]^{\eta}} \quad \text{for} \quad t \in Range(Q_T).$$

So

$$\int_{t_0}^T \frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t))]^{\eta}}\Delta t = \int_{-\int_{t_0}^T q(s)\Delta s}^0 \frac{dt}{(-t)^{\eta}} = \lim_{b\to 0^-} \frac{1}{1-\eta}\left[(-b)^{-\eta-1} - \left(\int_{t_0}^T q(s)ds\right)\right]^{-\eta+1}$$
$$= -\frac{1}{1-\eta}\left[\int_{t_0}^T q(s)ds\right]^{1-\eta}.$$

As $T \to \infty$, the assertion follows, in which $\nu(t) = -Q_T(t)$ and $f(t) = \frac{1}{[Q_T(\sigma(t)]^{\eta}}$ in Theorem 1.4. □

Lemma 3.5. *Let*

$$R_1(t) = 1 + \int_{t_0}^t r(s)\Delta s.$$

If $\eta > 1$, *then*

$$\int_{t_0}^{\infty} \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t < \infty.$$

Proof. Set $\nu(t) = R_1(t)$ and $f(t) = \frac{1}{R_1^{\eta}(t)}$ in Theorem 1.4. Since $R_1(t)$ is strictly increasing on $[t_0, T)_{\mathbb{T}}$ and $f \in C_{rd}([t_0, T)_{\mathbb{T}}, \mathbb{R})$ by Theorem 1.4, we have

$$\int_{t_0}^T \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t = \int_1^{1+\int_{t_0}^T r(s)\Delta s} \frac{dt}{\left[R_1(R_1^{-1}(t))\right]^{\eta}} \quad \text{for} \;\; t \in Range(R_1(t)).$$

So we have

$$\int_{t_0}^T \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t = \int_1^{1+\int_{t_0}^T r(s)\Delta s} \frac{dt}{t^{\eta}} = \frac{1}{1-\eta}\left[1 - \left(1 + \int_{t_0}^T r(s)\Delta s\right)^{-\eta+1}\right].$$

As $T \to \infty$, the assertion follows. □

Lemma 3.6. *Let* $\gamma > \lambda$. *If* $L_\lambda(r, q) = \infty$, *then* $M_\gamma(r, q) = \infty$.

Proof. Suppose that $\gamma > \lambda$. If (3.3) holds, the assertion follows. Since $L_\lambda(r,q) = \infty$, we can assume

$$\lim_{T\to\infty}\int_{t_0}^{T} r(t)\Delta t = \infty \quad \text{and} \quad \lim_{T\to\infty}\int_{t_0}^{T} q(t)\Delta t < \infty. \tag{3.5}$$

Denote

$$R_1(t) = 1 + R(t),$$

where

$$R(t) = \int_{t_0}^{t} r(s)\Delta s. \tag{3.6}$$

Consider two cases:
(i) $\gamma \geq 1$ and (ii) $0 < \gamma < 1$
Case (i): Let $t_1 \geq t_0$ be such that $R(t) > 1$ for $t \geq t_1$. Since Ł$_\lambda(r,q) = \infty$, we have

$$\int_{t_1}^{T} q(t)\left(\int_{t_0}^{t} r(s)\Delta s\right)^{\gamma}\Delta t \geq \int_{t_1}^{T} q(t)\left(\int_{t_0}^{t} r(s)\Delta s\right)^{\lambda}\Delta t.$$

As $T \to \infty$, the right hand side goes to infinity, so does the left hand side. Then by Lemma 3.3, we have $M_\gamma(r,q) = \infty$. This completes *Case* (i).
Case (ii)*:* By Theorem 1.2, we have

$$\int_{t_0}^{T} q(t)R_1^{\lambda}(t)\Delta t = Q(t_0) + \int_{t_0}^{T}\left(R_1^{\lambda}(t)\right)^{\Delta} Q_T(\sigma(t))\Delta t.$$

By Theorems 1.2, 1.3, and 1.6, we have

$$\begin{aligned}
\int_{t_0}^{T} q(t)R_1^{\lambda}(t)\Delta t &= Q_T(t_0) + \int_{t_0}^{T}\left\{\int_0^1 \lambda\left[R_1(t) + h\mu(t)R_1^{\Delta}(t)\right]^{\lambda-1} dh\right\} R_1^{\Delta}(t)Q_T(\sigma(t))\Delta t \\
&\leq Q_T(t_0) + \int_{t_0}^{T} \lambda\left[R_1(t)\right]^{\lambda-1} R_1^{\Delta}(t)Q_T(\sigma(t))\Delta t \\
&\leq Q_T(t_0) + \lambda\left[\int_{t_0}^{T} R_1^{\Delta}(t)Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t\right]^{\gamma}\left[\int_{t_0}^{T} R_1^{\Delta}(t)\left(R_1(t)\right)^{\frac{\lambda-1}{1-\gamma}}\Delta t\right]^{1-\gamma} \\
&= Q_T(t_0) + \lambda\left[\int_{t_0}^{T} R_1^{\Delta}(t)Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t\right]^{\gamma}\left[\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{\left[R_1(t)\right]^{\frac{1-\lambda}{1-\gamma}}}\Delta t\right]^{1-\gamma}.
\end{aligned}$$

Hence we have

$$\int_{t_0}^{T} q(t)R_1^{\lambda}(t)\Delta t \leq Q_T(t_0) + \lambda\left[\int_{t_0}^{T} R_1^{\Delta}(t)Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t\right]^{\gamma}\left[\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{\left[R_1(t)\right]^{\frac{1-\lambda}{1-\gamma}}}\Delta t\right]^{1-\gamma}.$$

Since

$$\int_{t_0}^{\infty} \frac{R_1^{\Delta}(t)}{\left[R_1(t)\right]^{\frac{1-\lambda}{1-\gamma}}}\Delta t < \infty$$

for $\frac{1-\lambda}{1-\gamma} > 1$, by Lemma 3.5 the assertion follows as $T \to \infty$. □

Lemma 3.7. *Let $\gamma < \lambda$. If $M_\gamma(r,q) = \infty$, then $L_\lambda(r,q) = \infty$*

Proof. It is clear that if (3.3) holds, there is nothing to show. So since $M_\gamma(r,q)=\infty$, as in the proof in Lemma 3.6, we can assume (3.5) holds.

We will consider two cases:
(i) $\gamma\le 1$ and (ii) $\gamma>1$.
Case (i): For $t_1\ge t_0$, we may suppose $R(t)>1$ for $t\ge t_1$. Since $M_\gamma(r,q)=\infty$, we have $L_\gamma(r,q)=\infty$ by Lemma 3.2. Hence, similar to the *Case* (i) in proof of Lemma 3.6, the assertion follows.
Case (ii): By (3.4), (3.6) and Theorem 1.2, we have

$$\int_{t_0}^{T} r(t)\left(Q_T(\sigma(t)\right)^{\frac{1}{\gamma}}\Delta t=-\int_{t_0}^{T}\left[\left(Q_T(t)\right)^{\frac{1}{\gamma}}\right]^{\Delta}R(t)\Delta t.$$

Finally, Theorems 1.3 and 1.6 yield

$$\begin{aligned}\int_{t_0}^{T} r(t)\left(Q_T(\sigma(t)\right)^{\frac{1}{\gamma}}\Delta t&=\frac{1}{\gamma}\int_{t_0}^{T}\left\{\int_0^1\left(Q_T(t)+h\mu(t)Q_T^{\Delta}(t)\right)^{\frac{1-\gamma}{\gamma}}dh\right\}q(t)R(t)\Delta t\\&\le\frac{1}{\gamma}\int_{t_0}^{T}\left(Q_T(\sigma(t))\right)^{\frac{1-\gamma}{\gamma}}q(t)R(t)\Delta t\\&\le\frac{1}{\gamma}\left[\int_{t_0}^{T}q(t)R^{\lambda}(t)\Delta t\right]^{\frac{1}{\lambda}}\left[\int_{t_0}^{T}-\frac{Q_T^{\Delta}(t)}{\left(Q_T(\sigma(t))\right)^{\xi}}\Delta t\right]^{\frac{\lambda-1}{\lambda}},\end{aligned}$$

where $\xi=\frac{(\gamma-1)\lambda}{\gamma(\lambda-1)}<1, w=q, f=R$ and $g=(Q^\sigma)^{\frac{1-\gamma}{\gamma}}$ in Theorem 1.6. Taking the limit as $T\to\infty$ and using Lemma 3.4 complete the proof. $\square$

4. **Examples**

In this section, we give two examples to highlight Theorem 2.5(b).

Example 4.1. Let $\mathbb{T}=\mathbb{R}$, $\alpha=1$, $\beta=\frac{1}{4}$, $a(t)=\frac{1+e^{-4t}}{2e^{-2t}}$ and $b(t)=4e^{\frac{-7t}{2}}$ in equation (1.1). Then we have

$$\lim_{T\to\infty}\left(\int_{\sigma(t)}^{T}A(s)ds\right)^{\beta}=\lim_{T\to\infty}\left(\int_t^T\frac{2e^{-2s}}{1+e^{-4s}}ds\right)^{\frac{1}{4}}<\left(\frac{\pi}{2}\right)^{\frac{1}{4}}$$

and so we obtain

$$\int_{t_0}^{T}b(t)\left(\int_{\sigma(t)}^{T}A(s)ds\right)^{\frac{1}{4}}dt=\int_{t_0}^{T}4e^{\frac{-7t}{2}}\left(\int_t^T\frac{2e^{-2s}}{1+e^{-4s}}ds\right)^{\frac{1}{4}}dt<\left(\frac{\pi}{2}\right)^{\frac{1}{4}}\frac{8}{7}e^{\frac{-7t_0}{2}}.$$

As $T\to\infty$, we have $K_2<\infty$. One can also easily show that $x(t)=e^{-2t}$ is a solution of

$$\left[\frac{1+e^{-4t}}{2e^{-2t}}|x'|\operatorname{sgn}x'\right]'=4e^{\frac{-7t}{2}}|x|^{\frac{1}{4}}\operatorname{sgn}x$$

such that $\lim_{t\to\infty}x(t)=0$ and $\lim_{t\to\infty}x^{[1]}(t)=-1$, i.e., $M_{0,B}^{-}\ne\emptyset$.

Example 4.2. Let $\mathbb{T}=\mathbb{Z}$, $\alpha=1$, $\beta<1$, $t_0\geq 1$, $a_n=\frac{3}{2}(3^n+1)$ and $b_n=2(3^{n+1})^{\beta-1}$ in equation (1.1). Letting $t=n$ and $s=m$ gives us

$$\int_{t_0}^{T} b(t)\left(\int_{\sigma(t)}^{T} A(s)\Delta s\right)^{\beta}\Delta t=\sum_{n=1}^{T-1}2(3^{n+1})^{\beta-1}\left(\sum_{m=n+1}^{T-1}\frac{2}{3(3^m+1)}\right)^{\beta}$$
$$\leq \frac{2}{3}\sum_{n=1}^{T-1}\left(\frac{1}{3^{1-\beta}}\right)^{n}.$$

Hence, we have $K_2<\infty$ as $T\to\infty$. One can show that $x_n=3^{-n}$ is a solution of

$$\Delta\left[\frac{3}{2}(3^n+1)|\Delta x_n|\operatorname{sgn}\Delta x_n\right]=2(3^{n+1})^{\beta-1}|x_{n+1}|^{\beta}\operatorname{sgn}x_{n+1}$$

such that $\lim_{n\to\infty}x_n=0$ and $\lim_{n\to\infty}x_n^{[1]}=-1$, i.e., $M_{0,B}^{-}\neq\emptyset$.

5. **Conclusions**

In this section, one can obtain the co-existence and non-coexistence of solutions of (1.1) in sub-classes of M^- and M^+ in each of the cases $\alpha=\beta$, $\alpha>\beta$ and $\alpha<\beta$.

The following integral relationships among J_1, K_1, J_2 and K_2 follow directly from Lemmas 3.1–3.3 and 3.6–3.7.

Lemma 5.1. *We have the followings:*
(a) *If* $\alpha=\beta=1$, *then* $J_1=K_2$ *and* $J_2=K_1$.
(b) *If* $\alpha=\beta\leq 1$, *then* $J_2=\infty\Longrightarrow K_1=\infty$ *and* $J_1=\infty\Longrightarrow K_2=\infty$.
(c) *If* $\alpha=\beta\geq 1$, *then* $K_1=\infty\Longrightarrow J_2=\infty$ *and* $K_2=\infty\Longrightarrow J_1=\infty$.
(d) *If* $\alpha>\beta$, *then* $K_1=\infty\Longrightarrow J_2=\infty$ *and* $J_1=\infty\Longrightarrow K_2=\infty$.
(e) *If* $\alpha<\beta$, *then* $J_2=\infty\Longrightarrow K_1=\infty$ *and* $K_2=\infty\Longrightarrow J_1=\infty$.

In the light of Lemma 5.1, there exist eight cases:

$$(C_1): J_1=J_2=K_1=K_2,$$
$$(C_2): J_1=K_2=\infty,\ J_2<\infty,\ K_1<\infty,$$
$$(C_3): J_1<\infty,\ K_2<\infty,\ J_2=K_1=\infty,$$
$$(C_4): J_1<\infty,\ K_1<\infty,\ J_2<\infty,\ K_2<\infty,$$
$$(C_5): J_1=J_2=K_2=\infty,\ K_1<\infty,$$
$$(C_6): J_1=J_2=K_1=\infty,\ K_2<\infty,$$
$$(C_7): J_1=K_1=K_2=\infty,\ J_2<\infty,$$
$$(C_8): K_1=K_2=J_2=\infty,\ J_1<\infty.$$

Note that Cases (C_i), $i=(1)$–(4) occur for any $\alpha>0$ and $\beta>0$ while (C_5) occurs only for $\alpha=\beta>1$ or $\alpha>\beta$, (C_6) occurs only for $\alpha=\beta>1$ or $\alpha<\beta$, (C_7) occurs only for $\alpha<\beta$ or $\alpha=\beta<1$ and (C_8) occurs only for $\alpha>\beta$ or $\alpha=\beta<1$.

We now investigate the co-existence and co-nonexistence of solutions of (1.1) by using the cases (C_i), $i = (1)$–(8) and Theorems (2.3), (2.4), (2.5) and (2.6) in the following theorems.

Theorem 5.2. *Let $\alpha = \beta$. For solutions of equation (1.1) in M^+ and M^-, we have the followings:*
(a) *If (C_1) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^- = M^-_{0,0}$.*
(b) *If (C_2) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*
(c) *If (C_3) holds, then $M^+_{B,\infty} \neq \emptyset$, $M^+_{B,B} = M^+_{\infty,B} = \emptyset$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore $M^- = M^-_{0,B}$.*
(d) *If (C_4) holds, then $M^+_{B,B} \neq \emptyset$, $M^+_{B,\infty} = M^+_{\infty,B} = \emptyset$ and $M^-_{0,B} \neq \emptyset$, $M^-_{0,0} = M^-_{B,0} = \emptyset$.*
(e) *If (C_5) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^- = M^-_{0,0}$.*
(f) *If (C_6) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore, $M^- = M^-_{0,B}$.*
(g) *If (C_7) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*
(h) *If (C_8) holds, then $M^+_{B,\infty} \neq \emptyset$, $M^+_{B,B} = M^+_{\infty,B} = \emptyset$ and $M^- = M^-_{0,0}$.*

Theorem 5.3. *Let $\alpha > \beta$. For solutions of equation (1.1) in M^+ and M^-, we have the followings:*
(a) *If (C_1) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^- = M^-_{0,0}$.*
(b) *If (C_2) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*
(c) *If (C_3) holds, then $M^+_{B,\infty} \neq \emptyset$, $M^+_{B,B} = M^+_{\infty,B} = \emptyset$ and $M^-_{0,B} \neq \emptyset$, $M^-_{B,B} = \emptyset$.*
(d) *If (C_4) holds, then $M^+_{B,B} \neq \emptyset$, $M^+_{B,\infty} = M^+_{\infty,B} = \emptyset$, and $M^-_{0,B} \neq \emptyset$.*
(e) *If (C_5) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^- = M^-_{0,0}$.*
(f) *If (C_8) holds, then $M^+_{\infty,\infty} \neq \emptyset$, $M^+_{B,B} = \emptyset$, $M^+_{\infty,B} = \emptyset$ and $M^- = M^-_{0,0}$.*

Theorem 5.4. *Let $\alpha < \beta$. For solutions of equation (1.1) in M^+ and M^-, we have the followings:*
(a) *If (C_1) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^- = M^-_{0,0}$.*
(b) *If (C_2) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*
(c) *If (C_3) holds, then $M^+_{B,\infty} \neq \emptyset$, $M^+_{B,B} = M^+_{\infty,B} = \emptyset$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore $M^- = M^-_{0,B}$.*
(d) *If (C_4) holds, then $M^+_{B,B} \neq \emptyset$, $M^+_{B,\infty} = M^+_{\infty,B} = \emptyset$ and $M^-_{0,B} \neq \emptyset$, $M^-_{0,0} = M^-_{B,0} = \emptyset$.*
(e) *If (C_6) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore, $M^- = M^-_{0,B}$.*
(f) *If (C_7) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*

Our goal for the entire paper has been to classify nonoscillatory solutions of (1.1) depending on J_1, K_1, J_2 and K_2. However, we would like to indicate the following remarks.

Remark 5.5. When $J_1 = \infty$ and $K_1 < \infty$, we have to assume that

(5.1) $\mu(t)$ is differentiable such that $\mu^{\Delta}(t) \geq 0$ and $a^{\sigma}(t) \geq a(t)$ for $t \geq t_1$

to be able to obtain $M^{+}_{\infty,B} \neq \emptyset$, which follows from [2, Theorem 3.1] and [2, Corollary 5.1]. On the other hand, in case (C_2) or (C_5) holds with $\alpha \geq \beta$, or (C_2) holds with $\alpha < \beta$, we obtain $M^{+}_{\infty,B} \neq \emptyset$ as well. If $\mathbb{T} = \mathbb{R}$, then (5.1) holds automatically. So our result corresponds with the continuous case. Of course, one can obtain that $M^{+}_{\infty,B} \neq \emptyset$ by assuming both conditions

$$J_1 = \infty, \qquad \text{and} \qquad \lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{t_0}^{t} A^{\sigma}(s)\Delta s \right)^{\beta} \Delta t < \infty$$

without (5.1) as in the discrete case, see [9].

Remark 5.6. When $J_1 < \infty$ or $K_1 < \infty$, we have to assume that

$$\int_{t_1}^{\infty} b(t)\mu^{\beta}(t) \left(\frac{1}{a(t)} \right)^{\frac{\beta}{\alpha}} \Delta t < \infty, \tag{5.2}$$

where $\alpha > \beta$ to be able to obtain $M^{+}_{\infty,\infty} = \emptyset$ by using [2, Theorem 3.2], Theorem 1.1, inequality (1.7), and Lemma 1.7(b). On the other hand, if we have one of the cases (C_2), (C_3), (C_4), (C_5) and (C_8) with $\alpha > \beta$, then $M^{+}_{\infty,\infty} = \emptyset$ as well. If $\mathbb{T} = \mathbb{R}$, then (5.2) holds automatically. So our result matches with the continuous case. Of course, one can show that $M^{+}_{\infty,\infty} = \emptyset$ by assuming

$$\lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{t_0}^{t} A^{\sigma}(s)\Delta s \right)^{\beta} \Delta t < \infty, \quad \alpha > \beta$$

without (5.2) as in the discrete case, see [9].

Another reasonable nonlinear dynamic equation is to consider

$$\left[a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}\right]^{\Delta} = -b(t)|x^{\sigma}(t)|^{\beta} \operatorname{sgn} x^{\sigma}(t) \tag{5.3}$$

as our new project because several questions arise. For example, what integral conditions might we have in order to obtain the existence of nonoscillatory solutions of (5.3)? And what sub-classes might occur for nonoscillatory solutions of (5.3) depending on the convergence/divergence of J_3 and K_3? Also what oscillation criteria do we need for (5.3)?

REFERENCES

[1] E. Akın-Bohner, Positive Decreasing Solutions of Quasilinear Dynamic Equations. *Math. and Comput. Modelling*, 43(3-4):283–293, 2006.

[2] E. Akın-Bohner, Positive Increasing Solutions of Quasilinear Dynamic Equations. *Math. Inequal. Appl.*, 10(1):99–110, 2007.

[3] E. Akın-Bohner, Regularly and Strongly Decaying Solutions for Quasilinear Dynamic Equations. *Adv. Dyn. Syst. Appl* 3(1), 15–24, 2008.

[4] E. Akın-Bohner, M. Bohner and Samir H. Saker, Oscillation Criteria for a certain class of second order Emden Fowler Dynamic Equations, *Electron. Trans. Numer. Anal.*, 27: 1–12, 2007.

[5] R. Bibi, M. Bohner, J. Pećarić, S. Varošanec, Minkowski and Beckenbach-Dresher Inequalities and Functionals on Time Scales. *J. Math. Inequal.*, Volume 7, Number 3, 299–312, 2013.

[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhäuser, Boston, 2001.

[7] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2003.

[8] M. Cecchi, Z. Došlá, M. Marini, and Ivo Vrkoč, Integral Conditions for Nonoscillation of Second Order Nonlinear Differential Equations. *Nonlinear Anal.* 64, 1278–1289, 2006.

[9] M. Cecchi, Z Došlá, and M. Marini, Unbounded Solutions of Quasilinear Difference Equations. *Comp. Math. Appl.*, 45(6-9):1113–11123, 2003.

[10] M. Cecchi, Z Došlá, and M. Marini, On oscillation and nonoscillation properties of Emden-Fowler difference equations. *Cent. Eur. J. Math.*, 7(2), 322–334, 2009.

[11] J.I. Diaz, Nonlinear partial differential equations and free boundaries, vol. 1: Elliptic equations. *Pitman Advanced Publ.*, 106, 1985.

[12] L. Erbe, J. Baoguo, A. Peterson, On the asymptotic behavior of solutions of Emden-Fowler equations on time scales. *Annali di Mathematica*, 191:205-17, 2012.

[13] M.R. Grossinho and P. Omari, A multiplicity result for a class of quasilinear elliptic and parabolic problems. *Electron. J. Diff. Eqns.*, 8:1–16, 1997.

[14] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Classical and New Inequalities in Analysis.* Kluwer Academic Publishers, 1993.

[15] T. Tanigawa, Existence and Asymptotic Behavior of Positive solutions of Second Order Quasilinear Differential Equations. *Adv. Math. Sci. Appl.*, Gakkotosho, Tokyo, Vol. 9, No. 2, pp. 907–938, 1999.