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IMPROPER INTEGRALS ON TIME SCALES

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ABSTRACT. In this paper we study improper integrals on time scales. We also give some mean value theorems for integrals on time scales, which are used in the proof of an analogue of the classical Dirichlet–Abel test for improper integrals.

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1. INTRODUCTION

The calculus on time scales has been introduced by Aulbach and Hilger [1, 6] in order to unify discrete and continuous analysis. In [1, 6] (see also [2]) the concept of integral on time scales is defined by means of an antiderivative (or pre-antiderivative) of a function and is called the Cauchy integral. In [7] the Darboux and in [3, 4, 5] the Riemann definitions of the integral on time scales are introduced and main theorems of the integral calculus are established. In the present paper, in addition to [3, 4, 5, 7], we consider some versions of mean value theorems for integrals on time scales and investigate improper integrals on time scales which are important in the study of dynamic systems on infinite intervals.

The set up of this paper is as follows. In Section 2 following [3] we briefly introduce the concept of Riemann integration on time scales and formulate without proof some properties of the integral, which are used in the subsequent sections. In Section 3 we prove two versions of mean value theorems for integrals on time scales. Those theorems are used in Section 4, where improper integrals of first kind are studied. Some examples of improper integrals of first kind are supplied in Section 5. Finally, in Section 6, we deal with improper integrals of second kind on time scales.

In this paper we mainly consider delta integrals. Nabla integrals may be treated in a similar way.

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2. PRELIMINARIES ON RIEMANN INTEGRATION

A *time scale* $\mathbb{T}$ (which is a special case of a *measure chain*) is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and therefore it is a complete metric space with the metric $d(t,s) = |t-s|$. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $\sigma(t) = t$, then t is called *right-dense*, and if $\rho(t) = t$, then t is called *left-dense*. For $a, b \in \mathbb{T}$ with $a \le b$ we define the closed interval $[a,b]$ in $\mathbb{T}$ by

$$[a,b] = \{t \in \mathbb{T} : \ a \le t \le b\}.$$

Open intervals and half-open intervals etc. are defined accordingly. Throughout this paper all considered intervals will be intervals in $\mathbb{T}$.

Let $a < b$ be points in $\mathbb{T}$. A *partition* of $[a,b]$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b], \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_n = b.$$

The number n depends on the particular partition. We denote the set of all partitions of $[a,b]$ by $\mathcal{P} = \mathcal{P}(a,b)$. Let f be a real-valued bounded function on $[a,b]$ and set

$$M = \sup\{f(t) : \ t \in [a,b)\}, \quad m = \inf\{f(t) : \ t \in [a,b)\},$$

and

$$M_i = \sup\{f(t) : \ t \in [t_{i-1}, t_i)\}, \quad m_i = \inf\{f(t) : \ t \in [t_{i-1}, t_i)\}.$$

The *upper Darboux sum* $U(f,P)$ and the *lower Darboux sum* $L(f,P)$ of f with respect to P are defined by

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

Obviously,

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

The *upper Darboux integral* $U(f)$ and the *lower Darboux integral* $L(f)$ of f from a to b are defined by

$$U(f) = \inf\{U(f,P) : \ P \in \mathcal{P}(a,b)\} \quad \text{and} \quad L(f) = \sup\{L(f,P) : \ P \in \mathcal{P}(a,b)\}.$$

The numbers $U(f)$ and $L(f)$ are finite, and $L(f) \le U(f)$.

Definition 2.1. We say that f is *integrable* (or *delta integrable*) from a to b (or on $[a,b]$) provided $L(f) = U(f)$. In this case we write $\int_a^b f(t)\Delta t$ for this common value. We call this integral the *Darboux integral*.

Riemann's definition of the integral is a little different (see Definition 2.5 below), but we will see in Theorem 2.6 that the definitions are euqivalent. For this reason, we call the integral defined above also the *Riemann integral*.

Lemma 2.2. *For every $\delta > 0$ there exists a partion $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(a,b)$ such that for each $i \in \{1, 2, \dots, n\}$ either $t_i - t_{i-1} \le \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$.*

Definition 2.3. We denote by $\mathcal{P}_\delta = \mathcal{P}_\delta(a,b)$ the set of all partitions $P \in \mathcal{P}$ possessing the property indicated in Lemma 2.2.

The next theorem gives a "Cauchy criterion" for integrability.

Theorem 2.4. *A bounded function f on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P \in \mathcal{P}_\delta \quad \textit{implies} \quad U(f,P) - L(f,P) < \varepsilon.$$

We now give Riemann's definition of integrability.

Definition 2.5. Let f be a bounded function on $[a,b]$ and let $P = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}(a,b)$. In each interval $[t_{i-1}, t_i)$, where $1 \le i \le n$, choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

We call S a *Riemann sum* of f corresponding to $P \in \mathcal{P}$. We say that f is *Riemann integrable* from a to b (or on $[a,b]$) provided there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann sum S of f corresponding to a partition $P \in \mathcal{P}_\delta$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)$, $1 \le i \le n$. It is easy to see that such a number I is unique. The number I is called the *Riemann integral* of f from a to b.

Theorem 2.6. *A bounded function f on $[a,b]$ is Riemann integrable if and only if it is Darboux integrable, in which case the values of the integrals are equal.*

In our definition of $\int_a^b f(t)\Delta t$ we assumed $a < b$. We remove that restriction with the definitions

$$\int_a^a f(t)\Delta t = 0 \quad \text{and} \quad \int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t \text{ for } a > b.$$

Now we present some properties of the Riemann integral, that will be used later in this paper.

Theorem 2.7. *Every constant function $f(t) \equiv c$ is integrable from a to b and $\int_a^b f(t)\Delta t = c(b-a)$.*

Theorem 2.8. *Every monotone function on $[a,b]$ is integrable.*

Theorem 2.9. *Every continuous function on $[a,b]$ is integrable.*

Theorem 2.10. *Let f be a bounded function that is integrable on $[a,b]$. Then f is integrable on every subinterval $[c,d]$ of $[a,b]$.*

Theorem 2.11. *Let f and g be integrable functions on $[a,b]$ and let $\alpha \in \mathbb{R}$. Then*

(i) *αf is integrable and $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;*
(ii) *$f+g$ is integrable and $\int_a^b (f+g)(t)\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.*

Theorem 2.12. *If f and g are integrable on $[a,b]$, then so is their product fg.*

Theorem 2.13. *Let f be a function defined on $[a,b]$ and let $c \in \mathbb{T}$ with $a < c < b$. If f is integrable from a to c and from c to b, then f is integrable from a to b and $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.*

Theorem 2.14. *If f and g are integrable on $[a,b]$ and $f(t) \le g(t)$ for all $t \in [a,b)$, then $\int_a^b f(t)\Delta t \le \int_a^b g(t)\Delta t$.*

Theorem 2.15. *If f is integrable on $[a,b]$, then so is $|f|$ and*

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b |f(t)|\Delta t.$$

Theorem 2.16. *Let $a < b$ be points in the time scale $\mathbb{T}$ such that there is no point of $\mathbb{T}$ between a and b. Then every function $f : \mathbb{T} \to \mathbb{R}$ is delta and nabla integrable from a to b with*

$$\int_a^b f(t)\Delta t = f(a)(b-a) \quad \text{and} \quad \int_a^b f(t)\nabla t = f(b)(b-a).$$

3. MEAN VALUE THEOREMS FOR INTEGRALS

Theorem 3.1 (First Mean Value Theorem)**.** *Let f and g be bounded and integrable functions on $[a,b]$, and let g be nonnegative (or nonpositive) on $[a,b]$. Let us set*

$$m = \inf\{f(t) : \ t \in [a,b)\} \quad \text{and} \quad M = \sup\{f(t) : \ t \in [a,b)\}.$$

Then there exists a real number Λ satisfying the inequalities $m \le \Lambda \le M$ such that

$$\int_a^b f(t)g(t)\Delta t = \Lambda \int_a^b g(t)\Delta t. \tag{3.1}$$

Proof. We have

$$m \le f(t) \le M \quad \text{for all} \quad t \in [a,b). \tag{3.2}$$

Suppose $g(t) \ge 0$. Multiplying (3.2) by $g(t)$, we get

$$mg(t) \le f(t)g(t) \le Mg(t) \quad \text{for all} \quad t \in [a,b).$$

Besides, each of the functions mg, Mg, and fg is integrable from a to b by Theorem 2.11 and Theorem 2.12. Therefore, we obtain from these inequalities, by using Theorem 2.14,

$$m\int_a^b g(t)\Delta t \le \int_a^b f(t)g(t)\Delta t \le M\int_a^b g(t)\Delta t. \tag{3.3}$$

If $\int_a^b g(t)\Delta t = 0$, it follows from (3.3) that $\int_a^b f(t)g(t)\Delta t = 0$, and therefore equality (3.1) becomes obvious; if $\int_a^b g(t)\Delta t > 0$, then (3.3) implies

$$m \le \frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g(t)\Delta t} \le M.$$

So the middle term of these inequalities is equal to a number Λ satisfying the inequalities $m \le \Lambda \le M$, which yields the desired result (3.1). □

In particular, when $g(t) \equiv 1$, we get from Theorem 3.1 the following result.

Corollary 3.2. *Let f be an integrable function on $[a,b]$ and let m and M be the infimum and supremum, respectively, of f on $[a,b)$. Then there exists a number Λ between m and M such that*

$$\int_a^b f(t)\Delta t = \Lambda(b-a).$$

In what follows we will make use of the following fact, known as Abel's lemma.

Lemma 3.3. *Let the numbers p_i for $1 \le i \le n$ satisfy the inequalities $p_1 \ge p_2 \ge \ldots \ge p_n \ge 0$, and the numbers $S_k = \sum_{i=1}^k q_i$ for $1 \le k \le n$ satisfy the inequalities $m \le S_k \le M$ for all values of k, where q_i, m, and M are some numbers. Then $mp_1 \le \sum_{i=1}^n p_i q_i \le Mp_1$.*

Theorem 3.4 (Second Mean Value Theorem I)**.** *Let f be a bounded function that is integrable on $[a,b]$. Let further m_F and M_F be the infimum and supremum, respectively, of the function $F(t) = \int_a^t f(s)\Delta s$ on $[a,b]$. Then:*

(i) *If a function g is nonincreasing with $g(t) \ge 0$ on $[a,b]$, then there is some number Λ such that $m_F \le \Lambda \le M_F$ and*

$$\int_a^b f(t)g(t)\Delta t = g(a)\Lambda. \tag{3.4}$$

(ii) *If g is any monotone function on $[a,b]$, then there is some number Λ such that $m_F \le \Lambda \le M_F$ and*

$$\int_a^b f(t)g(t)\Delta t = [g(a)-g(b)]\Lambda + g(b)\int_a^b f(t)\Delta t. \tag{3.5}$$

Proof. To prove part (i) of the theorem, assume that g is nonincreasing and that $g(t) \geq 0$ for all $t \in [a, b]$. Consider an arbitrary $\varepsilon > 0$. Since f and fg are integrable on $[a, b]$, we can choose, by Theorem 2.4 and Definition 2.5, a partition $P = \{t_0 < t_1 < \ldots < t_n\} \in \mathcal{P}(a, b)$ such that

$$\sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) < \varepsilon \tag{3.6}$$

and

$$\left| \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1}) - \int_a^b f(t)g(t)\Delta t \right| < \varepsilon, \tag{3.7}$$

where m_i and M_i are the infimum and supremum, respectively, of f on $[t_{i-1}, t_i)$. Since $g(t_{i-1}) \geq 0$, we get from $m_i \leq f(t_{i-1})] \leq M_i$ that

$$\sum_{i=1}^{n} m_i g(t_{i-1})(t_i - t_{i-1}) \leq \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1}) \leq \sum_{i=1}^{n} M_i g(t_{i-1})(t_i - t_{i-1}) \tag{3.8}$$

holds. Next, by Corollary 3.2, there exist numbers Λ_i for $1 \leq i \leq n$ such that $m_i \leq \Lambda_i \leq M_i$ and

$$\int_{t_{i-1}}^{t_i} f(t)\Delta t = \Lambda_i(t_i - t_{i-1}).$$

Consider the numbers

$$S_k = \sum_{i=1}^{k} \Lambda_i(t_i - t_{i-1}) = \int_a^{t_k} f(t)\Delta t$$

for $1 \leq k \leq n$. Obviously, $m_F \leq S_k \leq M_F$, where m_F and M_F are the infimum and supremum, respectively, of F on $[a, b]$. Put

$$p_i = g(t_{i-1}) \quad \text{and} \quad q_i = \Lambda_i(t_i - t_{i-1})$$

for $1 \leq i \leq n$. Since g is nonincreasing and $g(t) \geq 0$, we have

$$p_1 \geq p_2 \geq \ldots \geq p_n \geq 0.$$

The numbers p_i, S_i, and q_i satisfy the conditions of Lemma 3.3. Therefore

$$m_F g(a) \leq \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_{i-1}) \leq M_F g(a). \tag{3.9}$$

On the other hand,

$$\sum_{i=1}^{n} m_i g(t_{i-1})(t_i - t_{i-1}) \leq \sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i - t_{i-1}) \leq \sum_{i=1}^{n} M_i g(t_{i-1})(t_i - t_{i-1}). \tag{3.10}$$

From (3.8) and (3.10) we have, taking into account the monotonicity of g and (3.6),

$$\begin{aligned}\left|\sum_{i=1}^{n} g(t_{i-1})\left[f(t_{i-1})-\Lambda_i\right](t_i-t_{i-1})\right| &\leq \sum_{i=1}^{n}(M_i-m_i)g(t_{i-1})(t_i-t_{i-1})\\ &\leq g(a)\sum_{i=1}^{n}(M_i-m_i)(t_i-t_{i-1})\\ &\leq g(a)\varepsilon.\end{aligned}$$

From this and (3.7) it follows that

$$\left|\int_a^b f(t)g(t)\Delta t-\sum_{i=1}^{n} g(t_{i-1})\Lambda_i(t_i-t_{i-1})\right|<\varepsilon+g(a)\varepsilon.$$

Hence, using (3.9), we obtain

$$-\varepsilon-g(a)\varepsilon+m_F g(a)<\int_a^b f(t)g(t)\Delta t<M_F g(a)+\varepsilon+g(a)\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get

$$m_F g(a)\leq\int_a^b f(t)g(t)\Delta t\leq M_F g(a). \tag{3.11}$$

If $g(a)=0$, it follows from (3.11) that $\int_a^b f(t)g(t)\Delta t=0$, and therefore equality (3.4) becomes obvious; if $g(a)>0$, then (3.11) implies

$$m_F\leq\frac{\int_a^b f(t)g(t)\Delta t}{g(a)}\leq M_F.$$

So the middle term of these inequalities is equal to a number Λ satisfying the inequalities $m_F\leq\Lambda\leq M_F$, wich yields the desired result (3.4).

Let now g be an arbitrary nonincreasing function on $[a,b]$. Then the function h defined by $h(t)=g(t)-g(b)$ is nonincreasing and $h(t)\geq 0$ on $[a,b]$. therefore, applying formula (3.4) to the function h, we can write

$$\int_a^b f(t)\left[g(t)-g(b)\right]\Delta t=\left[g(a)-g(b)\right]\Lambda,$$

which yields the formula (3.5) of part (ii) for nonincreasing functions g. If g is nondecreasing, then the function $g_1=-g$ is nonincreasing, and applying the obtained result to g_1, we get the same result for nondecreasing functions g as well. Thus, part (ii) is proved for all monotone functions g. □

The following theorem can be proved in a similar way as Theorem 3.4.

Theorem 3.5 (Second Mean Value Theorem II). *Let f be a bounded function that is integrable on $[a,b]$. Let further m_Φ and M_Φ be the infimum and supremum, respectively, of the function $\Phi(t)=\int_t^b f(s)\Delta s$ on $[a,b]$. Then:*

(i) *If a function g is nonincreasing with $g(t) \geq 0$ on $[a, b]$, then there is some number Λ such that $m_\Phi \leq \Lambda \leq M_\Phi$ and*

$$\int_a^b f(t)g(t)\Delta t = g(b)\Lambda.$$

(ii) *If g is any monotone function on $[a, b]$, then there is some number Λ such that $m_\Phi \leq \Lambda \leq M_\Phi$ and*

$$\int_a^b f(t)g(t)\Delta t = [g(b) - g(a)]\Lambda + g(a)\int_a^b f(t)\Delta t.$$

4. IMPROPER INTEGRALS OF FIRST KIND

Let $\mathbb{T}$ be a time scale and $a \in \mathbb{T}$. Throughout this section we assume that there exists a subset

$$\{t_k : k \in \mathbb{N}_0\} \subset \mathbb{T} \quad \text{with} \quad a = t_0 < t_1 < t_2 < \dots \text{ and } \lim_{k\to\infty} t_k = \infty. \tag{4.1}$$

In other words, we assume that $\mathbb{T}$ is unbounded above. Now let us suppose that a real-valued function f is defined on $[a, \infty) = \{t \in \mathbb{T} : t \geq a\}$ and is (Riemann) integrable from a to any point $A \in \mathbb{T}$ with $A \geq a$. If the integral

$$F(A) = \int_a^A f(t)\Delta t \tag{4.2}$$

approaches a finite limit as $A \to \infty$, we call that limit the *improper integral of first kind* of f from a to ∞ and write

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty}\left\{\int_a^A f(t)\Delta t\right\}. \tag{4.3}$$

In such a case we say that the improper integral

$$\int_a^\infty f(t)\Delta t \tag{4.4}$$

exists or that it is *convergent*. If the limit (4.3) does not exist, we say that the improper integral (4.4) does not exist or that it is *divergent*.

Example 4.1. Let $\mathbb{T}$ satisfy (4.1) and assume $a > 0$. Then

$$\int_a^\infty \frac{\Delta t}{t\sigma(t)} = \frac{1}{a} \quad \text{and} \quad \int_a^\infty \frac{t + \sigma(t)}{(t\sigma(t))^2}\Delta t = \frac{1}{a^2}.$$

These two formulas follow from the equations

$$\left(\frac{1}{t}\right)^\Delta = -\frac{1}{t\sigma(t)} \quad \text{and} \quad \left(\frac{1}{t^2}\right)^\Delta = -\frac{t + \sigma(t)}{(t\sigma(t))^2}.$$

The existence of the limit $\lim_{A\to\infty} F(A)$ is equivalent to the conditions of Cauchy's criterion for the existence of the limit of a function, which reads: For any $\varepsilon > 0$ there exists $A_0 > a$ such that for all $A_1, A_2 \in \mathbb{T}$ with $A_1 > A_0$ and $A_2 > A_0$ the inequality $|F(A_1) - F(A_2)| < \varepsilon$ holds. So we can state the following Cauchy criterion for existence of an improper integral of first kind.

Theorem 4.2. *For the existence of the integral* (4.4) *it is necessary and sufficient that for any given* $\varepsilon > 0$ *there exists* $A_0 > a$ *such that*

$$\left| \int_{A_1}^{A_2} f(t)\Delta t \right| < \varepsilon$$

for any $A_1, A_2 \in \mathbb{T}$ *satisfying the inequalities* $A_1 > A_0$ *and* $A_2 > A_0$.

An integral of type (4.4) is said to be *absolutely convergent* provided the integral

$$\int_a^\infty |f(t)|\Delta t$$

of the modulus of the function f is convergent. If an integral is convergent, but not absolutely convergent, it is called *conditionally convergent.*

Theorem 4.3. *If the integral* (4.4) *is absolutely convergent, it is convergent.*

Proof. This follows from the inequality (see Theorem 2.15)

$$\left| \int_{A_1}^{A_2} f(t)\Delta t \right| \le \int_{A_1}^{A_2} |f(t)|\Delta t$$

and Theorem 4.2. □

A convergent improper integral may not be absolutely convergent. But, of course, a convergent improper integral of a nonnegative function is always absolutely convergent.

Let us now consider the integral (4.4) with a nonnegative function f. In this case the function F defined by (4.2) is obviously nondecreasing. Therefore, if it is bounded, i.e., if $F(A) \le M$ $(A \ge a)$ for some $M > 0$, then integral (4.4) is convergent:

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty} \int_a^A f(t)\Delta t \le M.$$

If F is unbounded, then integral (4.4) is divergent:

$$\int_a^\infty f(t)\Delta t = \lim_{A\to\infty} \int_a^A f(t)\Delta t = \infty.$$

Hence we have the following result.

Theorem 4.4. *An integral* (4.4) *with* $f(t) \geq 0$ *for all* $t \geq a$ *is convergent if and only if there exists a constant* $M > 0$ *such that*

$$\int_a^A f(t)\Delta t \leq M \quad \textit{whenever} \quad A \geq a.$$

The value of the improper integral is then not greater than M*.*

Now we present the following comparison test.

Theorem 4.5. *Let the inequalities* $0 \leq f(t) \leq g(t)$ *be satisfied for all* $t \in [a, \infty)$*. Then the convergence of the improper integral*

$$\int_a^\infty g(t)\Delta t \tag{4.5}$$

implies the convergence of the improper integral

$$\int_a^\infty f(t)\Delta t \tag{4.6}$$

and the inequality

$$\int_a^\infty f(t)\Delta t \leq \int_a^\infty g(t)\Delta t,$$

while the divergence of integral (4.6) *implies the divergence of integral* (4.5)*.*

Proof. This follows from the inequality

$$\int_a^A f(t)\Delta t \leq \int_a^A g(t)\Delta t \quad \text{for} \quad A \geq a$$

by Theorem 4.4. □

To avoid troublesome details of working with inequalities in practice, it is often convenient to use the following theorem rather than to use the comparison test directly.

Theorem 4.6. *Suppose* $\int_a^\infty f(t)\Delta t$ *and* $\int_a^\infty g(t)\Delta t$ *are improper integrals of the first kind with positive integrands, and suppose that the limit*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L \tag{4.7}$$

exists (finite) and is not zero. Then the integrals are simultaneously convergent or divergent.

Proof. It follows from (4.7) that for any $\varepsilon \in (0, L)$ there exists $A_0 \in \mathbb{T}$ with $A_0 > a$ such that

$$L - \varepsilon < \frac{f(t)}{g(t)} < L + \varepsilon \quad \text{for all} \quad t > A_0$$

and, since $g(t) > 0$, we have

$$(L - \varepsilon)g(t) < f(t) < (L + \varepsilon)g(t) \quad \text{for all} \quad t > A_0. \tag{4.8}$$

The convergence of the integral $\int_a^\infty g(t)\Delta t$ implies the convergence of the integral $\int_{A_0}^\infty g(t)\Delta t$ and hence also the convergence of the integral $\int_{A_0}^\infty (L+\varepsilon)g(t)\Delta t$. Therefore, by virtue of Theorem 4.5, the integral $\int_{A_0}^\infty f(t)\Delta t$ also converges, and, together with it, so does the integral $\int_a^\infty f(t)\Delta t$. Conversely, if $\int_a^\infty f(t)\Delta t$ is convergent, then $\int_a^\infty g(t)\Delta t$ is also convergent, which can be proved similarly using the left part of the inequalities (4.8). $\square$

Remark 4.7. For the integrals described in Theorem 4.6, suppose that $L = 0$. Then, if $\int_a^\infty g(t)\Delta t$ is convergent, so is $\int_a^\infty f(t)\Delta t$. Or, alternatively, suppose that $L = \infty$. Then, if $\int_a^\infty g(t)\Delta t$ is divergent, so is $\int_a^\infty f(t)\Delta t$.

From Theorem 4.3 and Theorem 4.5, the following result is straightforward.

Corollary 4.8. *Let $|f(t)| \le g(t)$ for all $t \in \mathbb{T}$ with $t \ge a$. Then the convergence of the integral $\int_a^\infty g(t)\Delta t$ implies the convergence of the integral $\int_a^\infty f(t)\Delta t$.*

If an integral is conditionally convergent, the demonstration of its convergence is usually a more delicate matter. Many of the instances of practical importance can be handled by the following theorem, which is an analogue of the well-known Dirichlet–Abel test for improper integrals on $\mathbb{R}$.

Theorem 4.9. *Let the following conditions be satisfied.*

(i) *f is integrable from a to any point $A \in \mathbb{T}$ with $A \ge a$, and the integral*

$$F(A) = \int_a^A f(t)\Delta t$$

is bounded for all $A \ge a$.

(ii) *g is monotone on $[a,\infty)$ and $\lim_{t\to\infty} g(t) = 0$.*

Then the improper integral of first kind of the form

$$\int_a^\infty f(t)g(t)\Delta t \tag{4.9}$$

is convergent.

Proof. Applying the mean value theorem (Theorem 3.4 (ii)), we can write, for any $A_1, A_2 \in \mathbb{T}$ with $A_2 > A_1 \ge a$,

$$\int_{A_1}^{A_2} f(t)g(t)\Delta t = [g(A_1) - g(A_2)]\,\Lambda + g(A_2)\int_{A_1}^{A_2} f(t)\Delta t, \tag{4.10}$$

where Λ is between $\inf_{A\in[A_1,A_2]} F(A)$ and $\sup_{A\in[A_1,A_2]} F(A)$. Let us suppose that K is a bound for $|F(A)|$ on $[a,\infty)$, that is, $|F(A)| \le K$. Then we have from (4.10), taking into account that the integral on the right hand side of (4.10) is equal to $F(A_2) - F(A_1)$,

$$\left|\int_{A_1}^{A_2} f(t)g(t)\Delta t\right| \le K\left[|g(A_1)| + 3|g(A_2)|\right]. \tag{4.11}$$

Using this inequality, it is not difficult to complete the proof. Let $\varepsilon > 0$ be arbitrary. Since $g(t) \to 0$ as $t \to \infty$, we can choose a number $A_0 > a$ such that $|g(t)| < \frac{\varepsilon}{4K}$ for all $t > A_0$. Hence, and from (4.11), it follows that for all A_1 and A_2 with $A_1 > A_0$ and $A_2 > A_0$ the inequality

$$\left| \int_{A_1}^{A_2} f(t)g(t)\Delta t \right| < \varepsilon$$

holds. Consequently, by the Cauchy criterion (Theorem 4.2) the integral (4.9) is convergent. □

Remark 4.10. Integrals with $-\infty$ as a limit of integration may be treated by methods parallel to those given above.

5. EXAMPLES

In the first part of this section we consider time scales of the form

$$\mathbb{T} = \{t_k : k \in \mathbb{N}_0\} \quad \text{with} \quad 0 < t_0 < t_1 < t_2 < \dots \text{ and } \lim_{k\to\infty} t_k = \infty. \tag{5.1}$$

Note that any time scale of the form (5.1) satisfies our general assumption (4.1). The following auxiliary result will be useful.

Lemma 5.1. *Let the time scale $\mathbb{T}$ satisfy* (5.1). *If $f : [t_0, \infty) \to \mathbb{R}$ is a function, then*

$$\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(t_k)(t_{k+1} - t_k) \quad \textit{and} \quad \int_{t_0}^{\infty} f(t)\nabla t = \sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1} - t_k).$$

If in addition f is nonincreasing on $[t_0, \infty)$, then

$$\int_{t_0}^{\infty} f(t)\nabla t \le \int_{t_0}^{\infty} f(t)dt \le \int_{t_0}^{\infty} f(t)\Delta t,$$

where the first and last integrals are taken over $\mathbb{T}$, while the middle integral is taken over the interval $[t_0, \infty)$ of $\mathbb{R}$.

Proof. The first part of the lemma follows from Theorem 2.13 and Theorem 2.16. Then the second part is obtained by summing the inequalities

$$f(t_{k+1})(t_{k+1} - t_k) \le \int_{t_k}^{t_{k+1}} f(t)dt \le f(t_k)(t_{k+1} - t_k)$$

from $k = 0$ to ∞. □

From Lemma 5.1 we immediately find the following convergence criteria.

Theorem 5.2. *Let $\mathbb{T}$ be a time scale satisfying* (5.1) *and suppose that $f : [t_0, \infty) \to \mathbb{R}_+$ is nonincreasing, where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.*

(i) *If $\int_{t_0}^{\infty} f(t)dt = \infty$, then $\int_{t_0}^{\infty} f(t)\Delta t = \infty$;*
(ii) *if $\int_{t_0}^{\infty} f(t)dt < \infty$, then $\int_{t_0}^{\infty} f(t)\nabla t < \infty$.*

Example 5.3. Let $\mathbb{T}$ be a time scale satisfying (5.1). Then

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} = \infty \quad \text{if} \quad 0 \le p \le 1$$

and

$$\int_{t_0}^{\infty} \frac{\nabla t}{t^p} < \infty \quad \text{if} \quad p > 1.$$

Example 5.4. Let $q > 1$ and $\mathbb{T} = \{q^k : k \in \mathbb{N}_0\}$. Then $\int_1^\infty \frac{\Delta t}{t} = \infty$. In fact,

$$\int_1^{q^n} \frac{\Delta t}{t} = \sum_{k=0}^{n-1} \frac{\mu(q^k)}{q^k} = \sum_{k=0}^{n-1} (q-1) = (q-1)n$$

and hence for $t \in \mathbb{T}$

$$\int_1^t \frac{\Delta \tau}{\tau} = (q-1)\log_q t \to \infty \quad \text{as} \quad t \to \infty.$$

Theorem 5.5. *Let $\mathbb{T}$ be a time scale satisfying* (5.1). *Then*

$$\int_{t_0}^{\infty} \frac{\nabla t}{t^p} = \infty \quad \textit{if} \quad 0 \le p \le 1.$$

Proof. We only show the statement for $p = 1$ as the remainder follows by comparison. We cannot use Theorem 5.2 to obtain this result. From Lemma 5.1 we find that

$$\int_{t_0}^{\infty} \frac{\nabla t}{t} = \sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}}. \tag{5.2}$$

We know from Lemma 5.1 and Example 5.3 (and also directly) that

$$\int_{t_0}^{\infty} \frac{\Delta t}{t} = \sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_k} = \infty. \tag{5.3}$$

To obtain a contradiction, let us now assume that

$$\sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}} < \infty. \tag{5.4}$$

But then

$$\frac{t_{k+1}}{t_k} \to 1 \quad \text{as} \quad k \to \infty.$$

Hence there exists an index $N \in \mathbb{N}_0$ such that

$$\frac{t_{k+1}}{t_k} \le 2 \quad \text{for all} \quad k \ge N.$$

It follows that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_k} &= \sum_{k=0}^{N-1} \frac{t_{k+1} - t_k}{t_k} + \sum_{k=N}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}} \frac{t_{k+1}}{t_k} \\ &\le \sum_{k=0}^{N-1} \frac{t_{k+1} - t_k}{t_k} + 2 \sum_{k=N}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}} \\ &< \infty, \end{aligned}$$

contradicting (5.3). Hence the assumption (5.4) is wrong and the result follows using (5.2). □

We now give an example which is not expected from the usual calculus of integrals and sums.

Example 5.6. Let $p > 1$. Define

$$t_k = 2^{(p^k)} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Now we consider the time scale $\mathbb{T} = \{t_k : \ k \in \mathbb{N}_0\}$, which satisfies our assumption (5.1). For this time scale we have

$$\sigma(t) = t^p \quad \text{and} \quad \mu(t) = t^p - t.$$

We use Lemma 5.1 to calculate

$$\begin{aligned} \int_{t_0}^{\infty} \frac{\Delta t}{t^p} &= \sum_{k=0}^{\infty} \frac{\mu(t_k)}{t_k^p} \\ &= \sum_{k=0}^{\infty} \frac{t_k^p - t_k}{t_k^p} \\ &= \sum_{k=0}^{\infty} \left[1 - \frac{1}{t_k^{p-1}}\right] \\ &= \infty, \end{aligned}$$

because the general term of the last series tends to 1 as $k \to \infty$.

In view of Example 5.6,

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} < \infty \quad \text{for} \quad p > 1 \tag{5.5}$$

does not hold in general. The question arises for which time scales (5.5) does in fact hold. We now elaborate on this problem.

Theorem 5.7. *Let $\mathbb{T}$ be a time scale satisfying* (5.1)*. Suppose that $f : [t_0, \infty) \to \mathbb{R}_+$ is nonincreasing with $\int_{t_0}^{\infty} f(t)dt < \infty$. If $g : \mathbb{T} \to \mathbb{R}_+$ satisfies*

$$g(t_k) \le K f(t_{k+1}) \quad \textit{for all} \quad k \in \mathbb{N}_0, \tag{5.6}$$

where $K > 0$ may be an arbitrary constant, then $\int_{t_0}^{\infty} g(t)\Delta t < \infty$.

Proof. We use again Lemma 5.1 to calculate

$$\begin{aligned} \int_{t_0}^{\infty} g(t)\Delta t &= \sum_{k=0}^{\infty} \mu(t_k)g(t_k) \\ &\leq K\sum_{k=0}^{\infty} \mu(t_k)f(t_{k+1}) \\ &\leq K\int_{t_0}^{\infty} f(t)dt \\ &< \infty, \end{aligned}$$

and hence $\int_{t_0}^{\infty} g(t)\Delta t$ converges. □

From Theorem 5.7 we have the following result.

Theorem 5.8. *Suppose that* $\mathbb{T}$ *is a time scale satisfying* (5.1). *Let* $p > 1$. *If there exists* $\alpha \in [1, p)$ *with*

$$\sigma(t) = \mathrm{O}(t^{\alpha}) \quad as \quad t \to \infty,$$

then (5.5) *holds, i.e.,*

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} < \infty.$$

Proof. Let $p > 1$ and $1 \leq \alpha < p$ such that $\sigma(t) = \mathrm{O}(t^{\alpha})$ as $t \to \infty$. Then there exists a constant $K > 0$ such that

$$t_{k+1} = \sigma(t_k) \leq Kt_k^{\alpha} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

We put $q = \frac{p}{\alpha}$ so that $q > 1$. Then

$$t_{k+1}^q \leq K^q t_k^p \quad \text{for all} \quad k \in \mathbb{N}_0$$

and therefore

$$\frac{1}{t_k^p} \leq K^q \frac{1}{t_{k+1}^q} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Hence (5.6) is satisfied with $f(t) = 1/t^q$ and $g(t) = 1/t^p$. In addition we have that f is nonincreasing on $[t_0, \infty)$ and that $\int_{t_0}^{\infty} f(t)dt < \infty$. By Theorem 5.7 we conclude that $\int_{t_0}^{\infty} g(t)\Delta t < \infty$, i.e., $\int_{t_0}^{\infty} \frac{\Delta t}{t^p} < \infty$. □

Example 5.9. We let

$$t_k = 2^{(2^k)} \quad \text{for all} \quad k \in \mathbb{N}_0$$

and consider the time scale $\mathbb{T} = \{t_k : k \in \mathbb{N}_0\}$, which satisfies our assumption (5.1). For this time scale we have

$$\sigma(t) = t^2 = \mathrm{O}(t^2) \quad \text{as} \quad t \to \infty.$$

We use Theorem 5.8 to conclude that

$$\int_{2}^{\infty} \frac{\Delta t}{t^3} < \infty.$$

For the second part of this section, we now consider arbitrary time scales that are unbounded above, i.e., we assume (4.1).

Lemma 5.10. *Let the time scale* $\mathbb{T}$ *satisfy* (4.1)*. If* $f : [t_0, \infty) \to \mathbb{R}$ *is nonincreasing, then*

$$\sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1} - t_k) \le \int_{t_0}^{\infty} f(t)\Delta t \le \sum_{k=0}^{\infty} f(t_k)(t_{k+1} - t_k).$$

Proof. The estimate

$$\begin{aligned} \int_{t_0}^{\infty} f(t)\Delta t &= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} f(t)\Delta t \\ &\ge \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} f(t_{k+1})\Delta t \\ &= \sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1} - t_k) \end{aligned}$$

shows the first inequality, and the second one can be verified similarly. □

Theorem 5.11. *Suppose* $\mathbb{T}$ *satisfies* (4.1)*. Let* $0 \le p \le 1$*. Then we have*

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} = \infty.$$

Proof. We only show the result for $p = 1$, as then the rest follows by comparison. By the first inequality of Lemma 5.10 we find

$$\int_{t_0}^{\infty} \frac{\Delta t}{t} \ge \sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_{k+1}}. \tag{5.7}$$

Because

$$\sum_{k=0}^{\infty} \frac{t_{k+1} - t_k}{t_k}$$

is an upper sum for the divergent integral $\int_{t_0}^{\infty} \frac{dt}{t}$, the value of this sum is infinity, and exactly as in the proof of Theorem 5.5 we find that the sum on the right hand side of (5.7) is equal to infinity as well. □

Now we again turn to the problem of the convergence of

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} \quad \text{if} \quad p > 1.$$

Theorem 5.12. *Let* $\mathbb{T}$ *be a time scale satisfying* (4.1)*. Suppose that* $f : [t_0, \infty) \to \mathbb{R}_+$ *is nonincreasing with* $\int_{t_0}^{\infty} f(t)dt < \infty$*. If* $g : [t_0, \infty) \cap \mathbb{T} \to \mathbb{R}_+$ *is nonincreasing and satisfies*

$$g(t_k) \le K f(t_{k+1}) \quad \textit{for all} \quad k \in \mathbb{N}_0, \tag{5.8}$$

where $K > 0$ *may be an arbitrary constant, then* $\int_{t_0}^{\infty} g(t)\Delta t < \infty$*.*

Proof. We use the second inequality of Lemma 5.10 and (5.8) to find

$$\begin{aligned}\int_{t_0}^{\infty} g(t)\Delta t &\leq \sum_{k=0}^{\infty} g(t_k)(t_{k+1}-t_k)\\ &\leq K\sum_{k=0}^{\infty} f(t_{k+1})(t_{k+1}-t_k)\\ &\leq K\int_{t_0}^{\infty} f(t)dt\\ &< \infty,\end{aligned}$$

where we also used again the fact that a lower sum for an integral is less than or equal to that integral. □

Using Theorem 5.12 we now obtain the following result.

Theorem 5.13. *Suppose that* $\mathbb{T}$ *satisfies* (4.1). *Let* $p > 1$. *If there exists* $\alpha \in [1, p)$ *such that*

$$t_{k+1} = \mathrm{O}(t_k^{\alpha}) \quad as \quad k \to \infty,$$

then

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} < \infty.$$

Proof. Let $p > 1$ and $\alpha \in [1, p)$ so that $t_{k+1} = \mathrm{O}(t_k^{\alpha})$ as $k \to \infty$. Then there exists a constant $K > 0$ such that

$$t_{k+1} \leq K t_k^{\alpha} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

We put $q = \frac{p}{\alpha}$ so that $q > 1$. Now the proof may be finished in exactly the same way as the proof of Theorem 5.8. □

Corollary 5.14. *Let* $\mathbb{T}$ *be a time scale that is unbounded above. Let* $p > 1$. *If there exists* $\alpha \in [1, p)$ *such that* $\sigma(t) = \mathrm{O}(t^{\alpha})$ *as* $t \to \infty$, *then*

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} < \infty.$$

Proof. We will construct a sequence $\{t_k : \ k \in \mathbb{N}_0\}$ satisfying (4.1) such that the assumptions of Theorem 5.13 are satisfied. Hence the claim will follow by Theorem 5.13. Let $p > 1$ and suppose $\alpha \in [1, p)$ is such that

$$\sigma(t) \leq K t^{\alpha} \quad \text{for all} \quad t \in \mathbb{T}, \tag{5.9}$$

where $K > 1$ without loss of generality. Now consider the sets

$$A(t) = \{s \in \mathbb{T} : \ t < s \leq K t^{\alpha}\} \quad \text{for} \quad t \in \mathbb{T} \text{ with } t > 1.$$

Because of (5.9), we have $A(t) \neq \emptyset$ for all $t \in \mathbb{T}$ and (observe that $\mathbb{T}$ is closed) $\max A(t)$ exists and is an element of $\mathbb{T}$ for all $t \in \mathbb{T}$. Now let $t_0 \in \mathbb{T}$ with $t_0 > 1$ and define a sequence $\{t_k : k \in \mathbb{N}_0\}$ recursively by

$$t_{k+1} = \max A(t_k) \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Hence we have

$$t_k < t_{k+1} \leq K t_k^{\alpha} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Therefore the sequence $\{t_k : k \in \mathbb{N}_0\}$ is increasing and satisfies the assumptions of Theorem 5.13. To complete the proof, we only need to show that

$$\lim_{k\to\infty} t_k = \infty. \tag{5.10}$$

We assume the opposite of (5.10), i.e., that $\{t_k : k \in \mathbb{N}_0\}$ is bounded. Then the limit $\lim_{k\to\infty} t_k$ exists, say is equal to M, and $M \in \mathbb{T}$ as $\mathbb{T}$ is closed. Now note that the length $l(t)$ (for $t \geq t_0$) of the interval $A(t)$ satisfies

$$l(t) = Kt^{\alpha} - t \geq Kt - t \geq (K-1)t_0 =: \delta > 0.$$

We pick an index $N \in \mathbb{N}_0$ such that $M - t_N < \delta$. But then, by the definition of t_{N+1}, we must have $t_{N+1} \geq M$, and because of (5.9), $t_{N+2} > M$, a contradiction. This shows that (5.10) holds and the proof is complete. □

Example 5.15. Let $\mathbb{T}$ be a time scale that is unbounded above. Suppose that the graininess of $\mathbb{T}$ is bounded above. Then the improper integral

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^p} \tag{5.11}$$

is divergent if $p \leq 1$ and convergent if $p > 1$. This follows from Theorem 5.11 and Corollary 5.14 (with $\alpha = 1$).

6. IMPROPER INTEGRALS OF SECOND KIND

Let $\mathbb{T}$ be a time scale, $a < b$ be fixed points in $\mathbb{T}$, and let b be left-dense. Let a function f be defined in the interval $[a, b)$. We suppose that f is integrable on any interval $[a, c]$ with $c < b$ and is unbounded on $[a, b)$. Then the ordinary Riemann integral of f on $[a, b]$ cannot exist since a Riemann integrable (in the proper sense) function from a to b must be bounded on $[a, b)$. The formal expression

$$\int_a^b f(t)\Delta t \tag{6.1}$$

is called an improper integral of *second kind*. We say that the integral (6.1) is improper at $t = b$. Sometimes we say that f has a *singularity* at $t = b$. If the left sided limit

$$\lim_{c\to b^-} \int_a^c f(t)\Delta t \tag{6.2}$$

exists as a finite number, we say that the improper integral (6.1) exists or that it is convergent. In such a case we call this limit the value of the improper integral (6.1) and write

$$\int_a^b f(t)\Delta t = \lim_{c\to b^-}\int_a^c f(t)\Delta t.$$

If the limit (6.2) does not exist, we say that the integral (6.1) does not exist or that it is divergent.

All theorems of Section 4 have exact analogues for the improper integrals of second kind whose wordings differ only slightly from the statements given in Section 4.

For the existence of the integral (6.1) it is necessary and sufficient that the conditions of Cauchy's criterion hold: Given any $\varepsilon > 0$, there is $b_0 < b$ such that

$$\left|\int_{c_1}^{c_2} f(t)\Delta t\right| < \varepsilon$$

for any $c_1, c_2 \in \mathbb{T}$ satisfying the inequalities $b_0 < c_1 < b$ and $b_0 < c_2 < b$.

Suppose in integral (6.1) we have $f(t) \geq 0$. Then, for $c \in [a, b)$,

$$F(c) = \int_a^c f(t)\Delta t$$

does not decrease as c increases, and the integral (6.1) is convergent if and only if F is bounded, in which case the value of the integral is $\lim_{c\to b^-} F(c)$. This result enables us to prove a comparison test strictly parallel to Theorem 4.5, from which in turn we deduce limit tests in which the convergence or divergence of two integrals of the same type,

$$\int_a^b f(t)\Delta t \quad \text{and} \quad \int_a^b g(t)\Delta t$$

are related by an examination of the limit

$$\lim_{t\to b^-}\frac{f(t)}{g(t)}.$$

Similar definitions are made and entirely similar results are obtained for integrals of the second kind improper at the lower limit of integration.

Remark 6.1. Many improper integrals occurring in practice are of mixed type. If singularities occur within the interval of integration, or at both ends of an interval of integration, the integral must be separated into several integrals, each of which is a pure type of either first or second kind. The integral is called divergent if any one of the constituent pure types is divergent.

As an example, we consider the improper integral

$$\int_a^b \frac{\Delta t}{(b-t)^p}. \tag{6.3}$$

Theorem 6.2. *Let $\mathbb{T}$ be an arbitrary time scale, $a, b \in \mathbb{T}$ with $a < b$, and suppose that b is left-dense. Let $p \geq 1$. Then the improper integral* (6.3) *is divergent.*

Proof. We only show the result for $p = 1$; the result for $p > 1$ then follows by comparison. Let us choose points $t_n \in \mathbb{T}$ for $n \in \mathbb{N}_0$ such that

$$a = t_0 < t_1 < t_2 < \ldots < b \quad \text{and} \quad \lim_{n\to\infty} t_n = b.$$

Setting

$$\tau_n = \frac{1}{b - t_n} \quad \text{for all} \quad n \in \mathbb{N}_0,$$

we have $\tau_0 = \frac{1}{b-a} > 0$, $\tau_0 < \tau_1 < \tau_2 < \ldots$, and $\lim_{n\to\infty} \tau_n = \infty$. We also have

$$t_{n+1} - t_n = \frac{\tau_{n+1} - \tau_n}{\tau_n \tau_{n+1}} \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Now

$$\begin{aligned}
\int_a^b \frac{\Delta t}{b-t} &= \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} \frac{\Delta t}{b-t} \\
&\geq \sum_{n=0}^{\infty} \frac{1}{b-t_n} \int_{t_n}^{t_{n+1}} \Delta t \\
&= \sum_{n=0}^{\infty} \frac{t_{n+1} - t_n}{b - t_n} \\
&= \sum_{n=0}^{\infty} \frac{\tau_{n+1} - \tau_n}{\tau_{n+1}} \\
&= \infty,
\end{aligned}$$

where the last equality follows as in the proof of Theorem 5.5. □

Theorem 6.3. *Let $\mathbb{T}$ be a time scale for which the conditions of Theorem 6.2 are satisfied. Suppose that there exist points $t_n \in \mathbb{T}$ for $n \in \mathbb{N}_0$ such that*

$$a = t_0 < t_1 < t_2 < \ldots < b \quad \text{and} \quad \lim_{n\to\infty} t_n = b.$$

Let $p < 1$ and suppose that for some $\alpha \in \left[1, \frac{1}{p}\right)$,

$$\frac{1}{b - t_{k+1}} = \mathrm{O}\left(\frac{1}{(b-t_k)^\alpha}\right) \quad \text{as} \quad k \to \infty.$$

Then the improper integral (6.3) *is convergent.*

Proof. Defining τ_n for $n \in \mathbb{N}_0$ as in the proof of Theorem 6.2, we have

$$\tau_{k+1} = \mathrm{O}(\tau_k^\alpha) \quad \text{as} \quad k \to \infty,$$

i.e., $\tau_{k+1} \le K\tau_k^\alpha$ for all $k \in \mathbb{N}_0$, where $K > 0$ is a constant. Therefore,

$$\begin{aligned}
\int_a^b \frac{\Delta t}{(b-t)^p} &= \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{\Delta t}{(b-t)^p} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(b-t_{k+1})^p} \int_{t_k}^{t_{k+1}} \Delta t \\
&= \sum_{k=0}^{\infty} \frac{t_{k+1}-t_k}{(b-t_{k+1})^p} \\
&= \sum_{k=0}^{\infty} \frac{\tau_{k+1}-\tau_k}{\tau_k \tau_{k+1}^{1-p}} \\
&\le K^{1/\alpha} \sum_{k=0}^{\infty} \frac{\tau_{k+1}-\tau_k}{\tau_{k+1}^{\frac{1}{\alpha}+1-p}} \\
&\le K^{1/\alpha} \sum_{k=0}^{\infty} \int_{\tau_k}^{\tau_{k+1}} \frac{dt}{t^{\frac{1}{\alpha}+1-p}} \\
&= K^{1/\alpha} \int_{\tau_0}^{\infty} \frac{dt}{t^{\frac{1}{\alpha}+1-p}} \\
&< \infty,
\end{aligned}$$

since from $\alpha \in \left[1, \frac{1}{p}\right)$ it follows that $\frac{1}{\alpha} + 1 - p > 1$. □

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