ABSTRACT. Alpha derivatives are studied on generalized time scales \( T \). We present a Liouville formula for an \( n \)th order linear vector alpha-dynamic equation on a generalized time scale. A criterion is given for a matrix function to be \( \alpha \)-regressive. As special cases, we get Liouville’s formula for the delta dynamic system and for the nabla dynamic system, and other examples are presented.

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1. INTRODUCTION

In this paper, we will assume that the reader is familiar with the common literature on dynamic equations on time scales (see, for example, Bohner and Peterson [3]). We first define generalized time scales and the \( \alpha \)-derivative as in Ahlbrandt, Bohner, and Ridenhour [1].

**Definition 1.1.** A generalized time scale \((T,\alpha)\) is a nonempty set \( T \subseteq \mathbb{R} \) such that every Cauchy sequence in \( T \) converges to a point in \( T \), except possibly Cauchy sequences which converge to a finite infimum or supremum of \( T \), and \( \alpha \) is a function mapping \( T \) into \( T \).

**Definition 1.2.** A function \( f : T \rightarrow \mathbb{R} \) is alpha differentiable at a point \( t \in T \) provided there is a unique number \( f(\alpha)(t) \), the alpha derivative of \( f \) at \( t \), with the property that for each \( \epsilon > 0 \) there exists a neighborhood \( U \) of \( t \) such that

\[
|f(\alpha)(t) - f(s) - f(\alpha)(t)(\alpha(t) - s)| \leq \epsilon|\alpha(t) - s|
\]

for all \( s \in U \), where \( f(\alpha) = f \circ \alpha \).

Note that if \( \alpha(t) = t \) and \( t \) is isolated, then for any function \( f \), we have \( f \) is not \( \alpha \)-differentiable at \( t \), as discussed in Bohner and Peterson [3]. When \( \alpha = \sigma \) and \( T \) is closed, we have the Hilger delta derivative [4]. For \( \alpha = \rho \) and a closed set \( T \), we have the Atıcı–Guseinov nabla derivative, which was introduced in Section 8.4 of Atıcı and Guseinov [2].
Definition 1.3. A scalar function \( p : \mathbb{T} \rightarrow \mathbb{R} \) is \( \alpha \)-regressive provided
\[
1 + p(t)\mu_\alpha(t) \neq 0 \quad \text{for all } \ t \in \mathbb{T},
\]
where \( \mu_\alpha(t) := \alpha(t) - t \) is the generalized graininess.

Definition 1.4. For two \( \alpha \)-regressive functions \( p \) and \( q \), we define circle-plus addition via
\[
(p \oplus_\alpha q)(t) = p(t) + q(t) + \mu_\alpha(t)p(t)q(t).
\]

Definition 1.5. A first order linear alpha dynamic equation is of the form
\[
y^{(\alpha)} = p(t)y, \quad \text{where } p \text{ is } \alpha \text{-regressive.}
\]

Definition 1.6. If the initial value problem
\[
y^{(\alpha)} = p(t)y, \quad y(t_0) = 1
\]
has a unique solution, we call the unique solution the generalized exponential function and denote it by \( e_p(t, t_0) \).

Note that this exponential function depends on both \( \mathbb{T} \) and \( \alpha \). We now introduce notation which is similar to notation used in Horn and Johnson [5]. Let \( \lambda_k \subseteq \{1, 2, \ldots, n\} \) be an indexed set with \( k \) elements. For an \( n \times n \) matrix-valued function \( A \), a principal submatrix of \( A \), denoted \( A(\lambda_k) \) is the submatrix that lies in the rows and columns of \( A(t) \) indexed by \( \lambda_k \). Note that \( A(\lambda_k) \) is \( k \times k \), and there are \( \binom{n}{k} \) different \( k \times k \) principal submatrices of \( A \). The determinant of a principal submatrix is called a principal minor of \( A(t) \). The sum of the \( \binom{n}{k} \) different \( k \times k \) principal minors of \( A(t) \) is denoted \( E_k(A(t)) \). We will usually suppress the \( t \) and just write \( E_k(A) \). As shown in Horn and Johnson [5], the characteristic polynomial for \( A(t) \),
\[
p_A(x) = \det(xI - A)
\]
can be written in the form
\[
p_A(x) = x^n - x^{n-1}E_1(A) + x^{n-2}E_2(A) + \ldots + (-1)^nE_n(A).
\]

Definition 1.7. We say \( A \) is \( \alpha \)-regressive provided \( I + \mu_\alpha(t)A(t) \) is invertible for \( t \in \mathbb{T} \).

For an \( n \times n \) matrix-valued function \( X \), \( X(\lambda_k, \alpha) \) is the \( n \times n \) matrix obtained from \( X(t) \) with alpha derivatives on the rows indexed by \( \lambda_k \) and the usual entries of \( X(t) \) on the remaining rows. Let \( X(\lambda_k, \alpha, j) \) for \( k \leq j \leq n \) denote that \( j \in \lambda_k \) and \( j \) is the largest number in \( \lambda_k \). Note that there are \( \binom{n}{k} \) different \( X(\lambda_k, \alpha) \) and \( \binom{j-1}{k-1} \) different \( X(\lambda_k, \alpha, j) \). Also, let \( D_k(X) \) denote the sum of the determinants of the \( X(\lambda_k, \alpha) \) and \( D_{k,j}(X) \) denote the sum of the determinants of the \( X(\lambda_k, \alpha, j) \).
2. MAIN RESULTS

**Lemma 2.1.** Let $A$ be an $n \times n$ matrix-valued function. $A$ is $\alpha$-regressive if and only if the scalar function $q$ defined by

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)$$

is $\alpha$-regressive.

**Proof.** If $\mu_\alpha(t) \neq 0$, we have

$$\det [I + \mu_\alpha(t)A(t)] = (-\mu_\alpha(t))^n \det \left[ \frac{-1}{\mu_\alpha(t)} I - A(t) \right]$$

$$= (-\mu_\alpha(t))^n p_A \left( \frac{-1}{\mu_\alpha(t)} \right)$$

$$= (-\mu_\alpha(t))^n \left[ \left( \frac{-1}{\mu_\alpha(t)} \right)^n - \left( \frac{-1}{\mu_\alpha(t)} \right)^{n-1} E_1(A) + \ldots + (-1)^n E_n(A) \right]$$

$$= 1 + \mu_\alpha(t) [E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)].$$

Thus,

$$\det [I + \mu_\alpha(t)A(t)] = 1 + \mu_\alpha(t)q(t)$$

where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

Note that this formula also holds for those $t$ where $\mu_\alpha(t) = 0$. Hence,

$$\det [I + \mu_\alpha(t)A(t)] \neq 0$$

if and only if

$$1 + \mu_\alpha(t)q(t) \neq 0.$$ 

Thus, $A(t)$ is an $\alpha$-regressive matrix if and only if $q(t)$ is $\alpha$-regressive where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

The proof is complete. \(\square\)

**Lemma 2.2.** Assume $A$ is an $n \times n$ matrix-valued function which is $\alpha$-regressive and $X(t)$ is a solution of the $n \times n$ matrix $\alpha$-dynamic equation $X^{(\alpha)} = A(t)X$ in a generalized time scale $\mathbb{T}$. Then, for an indexed set $\lambda_m \subseteq \{1, 2, \ldots, n\}$, we have

$$\det X(\lambda_m, \alpha) = \det A(\lambda_m) \det X(t).$$

**Proof.** Suppose $\lambda_m = \{i_1, i_2, i_3, \ldots, i_m\}$. Then, the $(i_k, j)$-component of $X(\lambda_m, \alpha)$ is

$$x_{i_k,j}^{(\alpha)} = \sum_{p=1}^n a_{i_k,p}x_{pj}.$$
The \((i, j)\)-component of \(X(\lambda_m, \alpha)\) where \(i \not\in \lambda_m\) is \(x_{ij}\). Since we can add \(-a_{ij} p\) times the \(p^{th}\) row of \(X(\lambda_m, \alpha)\) for \(p \not\in \lambda_m\) to the \(i^\text{th}\) row without changing the value of the determinant, we have the

\[
(i_k, j) - \text{component of } X(\lambda_m, \alpha) = \sum_{s=1}^{m} a_{ik}s x_{ij},
\]

Do this for each \(i_j\) row, \(1 \leq j \leq m\). Now, let \(C\) be the block diagonal matrix

\[
C = \begin{pmatrix}
I_{i_1-1} & 0 & 0 \\
0 & W & 0 \\
0 & 0 & I_{n-i_m}
\end{pmatrix}
\]

where \(I_j\) is the \(j \times j\) identity matrix and

\[
W = \begin{pmatrix}
a_{i_1i_1} & 0 & \ldots & 0 & a_{i_1i_2} & 0 & \ldots & 0 & \ldots & 0 & a_{i_1i_m} \\
0 & 0 & & & \ldots & 0 & & & & & 0 \\
\cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & & & \ldots & 0 & & & & & 0 \\
\vdots & \vdots & & \ddots & & \cdots & & & & & \vdots \\
0 & 0 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{i_mi_1} & 0 & \ldots & 0 & a_{i_mi_2} & 0 & \ldots & 0 & \ldots & 0 & a_{i_mi_m}
\end{pmatrix}
\]

Then, we have

\[
\det X(\lambda_m, \alpha) = \det C \det X(t) = \det W \det X(t).
\]

We can calculate the determinant of \(W\) by expanding about the rows with 1’s along the diagonal first. Thus, we see

\[
\det X(\lambda_m, \alpha) = \begin{vmatrix}
a_{i_1i_1} & a_{i_1i_2} & \ldots & a_{i_1i_m} \\
a_{i_2i_1} & a_{i_2i_2} & \ldots & a_{i_2i_m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_mi_1} & a_{i_mi_2} & \ldots & a_{i_mi_m}
\end{vmatrix} \det X(t).
\]

Therefore, \(\det X(\lambda_m, \alpha) = \det A(\lambda_m) \det X(t)\). \(\square\)

**Theorem 2.3.** Assume \(A\) is an \(n \times n\) matrix-valued function which is \(\alpha\)-regressive and \(X(t)\) is a solution of the \(n \times n\) matrix \(\alpha\)-dynamic equation \(X^{(\alpha)} = A(t)X\) in
a generalized time scale \( T \). Then, \( u(t) := \det X(t) \) satisfies the scalar \( \alpha \)-dynamic equation

\[
u^{(\alpha)} = q(t) u
\]

where

\[
q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).
\]

**Proof.** By Lemma 2.1, since \( A(t) \) is \( \alpha \)-regressive, we have \( q(t) \) is \( \alpha \)-regressive. Hence, we can consider the first order linear alpha-dynamic equation

\[
y^{(\alpha)} = q(t)y
\]

with

\[
q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).
\]

Now, if \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \) are the row vectors of \( X(t) \), then we have

\[
(det \ X(t))^{(\alpha)} = \begin{vmatrix}
\vec{x}_1^{(\alpha)} & \vec{x}_2^{(\alpha)} & \vec{x}_3^{(\alpha)} & \cdots & \vec{x}_n^{(\alpha)} \\
\vec{x}_2 & \vec{x}_2^{(\alpha)} & \vec{x}_3 & \cdots & \vec{x}_n \\
\vec{x}_3 & \vec{x}_3 & \vec{x}_3^{(\alpha)} & \cdots & \vec{x}_{n-1}^{(\alpha)} \\
& & & & \\
\vec{x}_n & \vec{x}_n & \vec{x}_n & \cdots & \vec{x}_n \\
\end{vmatrix}
\]

If \( B_j \) is the determinant of the matrix obtained from \( X(t) \) with \( \vec{x}_i^{(\alpha)} \) on the first \( j \) rows and \( \vec{x}_j^{(\alpha)} \) on the \( j + 1 \)th row, then

\[
(\det X(t))^{(\alpha)} = B_0 + B_1 + \ldots + B_{n-1}.
\]

Note that

\[
B_j = \begin{vmatrix}
\vec{x}_1 & \vec{x}_1 + \mu_\alpha(t)\vec{x}_1^{(\alpha)} \\
\vec{x}_2 & \vec{x}_2 + \mu_\alpha(t)\vec{x}_2^{(\alpha)} \\
& & \cdots \\
\vec{x}_j & \vec{x}_j + \mu_\alpha(t)\vec{x}_j^{(\alpha)} \\
\vec{x}_j^{(\alpha)} & \vec{x}_j^{(\alpha)} \\
& & \cdots \\
\vec{x}_n & \vec{x}_n \\
\end{vmatrix}
\]

In calculating this determinant, we obtain the sum of determinants of all possible combinations of \( \vec{x}_i \) and \( \mu_\alpha(t)\vec{x}_i^{(\alpha)} \) for rows \( i = 1, 2, \ldots, j \). Thus,

\[
B_j = D_{1,j+1}(X) + \mu_\alpha(t)D_{2,j+1}(X) + \mu_\alpha^2(t)D_{3,j+1}(X) + \ldots + \mu_\alpha^{j-1}(t)D_{j+1,j+1}(X).
\]

Hence, by (2.1) and (2.2), we have

\[
(\det X(t))^{(\alpha)} = D_1(X) + \mu_\alpha(t)D_2(X) + \mu_\alpha^2(t)D_3(X) + \ldots + \mu_\alpha^{n-1}(t)D_n(X).
\]

Using Lemma 2.2 and summing the determinants of the \( \binom{n}{m} \) different \( X(\lambda_m, \alpha) \), we have

\[
D_m(X) = E_m(A) \det X(t) \quad \text{for} \quad m \in \{1, 2, \ldots, n\}.
\]
Hence, by (2.3), we have

\[(\det X(t))^{(\alpha)} = (E_1(A) + \mu \alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)) \det X(t).\]

Thus for \(u(t) = \det X(t)\), we have

\[u^{(\alpha)} = q(t)u\]

where

\[q(t) = E_1(A) + \mu \alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).\]

The proof is complete. □

**Corollary 2.4.** Assume the initial value problem

\[(2.4) \quad y^{(\alpha)} = q(t)y, \quad y(t_0) = 1\]

has a unique solution, where

\[q(t) = \lambda_1 \oplus \alpha \lambda_2 \oplus \alpha \ldots \oplus \alpha \lambda_n\]

and \(\{\lambda_i : 1 \leq i \leq n\}\) are the eigenvalues of \(A(t)\). Suppose \(X(t)\) is a solution of the matrix \(\alpha\)-dynamic equation

\[X^{(\alpha)} = A(t)X\]

where \(A\) is \(\alpha\)-regressive. Then, \(X\) satisfies Liouville’s formula.

\[\det X(t) = e_q(t,t_0) \det X(t_0)\]

where

\[q(t) = \lambda_1 \oplus \alpha \lambda_2 \oplus \alpha \ldots \oplus \alpha \lambda_n.\]

**Proof.** First, a simple induction argument shows that

\[\lambda_1 \oplus \alpha \lambda_2 \oplus \alpha \ldots \oplus \alpha \lambda_n = S_1(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mu \alpha(t)S_2(\lambda_1, \lambda_2, \ldots, \lambda_n) + \ldots + \mu_\alpha^{n-1}(t)S_n(\lambda_1, \lambda_2, \ldots, \lambda_n)\]

where \(S_k\) is the elementary symmetric function, as defined in Horn and Johnson [5]. Also, from [5], we have

\[E_k(A) = S_k(\lambda_1, \lambda_2, \ldots, \lambda_n).\]

Hence,

\[E_1(A) + \mu \alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A) = \lambda_1 \oplus \alpha \lambda_2 \oplus \alpha \ldots \oplus \alpha \lambda_n,\]

and thus using the above theorem, Liouville’s formula holds. □
Remark 2.5. When \( \mu_\alpha(t) \equiv 0 \), then
\[
\lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n = \text{tr}(A(t)).
\]
This formula agrees with Liouville’s formula for differential equations, which states if \( X' = A(t)X \), then
\[
det X(t) = \exp \left\{ \int_{t_0}^{t} \text{tr}(A(\tau))d\tau \right\} \det X(t_0).
\]

Remark 2.6. If \( \alpha = \sigma \), \( T \) is a closed set, and \( f \) is rd-continuous and regressive, then the IVP (4) has a unique solution. Also, if \( \alpha = \rho \), \( T \) is a closed set, and \( f \) is ld-continuous and regressive, then the IVP (4) has a unique solution. There are many other examples in which (4) has a unique solution.

Example 2.7. For a closed generalized time scale \( T \) with \( \alpha(t) = \sigma(t) \), we have \( \mu_\alpha(t) = \mu(t) \) and the alpha derivative is the Hilger delta derivative [4]. Here, we require \( I + \mu(t)A(t) \) to be invertible and \( A(t) \) to be rd-continuous. Now, we see
\[
E_1(A) + \mu(t)E_2(A) + \mu^2(t)E_3(A) + \ldots + \mu^{n-1}(t)E_n(A)
\]
is also rd-continuous and from Lemma 2.1 is regressive. Hence, for \( X(t) \), a solution of \( X^\Delta = A(t)X \), we have from Corollary 2.4 that
\[
det X(t) = e_q(t, t_0) \det X(t_0)
\]
where
\[
q(t) = \lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_n.
\]

Example 2.8. For a closed generalized time scale \( T \) with \( \alpha(t) = \rho(t) \), we have \( \mu_\alpha(t) = -\nu(t) \) and the alpha derivative is the Atıcı–Guseinov nabla derivative [2]. Here, we require \( I - \nu(t)A(t) \) to be invertible and \( A(t) \) to be ld-continuous. Hence, we have
\[
E_1(A) - \nu(t)E_2(A) + \nu^2(t)E_3(A) + \ldots + (-1)^{n-1}\nu^{n-1}(t)E_n(A)
\]
is ld-continuous and from Lemma 2.1 is regressive. Thus, for \( X(t) \), as solution of \( X^\nabla = A(t)X \), we have from Corollary 2.4 that
\[
det X(t) = e_q(t, t_0) \det X(t)
\]
where
\[
q(t) = \lambda_1 \oplus_\nu \lambda_2 \oplus_\nu \ldots \oplus_\nu \lambda_n.
\]
Example 2.9. Fix a point $t_0 \in \mathbb{T}$, where $\mathbb{T}$ is a generalized time scale. Let

$$D := \{\alpha^n(t_0)|n \in \mathbb{N}_0\},$$

where

$$\alpha^0(t_0) = t_0 \quad \text{and} \quad \alpha^n = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha, \quad \text{n times}.$$ 

Assume that all points in $D$ are isolated and $\alpha(t) \neq t$ for all $t \in D$. Then, for $q$, an $\alpha$-regressive scalar function, the IVP

$$y^{(\alpha)} = q(t)y, \quad y(t_0) = 1$$

can be written

$$\frac{y^{\alpha}(t) - y(t)}{\mu_\alpha(t)} = q(t)y(t), \quad y(t_0) = 1.$$

Hence,

$$y^{\alpha}(t) = (1 + q(t)\mu_\alpha(t))y(t), \quad \text{for } t \in D.$$ 

Iterating, the above expression with $y(t_0) = 1$, we have

$$e_q(\alpha^n(t_0), t_0) = \prod_{s=0}^{n-1} [1 + q(\alpha^s(t_0))\mu_\alpha(\alpha^s(t_0))].$$

Note the this exponential is only defined on $D$. Thus, for $X(t)$, a solution of

$$X^{(\alpha)} = A(t)X,$$

where $A(t)$ is an $\alpha$-regressive matrix-valued function, we have

$$\det X(\alpha^n(t_0)) = \prod_{s=0}^{n-1} [1 + q(\alpha^s(t_0))\mu_\alpha(\alpha^s(t_0))] \det X(t_0) \quad \text{for } n \in \mathbb{N} \cup \{0\},$$

where

$$q(t) = \lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n.$$

REFERENCES


