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LIOUVILLE'S FORMULA ON TIME SCALES

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ABSTRACT. *Alpha derivatives* are studied on generalized time scales $\mathbb{T}$. We present a Liouville formula for an nth order linear vector alpha-dynamic equation on a generalized time scale. A criterion is given for a matrix function to be α-regressive. As special cases, we get Liouville's formula for the delta dynamic system and for the nabla dynamic system, and other examples are presented.

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1. INTRODUCTION

In this paper, we will assume that the reader is familiar with the common literature on dynamic equations on time scales (see, for example, Bohner and Peterson [3]). We first define generalized time scales and the α-derivative as in Ahlbrandt, Bohner, and Ridenhour [1].

Definition 1.1. A *generalized time scale* $(\mathbb{T}, \alpha)$ is a nonempty set $\mathbb{T} \subseteq \mathbb{R}$ such that every Cauchy sequence in $\mathbb{T}$ converges to a point in $\mathbb{T}$, except possibly Cauchy sequences which converge to a finite infimum or supremum of $\mathbb{T}$, and α is a function mapping $\mathbb{T}$ into $\mathbb{T}$.

Definition 1.2. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is *alpha differentiable* at a point $t \in \mathbb{T}$ provided there is a unique number $f^{(\alpha)}(t)$, the *alpha derivative* of f at t, with the property that for each $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t such that

$$|f^\alpha(t) - f(s) - f^{(\alpha)}(t)(\alpha(t) - s)| \leq \epsilon|\alpha(t) - s|$$

for all $s \in \mathcal{U}$, where $f^\alpha = f \circ \alpha$.

Note that if $\alpha(t) = t$ and t is isolated, then for any function f, we have f is not α-differentiable at t, as discussed in Bohner and Peterson [3]. When $\alpha = \sigma$ and $\mathbb{T}$ is closed, we have the Hilger delta derivative [4]. For $\alpha = \rho$ and a closed set $\mathbb{T}$, we have the Atıcı–Guseinov nabla derivative, which was introduced in Section 8.4 of Atıcı and Guseinov [2].

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Definition 1.3. A scalar function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is *α-regressive* provided

$$1 + p(t)\mu_\alpha(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T},$$

where $\mu_\alpha(t) := \alpha(t) - t$ is the *generalized graininess.*

Definition 1.4. For two α-regressive functions p and q, we define circle-plus addition via

$$(p \oplus_\alpha q)(t) = p(t) + q(t) + \mu_\alpha(t)p(t)q(t).$$

Definition 1.5. A *first order linear alpha dynamic equation* is of the form

$$y^{(\alpha)} = p(t)y, \quad \text{where} \quad p \quad \text{is} \quad \alpha\text{-regressive}.$$

Definition 1.6. If the initial value problem

$$y^{(\alpha)} = p(t)y, \quad y(t_0) = 1$$

has a unique solution, we call the unique solution the *generalized exponential function* and denote it by $e_p(t, t_0)$.

Note that this exponential function depends on both $\mathbb{T}$ and α. We now introduce notation which is similar to notation used in Horn and Johnson [5]. Let $\lambda_k \subseteq \{1, 2, \ldots, n\}$ be an indexed set with k elements. For an $n \times n$ matrix-valued function A, a *principal submatrix* of A, denoted $A(\lambda_k)$ is the submatrix that lies in the rows and columns of $A(t)$ indexed by λ_k. Note that $A(\lambda_k)$ is $k \times k$, and there are $\binom{n}{k}$ different $k \times k$ principal submatrices of A. The determinant of a principal submatrix is called a *principal minor* of $A(t)$. The sum of the $\binom{n}{k}$ different $k \times k$ principal minors of $A(t)$ is denoted $E_k(A(t))$. We will usually suppress the t and just write $E_k(A)$. As shown in Horn and Johnson [5], the characteristic polynomial for $A(t)$,

$$p_A(x) = \det(xI - A)$$

can be written in the form

$$p_A(x) = x^n - x^{n-1}E_1(A) + x^{n-2}E_2(A) + \ldots + (-1)^n E_n(A).$$

Definition 1.7. We say A is *α-regressive* provided $I + \mu_\alpha(t)A(t)$ is invertible for $t \in \mathbb{T}$.

For an $n \times n$ matrix-valued function X, $X(\lambda_k, \alpha)$ is the $n \times n$ matrix obtained from $X(t)$ with alpha derivatives on the rows indexed by λ_k and the usual entries of $X(t)$ on the remaining rows. Let $X(\lambda_k, \alpha, j)$ for $k \leq j \leq n$ denote that $j \in \lambda_k$ and j is the largest number in λ_k. Note that there are $\binom{n}{k}$ different $X(\lambda_k, \alpha)$ and $\binom{j-1}{k-1}$ different $X(\lambda_k, \alpha, j)$. Also, let $D_k(X)$ denote the sum of the determinants of the $X(\lambda_k, \alpha)$ and $D_{k,j}(X)$ denote the sum of the determinants of the $X(\lambda_k, \alpha, j)$.

2. MAIN RESULTS

Lemma 2.1. *Let A be an $n \times n$ matrix-valued function. A is α-regressive if and only if the scalar function q defined by*

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)$$

is α-regressive.

Proof. If $\mu_\alpha(t) \neq 0$, we have

$$\begin{aligned}\det[I + \mu_\alpha(t)A(t)] &= (-\mu_\alpha(t))^n \det[\frac{-1}{\mu_\alpha(t)}I - A(t)] \\ &= (-\mu_\alpha(t))^n p_A\left(\frac{-1}{\mu_\alpha(t)}\right) \\ &= (-\mu_\alpha(t))^n \left[\left(\frac{-1}{\mu_\alpha(t)}\right)^n - \left(\frac{-1}{\mu_\alpha(t)}\right)^{n-1} E_1(A) + \ldots + (-1)^n E_n(A)\right] \\ &= 1 + \mu_\alpha(t)\big[E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)\big].\end{aligned}$$

Thus,

$$\det[I + \mu_\alpha(t)A(t)] = 1 + \mu_\alpha(t)q(t)$$

where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

Note that this formula also holds for those t where $\mu_\alpha(t) = 0$. Hence,

$$\det[I + \mu_\alpha(t)A(t)] \neq 0$$

if and only if

$$1 + \mu_\alpha(t)q(t) \neq 0.$$

Thus, $A(t)$ is an α-regressive matrix if and only if $q(t)$ is α-regressive where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

The proof is complete. □

Lemma 2.2. *Assume A is an $n \times n$ matrix-valued function which is α-regressive and $X(t)$ is a solution of the $n \times n$ matrix α-dynamic equation $X^{(\alpha)} = A(t)X$ in a generalized time scale $\mathbb{T}$. Then, for an indexed set $\lambda_m \subseteq \{1, 2, \ldots, n\}$, we have*

$$\det X(\lambda_m, \alpha) = \det A(\lambda_m) \det X(t).$$

Proof. Suppose $\lambda_m = \{i_1, i_2, i_3, \ldots, i_m\}$. Then, the (i_k, j)-component of $X(\lambda_m, \alpha)$ is

$$x_{i_k,j}^{(\alpha)} = \sum_{p=1}^{n} a_{i_k p} x_{pj}.$$

The (i,j)-component of $X(\lambda_m,\alpha)$ where $i \notin \lambda_m$ is x_{ij}. Since we can add $-a_{i_k p}$ times the p^{th} row of $X(\lambda_m,\alpha)$ for $p \notin \lambda_m$ to the i_k^{th} row without changing the value of the determinant, we have the

$$(i_k, j) - \text{component of} \quad X(\lambda_m,\alpha) \quad \text{is} \quad \sum_{s=1}^{m} a_{i_k i_s} x_{i_s j}.$$

Do this for each i_j row, $1 \le j \le m$. Now, let C be the block diagonal matrix

$$C = \begin{pmatrix} \mathrm{I}_{i_1-1} & 0 & 0 \\ 0 & \mathrm{W} & 0 \\ 0 & 0 & \mathrm{I}_{n-i_m} \end{pmatrix}$$

where I_j is the $j \times j$ identity matrix and

$$W = \begin{pmatrix} a_{i_1 i_1} & 0\dots0 & a_{i_1 i_2} & 0\dots0 & \dots & 0\dots0 & a_{i_1 i_m} \\ 0 & & 0 & & & & 0 \\ \dots & \mathrm{I}_{i_2-i_1-1} & \dots & & & & \dots \\ 0 & & 0 & & & & 0 \\ a_{i_2 i_1} & 0\dots0 & a_{i_2 i_2} & 0\dots0 & \dots & 0\dots0 & a_{i_2 i_m} \\ 0 & & 0 & & & & 0 \\ \dots & & \dots & \mathrm{I}_{i_3-i_2-1} & & & \dots \\ 0 & & 0 & & & & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ 0 & & 0 & & & & 0 \\ \dots & & \dots & & & \mathrm{I}_{i_m-i_{m-1}-1} & \dots \\ 0 & & 0 & & & & 0 \\ a_{i_m i_1} & 0\dots0 & a_{i_m i_2} & 0\dots0 & \dots & 0\dots0 & a_{i_m i_m} \end{pmatrix}.$$

Then, we have

$$\det X(\lambda_m,\alpha) = \det C \det X(t) = \det W \det X(t).$$

We can calculate the determinant of W by expanding about the rows with 1's along the diagonal first. Thus, we see

$$\det X(\lambda_m,\alpha) = \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_m} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i_m i_1} & a_{i_m i_2} & \dots & a_{i_m i_m} \end{vmatrix} \det X(t).$$

Therefore, $\det X(\lambda_m,\alpha) = \det A(\lambda_m) \det X(t)$. □

Theorem 2.3. *Assume A is an $n \times n$ matrix-valued function which is α-regressive and $X(t)$ is a solution of the $n \times n$ matrix α-dynamic equation $X^{(\alpha)} = A(t)X$ in* *a generalized time scale* $\mathbb{T}$. *Then,* $u(t) := \det X(t)$ *satisfies the scalar* α*-dynamic equation*

$$u^{(\alpha)} = q(t)u$$

where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

Proof. By Lemma 2.1, since $A(t)$ is α-regressive, we have $q(t)$ is α-regressive. Hence, we can consider the first order linear alpha-dynamic equation

$$y^{(\alpha)} = q(t)y$$

with

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

Now, if $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are the row vectors of $X(t)$, then we have

$$(\det X(t))^{(\alpha)} = \begin{vmatrix} \vec{x}_1^{(\alpha)} \\ \vec{x}_2 \\ \vec{x}_3 \\ \cdots \\ \vec{x}_n \end{vmatrix} + \begin{vmatrix} \vec{x}_1^{\alpha} \\ \vec{x}_2^{(\alpha)} \\ \vec{x}_3 \\ \cdots \\ \vec{x}_n \end{vmatrix} + \begin{vmatrix} \vec{x}_1^{\alpha} \\ \vec{x}_2^{\alpha} \\ \vec{x}_3^{(\alpha)} \\ \cdots \\ \vec{x}_n \end{vmatrix} + \ldots + \begin{vmatrix} \vec{x}_1^{\alpha} \\ \vec{x}_2^{\alpha} \\ \cdots \\ \vec{x}_{n-1}^{\alpha} \\ \vec{x}_n^{(\alpha)} \end{vmatrix}.$$

If B_j is the determinant of the matrix obtained from $X(t)$ with $\vec{x}_i^{\alpha}$ on the first j rows and $\vec{x}_{j+1}^{(\alpha)}$ on the $j+1^{st}$ row, then

$$(\det X(t))^{(\alpha)} = B_0 + B_1 + \ldots + B_{n-1}. \tag{2.1}$$

Note that

$$B_j = \begin{vmatrix} \vec{x}_1^{\alpha} \\ \vec{x}_2^{\alpha} \\ \cdots \\ \vec{x}_j^{\alpha} \\ \vec{x}_{j+1}^{(\alpha)} \\ \cdots \\ \vec{x}_n \end{vmatrix} = \begin{vmatrix} \vec{x}_1 + \mu_\alpha(t)\vec{x}_1^{(\alpha)} \\ \vec{x}_2 + \mu_\alpha(t)\vec{x}_2^{(\alpha)} \\ \cdots \\ \vec{x}_j + \mu_\alpha(t)\vec{x}_j^{(\alpha)} \\ \vec{x}_{j+1}^{(\alpha)} \\ \cdots \\ \vec{x}_n \end{vmatrix}.$$

In calculating this determinant, we obtain the sum of determinants of all possible combinations of $\vec{x}_i$ and $\mu_\alpha(t)\vec{x}_i^{(\alpha)}$ for rows $i = 1, 2, \ldots, j$. Thus,

$$B_j = D_{1,j+1}(X) + \mu_\alpha(t)D_{2,j+1}(X) + \mu_\alpha^2(t)D_{3,j+1}(X) + \ldots + \mu_\alpha^j(t)D_{j+1,j+1}(X). \tag{2.2}$$

Hence, by (2.1) and (2.2), we have

$$(\det X(t))^{(\alpha)} = D_1(X) + \mu_\alpha(t)D_2(X) + \mu_\alpha^2(t)D_3(X) + \ldots + \mu_\alpha^{n-1}(t)D_n(X). \tag{2.3}$$

Using Lemma 2.2 and summing the determinants of the $\binom{n}{m}$ different $X(\lambda_m, \alpha)$, we have

$$D_m(X) = E_m(A) \det X(t) \quad \text{for} \quad m \in \{1, 2, \ldots, n\}.$$

Hence, by (2.3), we have

$$(\det X(t))^{(\alpha)} = (E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)) \det X(t).$$

Thus for $u(t) = \det X(t)$, we have

$$u^{(\alpha)} = q(t)u$$

where

$$q(t) = E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A).$$

The proof is complete. □

Corollary 2.4. *Assume the initial value problem*

$$y^{(\alpha)} = q(t)y, \quad y(t_0) = 1 \tag{2.4}$$

has a unique solution, where

$$q(t) = \lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n$$

and $\{\lambda_i : 1 \leq i \leq n\}$ *are the eigenvalues of* $A(t)$*. Suppose* $X(t)$ *is a solution of the matrix* α*-dynamic equation*

$$X^{(\alpha)} = A(t)X$$

where A *is* α*-regressive. Then,* X *satisfies Liouville's formula.*

$$\det X(t) = e_q(t, t_0) \det X(t_0)$$

where

$$q(t) = \lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n.$$

Proof. First, a simple induction argument shows that

$$\lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n = S_1(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mu_\alpha(t)S_2(\lambda_1, \lambda_2, \ldots, \lambda_n)$$
$$+ \ldots + \mu_\alpha^{n-1}(t)S_n(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where S_k is the elementary symmetric function, as defined in Horn and Johnson [5]. Also, from [5], we have

$$E_k(A) = S_k(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Hence,

$$E_1(A) + \mu_\alpha(t)E_2(A) + \mu_\alpha^2(t)E_3(A) + \ldots + \mu_\alpha^{n-1}(t)E_n(A)$$
$$= \lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n,$$

and thus using the above theorem, Liouville's formula holds. □

Remark 2.5. When $\mu_\alpha(t) \equiv 0$, then

$$\lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n = \operatorname{tr}(A(t)).$$

This formula agrees with Liouville's formula for differential equations, which states if $X' = A(t)X$, then

$$\det X(t) = \exp\left\{\int_{t_0}^{t} \operatorname{tr}(A(\tau))d\tau\right\} \det X(t_0).$$

Remark 2.6. If $\alpha = \sigma$, $\mathbb{T}$ is a closed set, and f is rd-continuous and regressive, then the IVP (4) has a unique solution. Also, if $\alpha = \rho$, $\mathbb{T}$ is a closed set, and f is ld-continuous and regressive, then the IVP (4) has a unique solution. There are many other examples in which (4) has a unique solution.

Example 2.7. For a closed generalized time scale $\mathbb{T}$ with $\alpha(t) = \sigma(t)$, we have $\mu_\alpha(t) = \mu(t)$ and the alpha derivative is the Hilger delta derivative [4]. Here, we require $I + \mu(t)A(t)$ to be invertible and $A(t)$ to be rd-continuous. Now, we see

$$E_1(A) + \mu(t)E_2(A) + \mu^2(t)E_3(A) + \ldots + \mu^{n-1}(t)E_n(A)$$

is also rd-continuous and from Lemma 2.1 is regressive. Hence, for $X(t)$, a solution of

$$X^\Delta = A(t)X,$$

we have from Corollary 2.4 that

$$\det X(t) = e_q(t, t_0) \det X(t_0)$$

where

$$q(t) = \lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_n.$$

Example 2.8. For a closed generalized time scale $\mathbb{T}$ with $\alpha(t) = \rho(t)$, we have $\mu_\alpha(t) = -\nu(t)$ and the alpha derivative is the Atıcı–Guseinov nabla derivative [2]. Here, we require $I - \nu(t)A(t)$ to be invertible and $A(t)$ to be ld-continuous. Hence, we have

$$E_1(A) - \nu(t)E_2(A) + \nu^2(t)E_3(A) + \ldots + (-1)^{n-1}\nu^{n-1}(t)E_n(A)$$

is ld-continuous and from Lemma 2.1 is regressive. Thus, for $X(t)$, as solution of

$$X^\nabla = A(t)X,$$

we have from Corollary 2.4 that

$$\det X(t) = e_q(t, t_0) \det X(t)$$

where

$$q(t) = \lambda_1 \oplus_\nu \lambda_2 \oplus_\nu \ldots \oplus_\nu \lambda_n.$$

Example 2.9. Fix a point $t_0 \in \mathbb{T}$, where $\mathbb{T}$ is a generalized time scale. Let

$$D := \{\alpha^n(t_0) | n \in \mathbb{N}_0\},$$

where

$$\alpha^0(t_0) = t_0 \quad \text{and} \quad \alpha^n = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha, \quad \text{n times.}$$

Assume that all points in D are isolated and $\alpha(t) \neq t$ for all $t \in D$. Then, for q, an α-regressive scalar function, the IVP

$$y^{(\alpha)} = q(t)y, \quad y(t_0) = 1$$

can be written

$$\frac{y^\alpha(t) - y(t)}{\mu_\alpha(t)} = q(t)y(t), \quad y(t_0) = 1.$$

Hence,

$$y^\alpha(t) = (1 + q(t)\mu_\alpha(t))y(t), \quad \text{for } t \in D.$$

Iterating, the above expression with $y(t_0) = 1$, we have

$$e_q(\alpha^n(t_0), t_0) = \prod_{s=0}^{n-1} [1 + q(\alpha^s(t_0))\mu_\alpha(\alpha^s(t_0))].$$

Note the this exponential is only defined on D. Thus, for $X(t)$, a solution of

$$X^{(\alpha)} = A(t)X,$$

where A(t) is an α-regressive matrix-valued function, we have

$$\det X(\alpha^n(t_0)) = \prod_{s=0}^{n-1} [1 + q(\alpha^s(t_0))\mu_\alpha(\alpha^s(t_0))] \det X(t_0) \quad \text{for } n \in \mathbb{N} \cup \{0\},$$

where

$$q(t) = \lambda_1 \oplus_\alpha \lambda_2 \oplus_\alpha \ldots \oplus_\alpha \lambda_n.$$

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