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ASYMPTOTIC BEHAVIOR OF NATURAL GROWTH ON TIME SCALES

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ABSTRACT. The asymptotic behavior of $x^\Delta = px$ is explored, with specific reference given to how the graininess of the time scale affects stability. In addition we prove a Perron type theorem for dynamic equations on time scales. The theorem gives sufficient conditions for exponential asymptotic stability of a critical point of an almost linear dynamic equation. Application to the dynamic logistic equation is given.

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1. INTRODUCTION

The fundamental solution of the most basic of differential equations

$$x' = px \tag{1.1}$$

is the exponential function, and even pre-calculus students know that it decays to zero asymptotically if $p < 0$. On the other hand, the corresponding solution $x(n) = (1+p)^n$ of the difference equation

$$x(n+1) - x(n) = px(n) \tag{1.2}$$

decays to zero only if $-2 < p < 0$. The exponential decay rate of the former is of course p, but it is $\log|1+p|$ for $p \neq -1$ for the latter. This paper explores the asymptotic behavior of the exponential function e_p on a general unbounded time scale $\mathbb{T}$, an arbitrary unbounded from above closed subset of the set of real numbers $\mathbb{R}$. Equations (1.1) and (1.2) are special cases of the dynamic equation

$$x^\Delta = px \tag{1.3}$$

on $\mathbb{T}$ whose fundamental solution is the time scale exponential function e_p. The time scale has a definite effect on which constants p yield decay to zero for e_p or, equivalently, asymptotic stability of (the zero solution of) equation (1.3). Pötzsche, Siegmund and Wirth in [9] have given the following characterization of stability.

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Theorem 1.1. *Let $p \in \mathbb{C}$. The scalar system*

$$x^{\Delta} = px, \quad x \in \mathbb{C}$$

is exponentially stable if and only if one of the following conditions is satisfied for arbitrary $t_0 \in \mathbb{T}$

(i) $\gamma(p) := \limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s\searrow\mu(t)} \frac{\log|1+sp|}{s} \Delta t < 0$,

(ii) *for all $T \in \mathbb{T}$, there exists $t \in \mathbb{T}$ with $t > T$ such that $1 + \mu(t)p = 0$,*

where we use the convention $\log 0 = -\infty$ *in (i).*

Although they provide some simplification results for the calculation of $\gamma(p)$, in general it is still quite difficult to apply. Thus in this paper we give a simpler estimate for the exponential rate of growth or decay of e_p when $p \in \mathbb{R}$, and a corresponding condition for exponential asymptotic stability of equation (1.3) which does not depend on $\gamma(p)$. As a consequence we also give a 1-dimensional Perron theorem for almost linear dynamic equations

$$x^{\Delta} = px + f(t, x). \tag{1.4}$$

This result gives conditions for local asymptotic stability of the zero critical point of equation (1.4) in terms of the linear approximation (1.3) of the equation at the critical point. The conditions explicitly include the graininess of the time scale. Finally, as an example, we apply this result to the dynamic logistic equation.

We begin with a preliminary section whose main objective is to define the time scales exponential function e_p and identify it as the fundamental solution of the scalar linear homogeneous dynamic equation (1.3). This section summarizes material in the recent book by Bohner and Peterson [1] and we include it here for completeness.

2. PRELIMINARIES

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$. For our purposes, we assume throughout that $\mathbb{T}$ is unbounded above.

Definition 2.1. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

A point $t \in \mathbb{T}$ is called right dense if $\sigma(t) = t$, and right scattered if $\sigma(t) > t$. It is convenient to have a graininess operator $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$.

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t)$ is the standard forward difference operator $f(t+1) - f(t)$. One can also define integration on an appropriate class of functions.

Definition 2.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$, then F is said to be an antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \quad \text{for} \quad a, b \in \mathbb{T}.$$

Theorem 2.4 (Chain Rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = g^\Delta(t) \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))\, dh$$

holds.

Definition 2.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called right dense continuous provided it is continuous at all right dense points of $\mathbb{T}$ and its left sided limit exists (finite) at left dense points of $\mathbb{T}$. The set of all right dense continuous functions on $\mathbb{T}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Remark 2.6. All right dense continuous functions are integrable.

Theorem 2.7. *Let $f \in C_{rd}$ with $a, b \in \mathbb{T}$. If $f(t) \geq 0$ for all $a \leq t < b$, then*

$$\int_a^b f(t)\,\Delta t \geq 0.$$

Definition 2.8. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)f(t) \neq 0$$

for all $t \in \mathbb{T}$. The set of all regressive and right dense continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.9. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} :\ 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

We next give the definition of the exponential function on a time scale. Then we list some useful properties of this exponential function.

Definition 2.10. For $h > 0$ we define the function

$$\xi_h(x) = \frac{1}{h}\mathrm{Log}(1 + xh),$$

for any real number x except $-\frac{1}{h}$, where Log is the principal logarithm function, i.e.,

$$\xi_h(x) = \frac{1}{h}\begin{cases} \log(1+xh) & \text{for } x > -\frac{1}{h} \\ \log|1+xh| + i\pi & \text{for } x < -\frac{1}{h} \end{cases}.$$

For $h = 0$, we define $\xi_0(x) = x$.

Definition 2.11. If $f : \mathbb{T} \to \mathbb{R}$ is regressive and right dense continuous, then we define the *exponential function* by

$$e_f(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}$.

Remark 2.12. Consider the dynamic initial value problem

$$x^\Delta = p(t)x, \quad x(t_0) = x_0$$

where $t_0 \in \mathbb{T}$ and $p \in \mathcal{R}$ is right dense continuous. Then the exponential function $x_0 e_p(t, t_0)$ is the unique solution to this initial value problem.

Theorem 2.13. *If $p, q \in \mathcal{R}$, then*

1. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$*;*
3. $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*
4. $e_p(t,s) = e_{\ominus p}(s,t)$ *where* $\ominus p = -\frac{p}{1+p\mu}$.

In Bohner and Peterson [1], the variation of parameters formula is proved for the dynamic equation

$$x^\Delta = p(t)x + f(t), \quad x(t_0) = x_0.$$

Similar to that, one may prove the following result.

Theorem 2.14 (Variation of Parameters)**.** *Suppose $p \in \mathcal{R}$, and assume $f(t,x)$ is a real-valued continuous function for $(t,x) \in \mathbb{T} \times \mathbb{R}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. If the initial value problem*

$$x^\Delta = p(t)x + f(t,x), \quad x(t_0) = x_0.$$

has a unique solution $x(t)$ defined for all t in the $\mathbb{T}$-interval I which contains t_0 in its interior, then $x(t)$ is given by

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau, x(\tau))\,\Delta\tau$$

for all $t \in I$.

Theorem 2.15. *Let $x, f \in C_{rd}$ and $p \in \mathcal{R}^+$. Then*

$$x^{\Delta}(t) \le p(t)x(t) + f(t) \text{ for all } t \in \mathbb{T}$$

implies

$$x(t) \le x(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\,\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Theorem 2.16 (Gronwall Inequality)**.** *Let $x, f \in C_{rd}$ and $p \in \mathcal{R}^+$, $p \ge 0$. Then*

$$x(t) \le f(t) + \int_{t_0}^{t} x(\tau)p(\tau)\,\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$x(t) \le f(t) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)p(\tau)\,\Delta\tau \text{ for all } t \in \mathbb{T}.$$

It is clear from the proofs of the last two results in Bohner and Peterson [1] that in each case, reversing the inequalities in the assumptions yields corresponding lower (instead of upper) estimates for the solution.

3. ESTIMATES FOR THE TIME SCALE EXPONENTIAL

In this section we give a simple estimate for the time scales exponential function $e_p(t,t_0)$ for any nonzero regressive real constant p in terms of a corresponding real continuous exponential function $e^{\alpha(t-t_0)}$. Toward this end, we define, for any such p, the (possibly extended) real-valued function

$$\alpha = \alpha_p(\mu) := \frac{\log|1+p\mu|}{\mu} \quad \text{for} \quad \mu > 0 \tag{3.1}$$

and set $\alpha_p(0) = p$. For the two standard examples, the continuous case, $\mu = 0$, and the discrete case, $\mu = 1$, we have

$$e_p(t,t_0) = e^{p(t-t_0)} = e^{\alpha_p(0)(t-t_0)},$$

and

$$|e_p(t,t_0)| = |1+p|^{(t-t_0)} = e^{(t-t_0)\log|1+p|} = e^{\alpha_p(1)(t-t_0)}$$

respectively. These examples suggest that the function α_p is the (logarithmic) growth rate of the dynamic exponential function e_p.

Hilger's explicit representation [5] for the modulus of $e_p(t,t_0)$

$$\begin{aligned} |e_p(t,t_0)| &= \exp\left(\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{\log|1+sp|}{s}\,\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \alpha_p(s)\,\Delta\tau\right) \end{aligned} \tag{3.2}$$

gives the relationship between α_p and e_p for a general time scale $\mathbb{T}$. Indeed, in the regressive case, continuity of the function α_p for $\mu \neq -1/p$ gives the further simplification of (3.2),

$$|e_p(t,t_0)| = \exp \int_{t_0}^{t} \alpha_p(\mu(\tau))\,\Delta\tau. \tag{3.3}$$

So what is required is an estimation of the time scales integral

$$\int_{t_0}^{t} \alpha_p(\mu(\tau))\,\Delta\tau.$$

Theorem 1.1 makes use of the time scales mean

$$\frac{1}{u-t_0}\int_{t_0}^{u} \alpha_p(\mu(\tau))\,\Delta\tau$$

of the function $\alpha_p(\mu(\tau))$, whereas our comparable result in the next section makes us of the estimate below which instead involves directly bounds on α_p as a function of the graininess μ on the time scale. The former gives a sharper result, but it appears that our result may be easier to apply.

First we document the behavior of α as a function of the graininess μ. For p positive, α is a decreasing continuous function on $[0,\infty)$ with

$$\lim_{\mu\to\infty} \alpha(\mu) = 0.$$

In particular for $p \geq 0$,

$$q = \sup\{\alpha(\mu)\} = p.$$

For p negative, α decreases from p to $-\infty$ on the interval $[0,-1/p)$ and increases from $-\infty$ to 0 on $(-1/p,-2/p]$. For $\mu > -2/p$, $\alpha(\mu) > 0$ first increases to its maximum value at $\mu = \mu^* \approx -4.59/p$, and then decreases to 0 as $\mu \to \infty$. Indeed, if $p < 0$ and $\mu > -2/p$, then

$$|1+p\mu| = -(1+p\mu) > 1 \quad \text{and} \quad \alpha_p'(\mu) = \frac{\frac{p\mu}{1+p\mu} - \log(-(1+p\mu))}{\mu^2}.$$

The numerator, considered as a function of $\nu = -(1+p\mu)$,

$$n(\nu) = \frac{\nu+1}{\nu} - \log\nu, \quad \nu > 1$$

is decreasing with a unique zero at approximately $\nu = 3.59$.

We are interested in the function α_p restricted to the subset of μ's that occur in the time scale $\mathbb{T}$. It is clear from the discussion above that if S is any subset of $[0,\infty)$,

$$q_S = \sup\{\alpha(\mu) : \mu \in S\} < \infty.$$

Note that in case $p < 0$ and $S = \{-1/p\}$, $q_S = -\infty$. In all other cases, q_S is finite. In addition let

$$r_S = \inf\{\alpha(\mu) : \mu \in S\}.$$

Our estimation result then is the following.

Theorem 3.1. *Let $p \in \mathcal{R}$ and $t_0, t \in \mathbb{T}$. Define $\mathbb{T}_0 = \mathbb{T} \cap [t_0, \infty)$,*

$$T_0 = \mu(\mathbb{T}_0) = \{\mu : \mu = \mu(t) : t \in \mathbb{T}_0\}$$

and

$$r_0 = r_{T_0}, \quad q_0 = q_{T_0}.$$

Then

$$e^{r_0(t-t_0)} \leq |e_p(t, t_0)| \leq e^{q_0(t-t_0)},$$

for all $t \in \mathbb{T}_0$.

Proof. We have, by linearity and the estimation property for time scales integrals (Theorem 2.7) and from the fact that t is a time scales antiderivative of 1,

$$\int_{t_0}^{t} \alpha_p(\mu(\tau))\, \Delta\tau \geq \int_{t_0}^{t} r_0\, \Delta\tau = r_0(t - t_0), \tag{3.4}$$

and

$$\int_{t_0}^{t} \alpha_p(\mu(\tau))\, \Delta\tau \leq \int_{t_0}^{t} q_0 \triangle\tau = q_0(t - t_0). \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) gives the result. □

4. ASYMPTOTIC BEHAVIOR OF DYNAMIC NATURAL GROWTH

We will now consider the dynamic natural growth model

$$x^{\Delta} = px \tag{4.1}$$

where $p \in \mathcal{R}$ is a nonzero constant. As mentioned in the Section 2, for any $t_0 \in \mathbb{T}$, the time scales exponential function $e_p(., t_0)$ determines the fundamental solution of equation (4.1). Corresponding to any initial value

$$x(t_0) = x_0, \tag{4.2}$$

the unique solution of the dynamic initial value problem (4.1), (4.2) is given by

$$x = x(t) = x_0 e_p(t, t_0)$$

which is exactly analogous to the continuous $\mathbb{T} = \mathbb{R}$ case. In particular for $x_0 = 0$, we obtain the constant zero solution of equation (4.1). We obtain the following asymptotic behavior result directly from Theorem 3.1.

Corollary 4.1. *Let*

$$\underline{r}_T = \liminf_{t\to\infty} \alpha_p(\mu(t)) \quad \text{and} \quad \overline{q}_T = \limsup_{t\to\infty} \alpha_p(\mu(t)),$$

and suppose $p \in \mathcal{R}$. If $\underline{r}_T > 0$, then every solution $x(t)$ of equation (4.1) satisfies

$$\lim_{t\to\infty} |x(t)| = \infty. \tag{4.3}$$

If $\overline{q}_T < 0$, *then every solution* $x(t)$ *of equation* (4.1) *satisfies*

$$\lim_{t\to\infty} x(t) = 0. \tag{4.4}$$

Proof. We verify (4.4), with (4.3) following similarly. We exhibit a positive constant K and a negative constant q such that the solution $x(t)$ of the initial value problem (4.1), (4.2), satisfies

$$|x(t)| \le |x_0| K e^{q(t-t_0)} \tag{4.5}$$

for all $t > t_0$. Toward determining K and q, first pick $t_1 \in \mathbb{T}$, $t_1 \ge t_0$, so that $q_{T_1} < 0$ where

$$T_1 = \mu(\mathbb{T} \cap [t_1, \infty)).$$

This is possible since $\overline{q}_T < 0$. Take $q = q_{T_1}$. Direct application of Theorem 3.1 now gives

$$|e_p(t, t_1)| \le e^{q(t-t_1)}$$

for all $t > t_1$. Now we take

$$K = \max_{t_0 \le t \le t_1} |e_p(t, t_0)| e^{-q(t_1 - t_0)}.$$

Then we have, first of all, for $t_0 \le t \le t_1$,

$$|x(t)| = |x_0||e_p(t, t_0)| \le |x_0| K e^{q(t_1 - t_0)} \le |x_0| K e^{q(t-t_0)}.$$

For $t > t_1$, we have

$$\begin{aligned} |x(t)| &= |x_0||e_p(t, t_0)| \\ &= |x_0||e_p(t, t_1)||e_p(t_1, t_0)| \\ &\le |x_0||e_p(t, t_1)| K e^{q(t_1 - t_0)} \\ &\le |x_0| e^{q(t-t_1)} K e^{q(t-t_0)} = |x_0| K e^{q(t-t_0)}. \end{aligned}$$

Finally, note that if $q_T < 0$ holds, we obtain (4.5) with $q = q_T$ and $K = 1$. □

This implies that if the maximal growth rate is negative, the exponential function decays to zero. The continuity of the function $\alpha(\mu)$ allows equivalent stability statements to be formulated explicitly in terms of the asymptotic graininess of the time scale. Continuity of α_p permits the following version of Corollary 4.1 in terms of the graininess of the time scale.

Corollary 4.2. *Let*

$$\overline{\mu} = \limsup_{t\to\infty} \mu(t) \quad \text{and} \quad \underline{\mu} = \liminf_{t\to\infty} \mu(t),$$

and suppose $p \in \mathcal{R}$. *Then* (4.3) *holds if either*

$$p > 0 \tag{4.6}$$

or

$$p < 0 \quad \textit{and} \quad \underline{\mu} > -\frac{2}{p}. \tag{4.7}$$

If

$$\overline{\mu} < -\frac{2}{p}, \tag{4.8}$$

then (4.4) *holds.*

Example 4.3. For the time scale $\mathbb{R}$,

$$\underline{\mu} = \overline{\mu} = \mu = 0 \quad \text{and} \quad \underline{r}_T = \overline{q}_T = \alpha_p(0) = p.$$

We see that (4.7) cannot hold, and (4.8) holds if and only if $p < 0$. Here

$$e_p(t, t_0) = e^{p(t-t_0)}$$

and Corollary 4.2 reduces to exactly the corresponding well-known result for the usual real exponential function.

Example 4.4. We consider completely scattered discrete time scales $\mathbb{T} = \{t_j : j \in \mathbb{N}\}$, $t_{j+1} > t_j$, for each j, and $t_j \to \infty$ as $j \to \infty$. In this case

$$\begin{aligned} e_p(t, t_0) &= \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p)\Delta\tau\right) \\ &= \exp\left(\int_{t_0}^{t} \frac{1}{\mu(\tau)} \mathrm{Log}(1 + \mu(\tau)p)\Delta\tau\right) \\ &= \exp\left(\sum_{\substack{t_0 \le \tau < t \\ \tau \in \mathbb{T}}} \mathrm{Log}(1 + \mu(\tau)p)\right). \end{aligned}$$

but the asymptotic behavior is not immediately apparent. We look at three special cases. In the difference equation case, $\mathbb{T} = \mathbb{Z}$, we have

$$\underline{\mu} = \overline{\mu} = \mu = 1 \quad \text{and} \quad \underline{r}_T = \overline{q}_T = \alpha_p(1) = \log|1 + p|.$$

Conditions (4.6), (4.7) hold in this case if and only if either $p > 0$ or $p < -2$, and Corollary 4.2 confirms that the exponential $e_p(t, t_0)$ explodes as $t \to \infty$; (4.8) is equivalent to $|1 + p| < 1$, and the corollary gives that the exponential asymptotically decays to zero in this case.

If $\mathbb{T} = \{q^n : n \in \mathbb{N}\}$ where $q > 1$, then $\overline{\mu} = \infty$. The corollary gives explosion for any (nonzero) value of p, as it is easily seen that either (4.6) or (4.7) always holds in this case.

Now we consider a special case of the time scale in [9, Example 5]. Let

$$s_0 = 0, \quad s_{k+1} = s_k + 3^{k+1} + 1, \quad k \in \mathbb{N}_0$$

and take

$$\begin{aligned}\mathbb{T} &= \{0,3,4,\dots,s_k,s_k+3,\dots,s_k+3^{k+1},s_{k+1},\dots\} \\ &= \{0,3,4,7,10,13,14,17,20,23,26,29,32,35,38,41,42,45,48,\dots\}.\end{aligned}$$

Here we have $\underline{\mu} = 1$ and $\overline{\mu} = 3$. The condition for explosion is the same as in the simple difference equation case, but the corollary only asserts that the exponential will asymptotically decay to zero if

$$-\frac{2}{3} < p < 0.$$

Example 4.5. We consider a time scale which is a disjoint union of closed intervals

$$\mathbb{T} = \bigcup_{j=1}^{\infty}[s_j, t_j]$$

with $s_j < t_j < s_{j+1}$ and $s_j \to \infty$ as $j \to \infty$. In this case the time scales exponential $e_p(t, t_0)$ is a product of real exponentials corresponding to the intervals in $\mathbb{T}$ between t and t_0 and factors of the form $1 + p(s_{j+1} - t_j)$ corresponding to the gaps (t_j, s_{j+1}) in $\mathbb{T}$ between t and t_0. It is clear that $\underline{\mu} = 0$, and

$$\overline{\mu} = \limsup_{t\to\infty}(s_{j+1} - t_j).$$

So Corollary 4.2 predicts explosion if $p > 0$ and asymptotic decay to zero if

$$-\frac{2}{\overline{\mu}} < p < 0.$$

Example 4.6. A time scale composed of the union of a sequence of identical Cantor sets $\mathcal{C}$ tending to infinity

$$\mathbb{T} = \bigcup_{n=1}^{\infty}\{t = n + c, c \in \mathcal{C}\}$$

causes no problem in applying the corollary. Density of the set implies $\underline{\mu} = 0$ while the size of the largest gap gives that $\overline{\mu} = 1/3$. So once again Corollary 4.2 predicts explosion of the time scales exponential if $p > 0$, with asymptotic decay to zero following if $-6 < p < 0$.

5. ASYMPTOTIC STABILITY OF ALMOST LINEAR DYNAMIC EQUATIONS

The estimates on the dynamic exponential function given in the last section allow us to give a straightforward proof of the Lyapunov stability result for almost linear dynamic equations in the scalar case. Dynamic equations are referred to as almost linear if they can be written in the form

$$x^{\Delta} = px + f(t, x) \tag{5.1}$$

with

$$\lim_{x\to 0}\frac{|f(t,x)|}{|x|}=0 \tag{5.2}$$

uniformly for $t \in \mathbb{T}$. We assume also throughout this section that f is continuous and at least in a neighborhood of $x = 0$, solutions of initial value problems (5.1),

$$x(t_0) = x_0 \tag{5.3}$$

exist for all $t > t_0$ and are unique.

The stability and instability results given in this section are dynamic equations versions of the original classical results established by A. M. Lyapunov for ordinary differential equations. The proofs given here are along the lines of the ones given in Coddington and Levinson [2], which are there attributed to Perron [8]. The stability result can be extended readily to the n-dimensional case by invoking a result in [2], but a complete treatment of the corresponding instability result requires more detail, and so for now for clarity we just consider the scalar case. The results themselves here are not new, but are rather contained in the more general Hartman–Grobman result for measure chains given by Hilger [6].

We start by recalling the definitions of Lyapunov stability and asymptotic stability of a particular solution of a dynamic equation. These definitions are exactly the same as for differential equations and difference equations.

Definition 5.1. The zero solution of (5.1) is stable if for any $t_0 \in \mathbb{T}$ and $\delta > 0$, there exists an $\epsilon = \epsilon(t_0, \delta) > 0$ such that if $x(t)$ is a solution of equation (5.1) with $|x(t_0)| < \delta$, then $|x(t)| < \epsilon$ for all $t \in \mathbb{T}$, $t \geq t_0$. If the zero solution is not stable, it is said to be unstable. If the zero solution of equation (5.1) is stable, and if also there exists a $\delta_0 > 0$ such that any solution of equation (5.1) with $|x(t_0)| < \delta_0$ satisfies

$$\lim_{t\to\infty} x(t) = 0,$$

then the zero solution is called asymptotically stable. In the latter case, the zero solution is called exponentially asymptotically stable if there exists constants δ_1, q, and K such that for any solution $x(t)$ of equation (5.1) with $|x(t_0)| < \delta_1$,

$$|x(t)| \leq Ke^{-q(t-t_0)}$$

for all $t \in \mathbb{T}$, $t \geq t_0$. If K can be chosen independent of t_0, then the zero solution is said to be uniformly exponentially asymptotically stable.

Remark 5.2. The condition that p is regressive is necessary for the exponential function on time scales to be defined, but is unnecessary for the zero solution to be exponentially asymptotically stable. Suppose p is nonregressive, i.e., $1 + p\mu(t_1) = 0$

for some $t_1 \in \mathbb{T}$. Then t_1 must be right scattered, and using the fact that e_p satisfies $x^\Delta = px$, we have

$$x^\Delta(t_1) = \frac{x(\sigma(t_1)) - x(t_1)}{\mu(t_1)}.$$

Thus

$$\frac{x(\sigma(t_1)) - x(t_1)}{\mu(t_1)} = px(t_1), \quad \text{hence} \quad x(\sigma(t_1)) = (1 + p\mu(t_1))x(t_1) = 0,$$

and solutions reach zero in finite time.

Theorem 5.3. *Let $\mathbb{T}$ be a time scale which is unbounded above, and let N be a neighborhood of $x = 0$. Assume $f(t,x)$ is a real-valued continuous function for $(t,x) \in \mathbb{T} \times N$ which satisfies the condition* (5.2) *uniformly in t on $\mathbb{T}$. If $p \in \mathcal{R}$ and $\overline{q}_T < 0$ holds on $\mathbb{T}$, then the zero solution of the almost linear dynamic equation* (5.1) *is exponentially asymptotically stable. Furthermore, if $q_T < 0$, the zero solution of equation* (5.1) *is uniformly exponentially asymptotically stable.*

Proof. Let $x(t) = x(t; t_0, x_0)$ denote the solution of the dynamic initial value problem (5.1), (5.3). By Theorem 2.14,

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau, x(\tau))\,\Delta\tau,$$

or

$$\begin{aligned} e_{\ominus p}(t,t_0)x(t) &= x_0 + \int_{t_0}^{t} e_{\ominus p}(t,t_0)e_p(t,\sigma(\tau))f(\tau,x(\tau))\,\Delta\tau \\ &= x_0 + \int_{t_0}^{t} e_{\ominus p}(t,t_0)e_{\ominus p}(\sigma(\tau),t)f(\tau,x(\tau))\,\Delta\tau \\ &= x_0 + \int_{t_0}^{t} e_{\ominus p}(\sigma(\tau),t_0)f(\tau,x(\tau))\,\Delta\tau. \end{aligned}$$

Thus, from (5.2), given $\epsilon > 0$, there is a $\delta > 0$ such that if $|x_0| < \delta$,

$$\begin{aligned} |e_{\ominus p}(t,t_0)x(t)| &\le |x_0| + \int_{t_0}^{t} |e_{\ominus p}(\sigma(\tau),t_0)||f(\tau,x(\tau))|\,\Delta\tau \\ &\le |x_0| + \epsilon\int_{t_0}^{t} |e_{\ominus p}(\sigma(\tau),t_0)||x(\tau)|\,\Delta\tau \\ &= |x_0| + \epsilon\int_{t_0}^{t} |e_{\ominus p}(\sigma(\tau),t_0)x(\tau)|\,\Delta\tau \end{aligned}$$

provided

$$|x(\tau)| < \delta \tag{5.4}$$

for all $\tau \in \mathbb{T} \cap [t_0, t)$. Note that $\epsilon > 0$ implies $\epsilon \in \mathcal{R}^+$. So by Theorem 2.16, we have

$$|e_{\ominus p}(t,t_0)x(t)| \le |x_0|e_\epsilon(t,t_0) \le |x_0||e_\epsilon(t,t_0)| \tag{5.5}$$

if (5.4) holds. Now, since $\epsilon > 0$,

$$|e_\epsilon(t, t_0)| \le e^{\epsilon(t-t_0)}$$

from Theorem 3.1 and the preceding discussion of the properties of the function α. Thus, from (5.5) and Corollary 4.1, we have, again under condition (5.4),

$$|x(t)| \le |x_0||e_p(t, t_0)|e^{\epsilon(t-t_0)} \le |x_0|Ke^{q(t-t_0)}e^{\epsilon(t-t_0)} = K|x_0|e^{(q+\epsilon)(t-t_0)} \tag{5.6}$$

for a positive constants K and $q < 0$. We choose $\epsilon < -q$. So the proof will be complete if it is shown that there is a $\delta_1 > 0$ such that if

$$|x_0| < \delta_1$$

then condition (5.6) holds for all $\tau \in \mathbb{T} \cap [t_0, \infty)$. Now by continuity of the solution $x(t)$,

$$|x(t)| < \delta$$

for all t in some interval $[t_0, t_1)$ if $|x_0| = |x(t_0)| < \delta$. If $K > 1$, we now make the choice

$$|x_0| < \delta/K. \tag{5.7}$$

So by continuity again, (5.6) holds on at least the interval $[t_0, t_1]$. Let t_1 be the largest such point in $\mathbb{T}$, i.e., let

$$t_1 = \sup\{t > t_0 : |x(t)| < \delta\}.$$

If $t_1 < \infty$, $|x(t_1)| = \delta$. But, by (5.6) and (5.7),

$$\delta = |x(t_1)| \le K|x_0|e^{(q+\epsilon)(t_1-t_0)} < K|x_0| < K(\delta/K) = \delta.$$

This contradiction means that (5.6) holds for all $t > t_0$, $t \in \mathbb{T}$, and $K|x_0| < \delta$. The choice of ϵ shows that the above exponential rate is negative. Hence the zero solution is exponentially asymptotically stable. As in Corollary 4.1, if $q_T < 0$ holds we obtain (5.6) with $q = q_T$ and $K = 1$: the zero solution is uniformly exponentially stable. □

Example 5.4. The regressivity of p is necessary in Theorem 5.3. Let $\{b_k\}$ be a sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{b_k}{3^k} = \infty$$

and define the recursive sequence s_k by

$$s_0 = 0, \quad s_{k+1} = s_k + 3b_k + 1 \text{ for } k \in \mathbb{N}.$$

Consider

$$x^\Delta = -x - x^3 \tag{5.8}$$

on the time scale $\mathbb{T} = \{\dots, s_k, s_k + 3, s_k + 3b_k, s_{k+1}, \dots\}$. Note that $p = -1$ is not a regressive constant on $\mathbb{T}$. Pötzsche, Siegmund and Wirth in [9] have shown that equation (5.8) is unstable on this time scale even though the linearized system $x^\Delta = -x$ is exponentially stable since trajectories reach zero in finite time. For this time scale, $\overline{\mu} = 3$. Thus if $-2/3 < p < 0$ then the zero solution of

$$x^\Delta = px - x^3$$

is uniformly exponentially stable.

Theorem 5.5. *Assume $f(t,x)$ is a real-valued continuous function for $(t,x) \in \mathbb{T} \times N$ which satisfies condition* (5.2) *uniformly in t on $\mathbb{T}$. If $p > 0$, then the zero solution of the almost linear dynamic equation* (5.1) *is unstable.*

Proof. By way of contradiction assume that equation (5.1) is stable. By the almost linearity property (5.2) of f, given $\epsilon > 0$, there is an $\eta > 0$ such that $|f(t,x)| \le \epsilon |x|$, uniformly in t, if $|x| \le \eta$. The assumed stability of the zero solution of (5.1) implies that, for this value η, there is a $\delta > 0$ such that if $x(t)$ is a solution of (5.1), (5.3) with $|x_0| \le \delta$, then $|x(t)| \le \eta$, for all $t \ge t_0$. Consider $x^2(t)$. Then by Theorem 2.4, we have

$$\begin{aligned}
\left(x^2(t)\right)^\Delta &= 2x^\Delta(t) \int_0^1 \left[x(t) + h\mu(t)x^\Delta(t)\right] dh \\
&= \mu(t)\left(x^\Delta(t)\right)^2 + 2x(t)x^\Delta(t) \\
&= \mu(t)\left(x^\Delta(t)\right)^2 + 2x(t)\left[px(t) + f(t,x)\right] \\
&\ge 2px^2(t) + 2x(t)f(t,x) \\
&\ge 2px^2(t) - 2\epsilon x^2(t) \\
&= (2p - 2\epsilon)x^2(t).
\end{aligned}$$

Since $p > 0$, one may choose ϵ sufficiently small so that $2p - 2\epsilon > 0$. Then by Corollary 4.2, $\lim_{t\to\infty} |x^2(t)| = \infty$, contradicting the assumption of stability. $\square$

Theorem 5.6. *Suppose there exists positive constant λ and ν such that*

$$-\infty < -\lambda < 1 + \mu(t)p < \nu < -1 \tag{5.9}$$

for all $t \in \mathbb{T}$, and let $f(t,x)$ satisfy condition (5.2). *Then the zero solution of the dynamic equation* (5.1) *is unstable.*

Proof. If the zero solution were stable, by (5.2) for any $\epsilon > 0$ there would be a $\delta > 0$ such that if $|x(t_0)| < \delta$, then both

$$|f(t,x(t))| < \epsilon |x(t)| \tag{5.10}$$

and $x(t)$ remains bounded for all $t \in \mathbb{T}$ with $t \ge t_0$. Note that under hypothesis (5.9), all points in $\mathbb{T}$ are scattered, and consequently equation (5.1) can be written in update difference equation form:

$$x(\sigma(t)) = (1 + \mu(t)p)x(t) + \mu(t)f(t,x(t)). \tag{5.11}$$

So we have

$$\begin{aligned} |x(\sigma(t))| &\geq ||(1+\mu(t)p)||x(t)| - \mu(t)|f(t,x(t))|| \\ &\geq \nu|x(t)| - \mu(t)\epsilon|x(t)| \end{aligned} \tag{5.12}$$

Now condition (5.9) also implies that the graininess μ is bounded:

$$\mu(t) < \frac{-1-\lambda}{p} = \beta.$$

Therefore (5.12) yields the estimate

$$|x(\sigma(t))| \geq (\nu - \beta\epsilon)|x(t)| \tag{5.13}$$

for all $t \geq t_0$. Since $\nu > 1$, we can now choose ϵ sufficiently small so that

$$\nu - \beta\epsilon > 1.$$

If $|x(t_0)| < \delta$, a value which corresponds to the above choice of ϵ according to (5.10), then, for any positive integer n, we have from (5.13)

$$|x(t)| \geq (\nu - \beta\epsilon)^n |x(t_0)|$$

eventually for some $t > t_0$. This contradicts stability of the zero solution. □

We now have the following corollaries that are analogous to Corollary 4.2.

Corollary 5.7. *Let f and p be as in Theorem 5.3. If*

$$\overline{\mu} < -\frac{2}{p}$$

then the zero solution of the almost linear dynamic equation (5.1) *is exponentially asymptotically stable. Furthermore, if*

$$\sup \mu(t) < -\frac{2}{p}$$

the zero solution of equation (5.1) *is uniformly exponentially asymptotically stable.*

Corollary 5.8. *Let f and p be as in Theorem 5.3 with $p < 0$. If*

$$\underline{\mu} > -\frac{2}{p} \tag{5.14}$$

then the zero solution of equation (5.1) *is unstable.*

Proof. Note, first of all, that condition (5.9) holding for all $t \in \mathbb{T}$ and t sufficiently large suffices for the conclusion of Theorem 5.6 to hold. Now, by (5.14), for such t,

$$\mu(t) > -\frac{2}{p} + \frac{\left(\underline{\mu} + \frac{2}{p}\right)}{2} = \frac{\underline{\mu}}{2} - \frac{1}{p}.$$

Therefore, since $p < 0$,

$$1 + p\mu(t) < \frac{p\underline{\mu}}{2} \tag{5.15}$$

for $t \in \mathbb{T}$ and t sufficiently large. Inequality (5.14) implies that

$$-\nu = \frac{p\mu}{2} < -1. \tag{5.16}$$

So combining inequalities (5.15) and (5.16) gives (5.9) and the result follows from Theorem 5.6. □

Example 5.9. We consider the dynamic logistic equation

$$x^\Delta = px(1-x). \tag{5.17}$$

where $p > 0$. The differential ($\mu \equiv 0$) and difference ($\mu \equiv 1$) cases are basic population models which feature density-dependent limitations on growth (see e.g., [4]). The differential equations case, also known as the Verhulst–Pearl growth model, is of course familiar to all differential equations students. The dynamic asymptotic behavior is robust and simple – for all positive constants p, the positive solutions of equation (5.17) are monotonic and tend toward the equilibrium $x_1 = 1$, and the other equilibrium $x_0 = 0$ is unstable. The difference case is a well-known example of a simple equation which can exhibit complex dynamics, as shown by the parameter regime $2 < p \leq 3$ – oscillations and chaotic dynamics characterized by interval attractors occur, a situation first observed by Robert May [7].

For all time scales, the equilibria of equation (5.17) are $x_0 = 0$ and $x_1 = 1$. Considering the equilibrium $x_0 = 0$, we note that equation (5.17) is in almost linear form

$$x^\triangle = px(1-x) = px - px^2,$$

and that the conditions of Theorem 5.5 are satisfied. In particular, since $p > 0$, the regressive assumption and the condition for instability of the zero solution hold on any time scale. Thus Theorem 5.5 indicates that this equilibrium solution is unstable on any time scale. Letting

$$z = 1 - x,$$

equation (5.17) transforms to

$$z^\Delta = -p(1-z)z = -pz + pz^2 \tag{5.18}$$

and we are interested in the equilibrium solution $z_0 = 1 - x_1 = 0$. It is clear that the conditions of Theorem 5.3 are satisfied. The regressive assumption requires that

$$\mu(t) \neq \frac{1}{p}$$

for any t in the time scale. Then Theorem 5.3 indicates that the zero solution of equation (5.18) (the $x_1 = 1$ solution of equation (5.17)) is exponentially stable if

$$\overline{q}_T = \limsup_{t\to\infty} \alpha(\mu(t)) = \limsup_{t\to\infty} \frac{\log|1 - p\mu(t)|}{\mu(t)} < 0 \tag{5.19}$$

when

$$\overline{\mu} = \limsup_{t\to\infty} \mu(t) > 0.$$

In this case, as in Corollary 5.7, (5.19) corresponds to

$$\overline{\mu} < \frac{2}{p}.$$

On the other hand, Corollary 5.8 indicates that the zero solution of equation (5.17) is unstable if

$$\underline{\mu} > -\frac{2}{p} \tag{5.20}$$

In this case the dynamic equation is a difference equation (with possibly nonconstant graininess), and so (5.15) can be written in the more standard update form

$$z_{n+1} = (1 - p\mu(t_n))z_n + p\mu(t_n)z_n^2.$$

Condition (5.20) indicates that for sufficiently large t,

$$1 - p\mu(t_n) > 3.$$

It is well known that the related logistic map

$$x_{n+1} = \gamma x_n(1 - x_n)$$

exhibits (at least) sustained positive oscillations precisely when $\gamma > 3$. Condition (5.20) gives an estimate for the loss of stability threshold of the positive equilibrium in terms of the graininess of the time scale, signaling the onset of more complicated qualitative behavior.

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