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POLYNOMIAL AND SERIES SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. Much work paralleling the standard theory of linear differential equations has been done recently for dynamic equations on time scales, providing a unified treatment of the continuous and discrete analysis in this area. One area which is less developed is the theory of series solutions, partly due to the lack of sufficient differentiability in a general time scale, and also due to the lack of an analogue for polynomials which enjoys all the properties of the polynomials over the real numbers. In this paper, we obtain results for series and polynomial solutions for certain classes of dynamic equations and/or certain time scales.

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1. INTRODUCTION

The study of dynamic equations on time scales is an active area of recent research, having as a major goal the creation of a theory which can unify continuous and discrete analysis. The idea of a calculus on a time scale was developed by S. Hilger [4] in 1988, and the best source for an introduction to dynamic equations on time scales is the 2001 book by M. Bohner and A. Peterson [3], which contains an extensive bibliography in addition to being an excellent treatment of the subject.

A time scale is a nonempty closed subset of the real numbers. Analogues of the derivative can be defined for functions on time scales, and these reduce to the usual derivative when the time scale is $\mathbb{R}$, the real numbers, or to the forward difference operator when the time scale is $\mathbb{Z}$, the integers. The specific derivative analogue used in this paper is the delta derivative, denoted by f^{Δ}. See [3] for details. An important, and sometimes frustrating, role in the delta derivative calculus is played by the forward jump operator σ, defined on a time scale $\mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

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A related function is the graininess function $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$, while for $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$ and $\mu(t) = 1$. In general, μ is not necessarily a constant. One of the troubles with $\sigma(t)$ is that σ is not, in general, a delta differentiable function, but σ appears frequently in derivatives of elementary functions, thus preventing the existence of higher derivatives. For example, if $f(t) = t^2$, the delta derivative is $f^{\Delta}(t) = t + \sigma(t)$, so even the standard polynomials are not differentiable more than once on a general time scale.

There is a Taylor's formula for sufficiently differentiable functions on a general time scale, which we will use later. See R. Agarwal and M. Bohner [2] or Section 1.6 of [3] for details. One of the difficulties with using this result for developing a theory of series solutions for linear dynamic equations is that products and quotients of the functions that play the role of powers of t are not easily found. If the time scale has constant graininess, we can make some progress towards a general theory of series solutions, and for general time scales, we can also get some interesting results when the operator factors in a certain way.

2. SERIES IN TIME SCALE CALCULUS

On a time scale $\mathbb{T}$, define the functions $\{h_k\}$ by

$$h_0(t, s) \equiv 0, \qquad h_{k+1}(t, s) = \int_s^t h_k(\tau, s) d\tau.$$

Integration on a time scale is covered in detail in [3], but the principal feature of the definition is that, for each fixed s, the delta derivative of h_k with respect to t satisfies

$$h_k^{\Delta}(t, s) = h_{k-1}(t, s), \quad k \geq 1.$$

These functions take the place of the $\frac{(t-s)^k}{k!}$ terms in Taylor series in $\mathbb{R}$. In fact, when $\mathbb{T} = \mathbb{R}$,

$$h_k(t, s) = \frac{(t-s)^k}{k!}$$

and when $\mathbb{T} = \mathbb{Z}$,

$$h_k(t, s) = \binom{t-s}{k}.$$

Also, for any time scale, $h_1(t, s) = t - s$, but for $k \geq 2$ the analogy to powers of t breaks down in general.

Another important function in time scale calculus is the generalized exponential function, introduced in Chapter 2 of [3]. With certain assumptions on the coefficient function p, the exponential function $e_p(t, t_0)$ can be defined as the unique solution of the initial value problem

$$y^{\Delta} = p(t)\, y, \qquad y(t_0) = 1 \tag{2.1}$$

on a time scale $\mathbb{T}$. It is easy to show that

$$e_1(t, 0) = \sum_{k=0}^{\infty} h_k(t, 0)$$

for any time scale. If $\mathbb{T} = \mathbb{R}$,

$$e_1(t, 0) = e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!},$$

and if $\mathbb{T} = \mathbb{Z}$,

$$e_1(t, 0) = 2^t = \sum_{k=0}^{\infty} \binom{t}{k} = \sum_{k=0}^{\infty} \frac{t^{(k)}}{k!}$$

where $t^{(0)} = 1$ and $t^{(k)} = t(t-1)(t-2)\cdots(t-k+1)$.

3. A MOTIVATING EXAMPLE IN $\mathbb{R}$

The following example, which can be found in the paper of T. A. Newton [5] (and also in some of the references therein), is an ordinary linear second order homogeneous differential equation having the property that the functions e^t and the Nth degree Maclaurin polynomial for e^t constitute a fundamental set of solutions. The equation is

$$t\, y'' - (t + N)\, y' + N\, y = 0. \tag{3.1}$$

Because this equation has a regular singular point at $t = 0$, the Frobenius method can be used to show that

$$y_1(t) = e^t \quad \text{and} \quad y_2(t) = \sum_{k=0}^{N} \frac{t^k}{k!}$$

are two linearly independent solutions. However, a different approach gives more insight into the relationship between the solutions.

Factor the left side of (3.1) as follows

$$t\, y'' - (t + N)\, y' + N\, y = (tD - N)(D - 1).$$

Clearly, one solution is $y_1(t) = e^t$, and if we consider a Maclaurin polynomial of degree M for e^t, where

$$y_2(t) = \sum_{k=0}^{M} \frac{t^k}{k!},$$

then $(D-1)y_2 = -t^M/M!$. Because the operator $(tD-N)$ is a Cauchy–Euler operator with kernel t^N, we see that $y_2(t)$ is also a solution of (3.1) when $M = N$.

4. **EXTENSION TO TIME SCALES**

The best way to generalize an ordinary differential equation to a general time scale is not always obvious. If we simply replace the derivatives in (3.1) by delta derivatives, we get

$$t\, y^{\Delta\Delta} - (t+N)\, y^{\Delta} + N\, y = 0. \tag{4.1}$$

Encouraged by the fact that both solutions of (3.1) were analytic at $t = 0$ even though $t = 0$ was a singular point, and also motivated by the fact that an extension of the Frobenius method to time scales has not been done yet, we try a formal solution of the form $y(t) = \sum_{k=0}^{\infty} c_k h_k(t,0)$. Substitution into (4.1) gives

$$\sum_{k=0}^{\infty} \left[(c_{k+2} - c_{k+1})\; t - (c_{k+1} - c_k)\; N \right]\; h_k(t,0) = 0. \tag{4.2}$$

The presence of t in the recurrence relation is not ideal, but it is clear that setting $c_k = 1$ for $k = 0, 1, 2, \ldots$ reduces (4.2) to an identity. Thus, as can be verified,

$$y(t) = \sum_{k=0}^{\infty} h_k(t,0) = e_1(t,0) \tag{4.3}$$

is a solution of (4.1). The reader is invited to consult [3] for the method of reduction of order and to use that method to find the second solution, but that solution is not $\sum_{k=0}^{N} h_k(t,0)$.

A modification of the factorization method will give better results. We begin with the factored form and then expand to get a better time scale version of (3.1). Let D_Δ represent the delta derivative operator. Recall from (2.1) that $e_1(t,0) = \sum_{k=0}^{\infty} h_k(t,0)$ is a solution of $(D_\Delta - 1)y = 0$ and then calculate

$$(D_\Delta - 1) \left(\sum_{k=0}^{N} h_k(t,0) \right) = -h_N(t,0).$$

Thus, the left factor in the operator must annihilate $h_N(t,0)$ in order for the nth degree "Maclaurin" polynomial to also be a solution. The operator $(tD_\Delta - N) = (h_1(t,0)D_\Delta - N)$ does not work in general because

$$(h_1(t,0)D_\Delta - N)h_N(t,0) = h_1(t,0)h_{N-1}(t,0) - Nh_N(t,0) \tag{4.4}$$

and products of the h_k functions are not known in general. Later we will see how to calculate the product $h_1 h_m$ for time scales with constant graininess, and even in that restricted setting, (4.4) will only be zero when $\mu = 0$, i.e., only when $\mathbb{T} = \mathbb{R}$. The operator that annihilates $h_N(t,0)$ is easily seen to be

$$h_N(t,0)D_\Delta - h_{N-1}(t,0) \tag{4.5}$$

and composing this operator on the left of $(D_\Delta - 1)$ leads to the dynamic equation

$$h_N(t,0)y^{\Delta\Delta} - (h_N(t,0) + h_{N-1}(t,0))y^\Delta + h_{N-1}(t,0)y = 0. \tag{4.6}$$

As a result of its construction, equation (4.6) has

$$y_1(t) = e_1(t,0) = \sum_{k=0}^{\infty} h_k(t,0) \quad \text{and} \quad y_2(t) = \sum_{k=0}^{N} h_k(t,0)$$

as two linearly independent solutions.

If $\mathbb{T} = \mathbb{R}$, (4.6) reduces to (3.1) as it should, and if $\mathbb{T} = \mathbb{Z}$, equation (4.6) becomes

$$(t - N + 1)\Delta^2 y - (t+1)\Delta y + Ny = 0. \tag{4.7}$$

A fundamental set of solutions of (4.7) is

$$y_1(t) = 2^t = \sum_{k=0}^{\infty} \frac{t^{(k)}}{k!} \quad \text{and} \quad y_2(t) = \sum_{k=0}^{N} \frac{t^{(k)}}{k!}.$$

The results in this section are the first we know of that extend the example of Newton [5] to other settings, such as difference equations. Without knowing the solutions to (4.7), and especially without its factored form, this equation would probably not be the first choice as a difference equations analog to (3.1). But because we did the analysis for a general time scale, we were guaranteed to get the correct form for any specific time scale.

5. MORE GENERAL EQUATIONS

The analysis leading to (4.6) and its solutions can, as in [5], be extended to certain other cases. Solutions of the second order linear dynamic equations with constant coefficients

$$y^{\Delta\Delta} + \alpha\, y^\Delta + \beta\, y = 0 \tag{5.1}$$

are completely described in [3]. If $\{c_k\}$ can be found so that

$$y_1(t) = \sum_{k=0}^{\infty} c_k h_k(t,0) \tag{5.2}$$

is a solution of (5.1), and if

$$y_2(t) = \sum_{k=0}^{N} c_k h_k(t,0)$$

then

$$(D_{\Delta\Delta} + \alpha D_\Delta + \beta)\, y_2 = -c_{N+1} h_{N-1}(t,0) + \beta c_N h_N(t,0). \tag{5.3}$$

If $\mathbb{T} = \mathbb{R}$, the right side of (5.3) is a linear combination of t^{N-1} and t^N, and a second order Cauchy–Euler operator can be used to annihilate both powers of t. For a general time scale, however, the role of powers of t is taken by functions of the form $e_{\frac{\lambda}{t}}(t, t_0)$, and we do not know a simple operator which will annihilate both h_{N-1} and h_N. This difficulty is avoided if the series in (5.2) consists of only even- or odd-subscripted terms. In this case, either $c_{N-1} = 0$ or $c_N = 0$ and we only need to annihilate one of $h_{N-1}(t, 0)$ or $h_N(t, 0)$, which can be done as in the last section. For example, consider

$$(h_{2M+1}(t,0)D_\Delta - h_{2M}(t,0))\,(D_{\Delta\Delta} - 1) = 0. \tag{5.4}$$

The two solutions of (5.4) from the right-hand factor are the generalized hyperbolic functions $\cosh_1(t, 0)$ and $\sinh_1(t, 0)$ defined by

$$\begin{aligned}
\cosh_1(t,0) &= \frac{e_1(t,0) + e_{-1}(t,0)}{2} = \sum_{k=0}^{\infty} h_{2k}(t,0) \\
\sinh_1(t,0) &= \frac{e_1(t,0) - e_{-1}(t,0)}{2} = \sum_{k=0}^{\infty} h_{2k+1}(t,0).
\end{aligned}$$

If we compare the left-hand factor of (5.4) with (4.5), we see that $N = 2M + 1$, and because $c_{N+1} = c_{2M+2} = 0$ in the series for $\sinh_1(t, 0)$, the third solution of (5.4) is

$$y_3 = \sum_{k=0}^{M} h_{2k+1}(t,0).$$

6. SERIES SOLUTIONS

In this section, we consider only time scales with constant graininess, i.e., $\mu \equiv c$, in order to ensure that all necessary delta derivatives exist. In this setting, we can determine what products of the h_k functions look like, and so can do series solutions of dynamic equations. There are two product rules for delta derivatives in [3], and the one we will use is

$$(f(t)g(t))^\Delta = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t). \tag{6.1}$$

Also, the following result from [3] (called "Peterson's favorite formula" by some) will be quite useful:

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \tag{6.2}$$

The following theorem can be proved by using (6.1), (6.2), Taylor's theorem for time scales, and the properties of the h_k functions to get the $n = 1$ case, and then using induction to get the general result.

Theorem 6.1. *For a time scale $\mathbb{T}$ with μ constant,*

$$\begin{aligned}
h_1(t,s)\,h_m(t,s) &= \mu\, m\, h_m(t,s) + (m+1)h_{m+1}(t,s), \\
h_n(t,s)\,h_m(t,s) &= \sum_{k=0}^{n} \frac{\mu^{n-k}(m+k)^{(n)}}{k!(n-k)!} h_{m+k}(t,s).
\end{aligned}$$

Proof. By the product rule,

$$\begin{aligned}[h_1(t,\alpha)\,h_m(t,\alpha)]^{\Delta} &= h_m(t,\alpha) + h_1(\sigma(t),\alpha)\,h_{m-1}(t,\alpha) \\ &= h_m(t,\alpha) + h_1(t,\alpha)h_{m-1}(t,\alpha) + \mu h_{m-1}(t,\alpha).\end{aligned}$$

Continuing, and suppressing the arguments of the h_k functions, we get, for $k = 0, 1, 2, \ldots, m$,

$$[h_1\,h_m]^{\Delta^k} = k\,h_{m-k+1} + h_1\,h_{m-k} + \mu k\,h_{m-k},$$

and, for $k = m+1$,

$$[h_1\,h_m]^{\Delta^{m+1}} = m+1$$

Derivatives of order $m+2$ and higher are identically zero, and all except the mth and $(m+1)$st derivatives are zero at $t = \alpha$, so by Taylor's formula for time scales,

$$h_1(t,\alpha)\,h_m(t,\alpha) = \mu\,m\,h_m(t,\alpha) + (m+1)h_{m+1}(t,\alpha). \tag{6.3}$$

Similarly, we can express $h_n(t,\alpha)h_m(t,\alpha)$ as a linear combination of h_m through h_{m+n}, but the calculations become quite cumbersome for n larger than 2. Instead, as an induction hypothesis, assume that

$$h_n(t,\alpha)\,h_m(t,\alpha) = \sum_{k=0}^{n} \frac{\mu^{n-k}(m+k)^{(n)}}{k!(n-k)!}\,h_{m+k}(t,\alpha). \tag{6.4}$$

From (6.3), for $m = n$,

$$h_{n+1}(t,\alpha) = \frac{1}{n+1}\,(h_1(t,\alpha) - \mu n)\,h_n(t,\alpha),$$

so

$$h_{n+1}(t,\alpha)h_m(t,\alpha) = \frac{1}{n+1}\,(h_1(t,\alpha) - \mu n)\,h_n(t,\alpha)h_m(t,\alpha).$$

All products on the right can be formed using (6.3) and (6.4), so we have, again suppressing the arguments,

$$\begin{aligned} h_{n+1}h_m &= \left[\frac{\mu^{n+1} m\,m^{(n)}}{(n+1)0!n!} - \frac{\mu^{n+1} n\,m^{(n)}}{(n+1)0!n!}\right] h_m \\ &+ \sum_{k=1}^{n} \frac{\mu^{n+1-k}(m+k)^{(n+1)}}{k!(n+1-k)!}\,h_{m+k} \\ &+ \frac{(m+n+1)^{(n+1)}}{(n+1)n!0!}\,h_{m+n+1} \\ &= \sum_{k=0}^{n+1} \frac{\mu^{n+1-k}(m+k)^{(n+1)}}{k!(n+1-k)!}\,h_{m+k}, \end{aligned}$$

and the induction is complete. □

Equation (6.3) and the fact that $h_1(t,0) = t$ immediately leads to

Corollary 6.2. *For a time scale $\mathbb{T}$ with μ constant,*

$$h_k(t,0) = \frac{t(t-\mu)(t-2\mu)\cdots(t-(k-1)\mu)}{k!} = \frac{\mu^k \left(\frac{t}{\mu}\right)^{(k)}}{k!}.$$

Theorem 6.1 allows us to find series solutions for linear dynamic equations with "h_k-polynomial" coefficients. For example, Airy's equation on a constant-graininess time scale $\mathbb{T}$ can be written

$$y^{\Delta\Delta} - h_1(t,0)\, y = 0. \tag{6.5}$$

If we seek a solution of the form $y = \sum_{k=0}^{\infty} c_k\, h_k(t,0)$, then (6.5) becomes

$$\sum_{k=0}^{\infty} c_{k+2} h_k(t,0) - \sum_{k=0}^{\infty} c_k h_1(t,0)\, h_k(t,0) = 0.$$

Using Theorem 6.1 we get

$$\sum_{k=0}^{\infty} c_{k+2} h_k(t,0) - \sum_{k=0}^{\infty} \mu\, k\, c_k h_k(t,0) - \sum_{k=1}^{\infty} k\, c_{k-1} h_k(t,0) = 0,$$

leading to

$$c_2 = 0 \quad \text{and} \quad c_{k+2} = k(\mu\, c_k + c_{k-1}),\ k = 1, 2, \ldots. \tag{6.6}$$

The coefficients c_0 and c_1 are arbitrary, and if we choose $c_0 = 1$, $c_1 = 0$, one series solution is (all h_k functions have argument $(t,0)$)

$$f(t) = 1 + h_3 + 3\mu h_5 + 4h_6 + 15\mu^2 h_7 + 42\mu h_8 + (105\mu^3 + 28)h_9 + \cdots.$$

Choosing $c_0 = 0$ and $c_1 = 1$ yields a second linearly independent solution,

$$g(t) = h_1 + \mu h_3 + 2h_4 + 3\mu^2 h_5 + 12\mu h_6 + (15\mu^3 + 10)h_7 + \cdots.$$

Now let $a_1 = \dfrac{3^{-2/3}}{\Gamma(\frac{2}{3})}$ and $a_2 = \dfrac{3^{-1/3}}{\Gamma(\frac{1}{3})}$, and form the fundamental set of solutions (compare with p. 446 of Abramowitz and Stegun [1])

$$\begin{aligned} Ai(t;\mathbb{T}) &= a_1 f(t) - a_2 g(t) \\ Bi(t;\mathbb{T}) &= \sqrt{3} a_1 f(t) + \sqrt{3} a_2 g(t). \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, we have $\mu = 0$ and $h_k(t,0) = \dfrac{t^k}{k!}$, making $Ai(t;\mathbb{R}) = Ai(t)$ and $Bi(t;\mathbb{R}) = Bi(t)$, the standard Airy functions.

7. DIRECTIONS FOR FURTHER STUDY

As noted, the results in the last section allow series solutions only for time scales with constant graininess. This essentially limits us to the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ (difference equations with step size h). Series solutions for time scales with differentiable graininess are probably possible. For example, the time scale $\overline{q^{\mathbb{Z}}} = \left\{q^k : k \in \mathbb{Z}\right\} \cup \{0\}$ for $q > 1$ has $\mu(t) = (q-1)t$. If we assume that μ is a linear function of t, which will cover this case, a formula like (6.3) can be found, but the coefficients are much more complicated functions of μ and μ^{Δ}, and the equivalent to (6.4) would be even worse.

Another direction for further study is to develop an analogue of the Frobenius method for equations with a singularity. We noted earlier that the Cauchy–Euler equation for time scales involves the functions $e_{\frac{\lambda}{t}}(t, t_0)$. Perhaps some combination of these generalized exponentials and a series in the h_k's could be made to work.

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