CHAPTER 1

Interest Rates

Definition 1.1 (Zero-coupon bond). A zero-coupon bond with maturity $T > 0$ is a contract that guarantees the holder a cash payment of one unit on the date $T$. The price at time $t \in [0, T]$ of a zero-coupon bond with maturity $T$ is denoted by $P(t, T)$. At time $t$, the time to maturity is $T - t$, or, more generally, when taking day-count conventions into account, $\tau(t, T)$.

Definition 1.2 (Interest rates). Let $0 \leq t \leq T < S$.

(i) The forward rate for the period $[T, S]$ as seen at time $t$ is defined as

$$R(t; T, S) = -\frac{\ln P(t, S) - \ln P(t, T)}{\tau(T, S)}.$$

(ii) The continuously-compounded spot interest rate with maturity $T$ prevailing at $t$ is defined as

$$R(t, T) = -\frac{\ln P(t, T)}{\tau(t, T)}.$$

(iii) The simply-compounded spot interest rate with maturity $T$ prevailing at $t$ is defined as

$$L(t, T) = \frac{1 - P(t, T)}{\tau(t, T)}.$$

(iv) The simply-compounded forward interest rate for the period $[T, S]$ as seen at time $t$ is defined as

$$F(t; T, S) = \frac{1}{\tau(T, S)} \left( \frac{P(t, T)}{P(t, S)} - 1 \right).$$

(v) The instantaneous forward interest rate with maturity $T$ at $t$ is defined as

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T}.$$

(vi) The instantaneous spot rate at time $t$ is defined as

$$r(t) = f(t, t).$$
For $\alpha, \beta \in \mathbb{N}$ with $\alpha < \beta$ and times $T = [T_\alpha, \ldots, T_\beta] = S$, the forward swap rate for $T = \{T_\alpha, \ldots, T_\beta\}$ at time $t$ is defined as

$$S_{\alpha, \beta}(t) = \frac{P(t, T) - P(t, S)}{\sum_{i=\alpha+1}^{\beta-1} \tau(T_{i-1}, T_i)P(t, T_i)}.$$ 

**Remark 1.3 (Interest rates).** Let $0 \leq t \leq T < S$.

(i) Suppose $R$ is the equivalent constant rate of interest over the period $[T, S]$. In order to exclude arbitrage, we should have

$$e^{R\tau(T, S)} = \frac{P(t, T)}{P(t, S)}.$$

(ii) We have

$$R(t, T) = R(t; t, T)$$

and

$$P(t, T) = e^{-R(t, T)\tau(t, T)}.$$ 

Thus $R(t, T)$ is the constant rate at which an investment of $P(t, T)$ units of currency at time $t$ accrues continuously to yield a unit amount of currency at maturity.

(iii) We have

$$P(t, T) = \frac{1}{1 + \tau(t, T)L(t, T)}.$$ 

Thus $L(t, T)$ is the constant rate at which an investment of $P(t, T)$ units of currency at time $t$ produces one unit amount of currency at maturity $T$, when accruing occurs proportionally to the investment time.

(iv) $F(t; T, S)$ is the value of the fixed rate that makes an FRA for the period between $T$ and $S$ a fair contract at time $t$. FRAs are introduced in Section 2.1. Note also

$$L(t, T) = F(t; t, T).$$

(v) If $\tau(T, S) = S - T$, then we have

$$f(t, T) = \lim_{S \to T^+} F(t; T, S) = \lim_{S \to T^+} R(t; T, S).$$

Note also that for $t \leq s \leq T$ we have

$$P(t, T) = P(t, s) \exp \left( - \int_s^T f(t, u)\,du \right)$$
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and in particular
\[ P(t, T) = \exp \left( - \int_t^T f(t, u) du \right). \]

(vi) The rate \( r(t) \), also briefly called the short rate, is the instantaneous rate at which the bank accrues, where the bank account is defined as
\[ B(t) = \exp \left( \int_0^t r(s) ds \right). \]

The short rate is also used to define the discount factor
\[ D(t, T) = \exp \left( - \int_t^T r(s) ds \right) = \frac{B(t)}{B(T)}. \]

If rates \( r(t) \) are deterministic, then so is \( D \) and we have necessarily \( D(t, T) = P(t, T) \). However, if rates \( r(t) \) are stochastic, \( D(t, T) \) is stochastic while \( P(t, T) \) is deterministic, so they cannot be the same. We will see later that \( P(t, T) \) can be viewed as the expectation of the random variable \( D(t, T) \) under a particular probability measure. Concerning stochastic rates \( r(t) \), we also note that when dealing with interest-rate products, the main variability that matters is clearly that of the interest rates themselves.

(vii) \( S_{\alpha,\beta}(t) \) is the value of the fixed rate that makes an IRS for the period between \( T \) and \( S \) a fair contract at time \( t \). IRSs are introduced in Section 2.2. Note also that
\[ S_{\beta-1,\beta}(t) = F(t; T_{\beta-1}, T_{\beta}). \]

Finally, put
\[ F_j(t) = F(t; T_{j-1}, T_j) \quad \text{and} \quad \tau_j = \tau(T_{j-1}, T_j) \]
and note that
\[ \frac{P(t, T_i)}{P(t, T_\alpha)} = \prod_{j=\alpha+1}^{i} \frac{1}{1 + \tau_j F_j(t)} \]
implies
\[ S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1 + \tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1 + \tau_j F_j(t)}}{}}. \]