CHAPTER 10

Pricing of Further Derivatives

10.1. In-Arrears Swaps

Definition 10.1 (In-arrears swaps). An in-arrears swap of payer type, depending on the notional value $N$, the fixed rate $K$, and the set of times $T$, is a contract, where its holder pays $NK\tau(T_i, T_{i+1})$ and receives $N\tau(T_i, T_{i+1})L(T_i, T_{i+1})$ units of currency at the same time $T_i$, for all $\alpha + 1 \leq i \leq \beta$.

Remark 10.2 (In-arrears swaps). An in-arrears swap is an IRS that resets at dates $T_{\alpha+1}, \ldots, T_{\beta}$ and pays at the same dates, with notional value $N$ and fixed-leg rate $K$. The discounted payoff of an in-arrears swap of payer type is

$$N \sum_{i=\alpha+1}^{\beta} (L(T_i, T_{i+1}) - K) \tau_{i+1} D(t, T_i).$$

Lemma 10.3. Assume $0 \leq t \leq T \leq S$. If $X$ is $\mathcal{F}(T)$-measurable, then

$$E(D(t,T)X|\mathcal{F}(t)) = E\left(D(t,S)X \middle| \mathcal{F}(t)\right).$$

Theorem 10.4 (Price of in-arrears swaps in the LFM model). In the LFM model, the price of an in-arrears swap of payer type with notional value $N$, fixed rate $K$, and the set of times $T$, is given by

$$\text{IAS}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} \left\{ \begin{array}{l} P(t, T_{i+1}) \left( 1 + 2\tau_{i+1} F_{i+1}^2(t) + \tau_{i+1}^2 F_{i+1}^2(t) e^{v_i^2(t)} \right) \\ -(1 + \tau_{i+1} K) P(t, T_i) \end{array} \right\},$$

where

$$v_i(t) = \sqrt{\int_t^{T_{i-1}} \sigma_u^2 \, du}, \quad \alpha + 1 \leq i \leq \beta.$$

Lemma 10.5. Define

$$B_i(T, S) = \frac{1 - e^{-k_i(S-T)}}{k_i} \quad \text{and} \quad B_{ij}(T, S) = \frac{1 - e^{-(k_i+k_j)(S-T)}}{k_i + k_j}$$

61
PRICING OF FURTHER DERIVATIVES

for \( i, j \in \{1, 2\} \). Then we have

\[
\int_t^{\hat{T}} (B_i(u, S) - B_i(u, T)) \left( B_j(u, S) - B_j(u, \hat{T}) \right) du = e^{-k_i(T-\hat{T})} e^{-k_j(\hat{T}-T)} B_i(T, S) B_j(\hat{T}, S) B_{ij}(t, \hat{T}).
\]

**Theorem 10.6** (Price of in-arrears swaps in the G2++ model). In the G2++ model, the price of an in-arrears swap of payer type with notional value \( N \), fixed rate \( K \), and the set of times \( T \), is given by

\[
IAS(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_i) \left( \frac{P(t, T_i) e^{V^2_{i+1}(t)}}{P(t, T_{i+1})} - (1 + K_{\tau_{i+1}}) \right),
\]

where

\[
V^2_i(t) = \sigma_1^2 B^2_1(T_{i-1}, T_i) B_{11}(t, T_{i-1}) + \sigma_2^2 B^2_2(T_{i-1}, T_i) B_{22}(t, T_{i-1})
+ 2\sigma_1\sigma_2 \rho B_1(T_{i-1}, T_i) B_2(T_{i-1}, T_i) B_{12}(t, T_{i-1}).
\]

### 10.2. In-Arrears Caps

**Definition 10.7** (In-arrears caps). An in-arrears cap of payer type, depending on the notional value \( N \), the cap rate \( K \), and the set of times \( T \), is a contract, where its holder pays \( N\tau(T_i, T_{i+1})K \) and receives \( N\tau(T_i, T_{i+1})L(T_i, T_{i+1}) \) units of currency at the same time \( T_i \), but only if \( L(T_i, T_{i+1}) > K \), for all \( \alpha + 1 \leq i \leq \beta \).

**Remark 10.8** (In-arrears caps). An in-arrears cap is composed by caplets resetting at dates \( T_{\alpha+1}, \ldots, T_{\beta} \) and paying at the same dates, with notional value \( N \) and cap rate \( K \). The discounted payoff of an in-arrears cap of payer type is

\[
N \sum_{i=\alpha+1}^{\beta} (L(T_i, T_{i+1}) - K)^+ \tau_{i+1} D(t, T_i).
\]

**Lemma 10.9.** Let \( K > 0 \). If \( Y \) is lognormally distributed such that \( E(\ln(Y)) = M \) and \( V(\ln(Y)) = V^2 \), then

\[
E(Y(Y - K)^+) = e^{M + \frac{V^2}{2}} \text{Bl} \left( K e^{-V^2}, e^{M + \frac{V^2}{2}}, V \right).
\]

**Theorem 10.10** (Price of in-arrears caps in the LFM model). In the LFM model, the price of an in-arrears cap of payer type with notional value \( N \), cap rate
10.3. Caps with Deferred Caplets

$K$, and the set of times $\mathcal{T}$, is given by

$$IAC(t, \mathcal{T}, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_{i+1}) \tau_{i+1} \left\{ \text{Bl} (K, F_{i+1}(t), v_{i+1}(t)) + \tau_{i+1} F_{i+1}(t) e^{V_{i+1}(t)} \text{Bl} \left( K e^{-v_{i+1}^2(t)}, F_{i+1}(t), v_{i+1}(t) \right) \right\},$$

where $v_i$ is as given in Theorem 10.4.

**Theorem 10.11 (Price of in-arrears caps in the G2++ model).** In the G2++ model, the price of an in-arrears cap of payer type with notional value $N$, fixed rate $K$, and the set of times $\mathcal{T}$, is given by

$$IAC(t, \mathcal{T}, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_i) \text{Bl} \left( 1 + K \tau_{i+1}, \frac{P(t, T_{i+1}) e^{V_{i+1}(t)}}{P(t, T_{i+1})}, V_{i+1}(t) \right),$$

where $V_i(t)$ is as in Theorem 10.6.

### 10.3. Caps with Deferred Caplets

**Definition 10.12 (Caps with deferred caplets).** A cap with deferred caplets of payer type, depending on the notional value $N$, the cap rate $K$, and the set of times $\mathcal{T}$, is a contract, where its holder pays $N \tau (T_{i-1}, T_i) K$ and receives $N \tau (T_{i-1}, T_i) L(T_{i-1}, T_i)$ units of currency at the same time $T_\beta$, but only if $L(T_{i-1}, T_i) > K$, for all $\alpha + 1 \leq i \leq \beta$.

**Remark 10.13 (Caps with deferred caplets).** Caps with deferred caplets are caps for which all caplets payments occur at the final time $T_\beta$. The discounted payoff of a cap with deferred caplets of payer type is

$$N \sum_{i=\alpha+1}^{\beta} (L(T_{i-1}, T_i) - K)^+ \tau_i D(0, T_\beta).$$

**Definition 10.14 (Forward-rate dynamics in the partially frozen LFM model).** In the partially frozen LFM model (with respect to the terminal measure), the simply-compounded forward interest rates $F_i$ is assumed to satisfy the stochastic differential equation

$$dF_i(t) = \sigma_i(t) F_i(t) dW^{T_\beta}(t) - \sum_{j=\alpha+1}^{\beta} \frac{\tau_j \sigma_j(t) \sigma_i(t) F_j(0) F_i(0)}{1 + \tau_j F_j(0)} dt,$$
where \( \sigma_j \) are deterministic and \( W^S \) is a Brownian motion under the \( S \)-forward measure.

**Remark 10.15.** The dynamics of the forward rate in the partially frozen LFM model is an approximation of the corresponding dynamics in the LFM model, by replacing \( F_j(t) \) twice by \( F_j(0) \) in the drift of the last formula of Theorem 8.6.

**Theorem 10.16 (Price of caps with deferred caplets in the partially frozen LFM model).** In the partially frozen LFM model, the price of a cap with deferred caplets of payer type with notional value \( N \), cap rate \( K \), and the set of times \( T \), is given by

\[
\text{CDC}(0, T, N, K) = N \sum_{i=\alpha+1}^{\beta} P(0, T_i) \times \\
\times B_i \left( K, F_i(0) \exp \left( - \sum_{j=i+1}^{\beta} \int_0^{T_i-1} \sigma_i(u) \sigma_j(u) \, du \right), v_i(0) \right),
\]

where \( v_i \) is as given in Theorem 10.4.

**Theorem 10.17 (Price of caps with deferred caplets in the G2++ model).** In the G2++ model, the price of an in-arrears swap of payer type with notional value \( N \), fixed rate \( K \), and the set of times \( T \), is given by

\[
\text{CDC}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_i) \times B_i \left( 1 + \tau_i K, \frac{P(t, T_i)}{P(t, T_i)}, V_i(t) \right),
\]

where \( V_i(t) \) is as in Theorem 10.6 and

\[
\tilde{V}_i^2(t) = \sigma_1^2 e^{-k_1(T_i-T_{i-1})} B_1(T_{i-1}, T_i) B_1(T_i, T_\beta) B_{11}(t, T_{i-1}) \\
+ \sigma_2^2 e^{-k_2(T_i-T_{i-1})} B_2(T_{i-1}, T_i) B_2(T_i, T_\beta) B_{22}(t, T_{i-1}) \\
+ \sigma_1 \sigma_2 \rho (B_1(T_{i-1}, T_i) B_2(T_i, T_\beta) + B_2(T_{i-1}, T_i) B_1(T_i, T_\beta)) B_{12}(t, T_{i-1}).
\]