CHAPTER 2

Interest Rate Derivatives

2.1. Forward Rate Agreements

**Definition 2.1 (FRA).** A *forward rate agreement*, briefly *FRA*, depending on the *notional value* \( N \), the *fixed rate* \( K \), the *expiry time* \( T \), and the *maturity time* \( S > T \), is a contract, where its holder receives \( N \tau(T,S)K \) and pays \( N \tau(T,S)L(T,S) \) units of currency at the same time \( S \).

**Remark 2.2 (FRA).** An FRA gives its holder an interest-rate payment for the period between \( T \) and \( S > T \). The contract allows to “lock in” the interest rate between \( T \) and \( S \) at the desired value \( K \). At maturity \( S \), a fixed payment based on a fixed rate \( K \) is exchanged against a floating payment based on the spot rate \( L(T,S) \), resetting in \( T \) and with maturity \( S \). The value of an FRA at time \( S \) is

\[
N \tau(T,S) (K - L(T,S)) = N\left(\tau(T,S)K - \frac{1}{P(T,S)} + 1\right).
\]

The value of an FRA at time \( t \leq T \) is

\[
\text{FRA}(t,T,S,N,K) = N(\tau(T,S)P(t,S)K - P(t,T) + P(t,S))
\]

\[
= N\tau(T,S)P(t,S)(K - F(t;T,S)).
\]

Hence \( F(t;T,S) \) is that value of \( K \) that makes the FRA a fair contract at time \( t \). We also see that in order to value an FRA, we can just replace \( L(T,S) \) by \( F(t;T,S) \) in the payoff at \( S \) and then take the present value at \( t \).

2.2. Interest Rate Swaps

**Definition 2.3 (IRS).** We consider two kinds of *interest rate swaps*, briefly *IRS*.

(i) A *receiver IRS*, briefly *RFS*, depending on the *notional value* \( N \), the *fixed rate* \( K \), and the *set of times* \( T \), is a contract, where its holder receives...
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\[ N\tau(T_{i-1}, T_i)K \] and pays \[ N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i) \] units of currency at the same time \( T_i \), for all \( \alpha + 1 \leq i \leq \beta \).

(ii) A \textit{payer IRS}, briefly \textit{PFS}, depending on the \textit{notional value} \( N \), the \textit{fixed rate} \( K \), and the \textit{set of times} \( T \), is a contract, where its holder pays \( NK\tau(T_{i-1}, T_i) \) and receives \( N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i) \) units of currency at the same time \( T_i \), for all \( \alpha + 1 \leq i \leq \beta \).

**Remark 2.4 (IRS).** An IRS exchanges interest payments starting from a future time instant \( T_{\alpha+1} \). At every instant \( T_i \), the holder of an RFS receives an amount corresponding to a fixed interest rate of the notional value (“fixed leg”) and pays an amount corresponding to the LIBOR rate that is reset at the previous time instant \( T_{i-1} \) (“floating leg”). The discounted payoff at time \( t \leq T_\alpha \) of an RFS is

\[ N \sum_{i=\alpha+1}^{\beta} (K - L(T_{i-1}, T_i)) \tau_i D(t, T_i) \]

and of a PFS is

\[ N \sum_{i=\alpha+1}^{\beta} (L(T_{i-1}, T_i) - K) \tau_i D(t, T_i). \]

The value of an RFS at time \( t \leq T_\alpha \) is

\[ \text{RFS}(t, T, N, K) = \sum_{i=\alpha+1}^{\beta} \text{FRA}(t, T_{i-1}, T_i, N, K) \]

\[ = NP(t, T_\beta) - NP(t, T_\alpha) + NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \]

\[ = N (K - S_{\alpha,\beta}(t)) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i). \]

Hence \( S_{\alpha,\beta}(t) \) is that value of \( K \) that makes the IRS a fair contract at time \( t \). We also see that an IRS can be viewed as a portfolio of a coupon-bearing bond (fixed leg) and a floating-rate note (floating leg).

**Definition 2.5 (Floating-rate note).** A \textit{floating-rate note}, briefly \textit{FRN}, depending on the set of future times \( T \) and the notional value \( N \), is a contract, where its holder receives \( N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i) \) units of currency at time \( T_i \) for all \( \alpha + 1 \leq i \leq \beta \). In addition, the holder also receives \( N \) units of currency at time \( T_\beta \).
Remark 2.6 (Floating-rate note). A floating-rate note ensures payments at future times $T_{\alpha+1}, \ldots, T_{\beta}$ of the LIBOR rates that reset at the previous instants $T_{\alpha}, \ldots, T_{\beta-1}$ and, moreover, it pays a last cash flow consisting of the reimbursement of the notional value of the note at the final time. The value of a floating-rate note at time $t \leq T_{\alpha}$ is

$$\text{FRN}(t, T, N) = NP(t, T_{\alpha}).$$

This means that a floating-rate note with notional value $N$ at its first reset date is always worth its notional value, i.e., “a floating-rate note trades at par”.

Definition 2.7 (Coupon-bearing bond). A coupon-bearing bond, depending on the set of future times $T$ and the deterministic cash flow $c = \{c_{\alpha+1}, \ldots, c_{\beta}\}$, is a contract, where its holder receives $c_i$ units of currency at time $T_i$ for all $\alpha + 1 \leq i \leq \beta$.

Remark 2.8 (Coupon-bearing bond). The value of a coupon-bearing bond at time $t \leq T_{\alpha}$ is

$$\text{CB}(t, T, c) = \sum_{i=\alpha+1}^{\beta} c_i P(t, T_i).$$

If we let $c_i = N \tau_i K$ for $\alpha + 1 \leq i \leq \beta - 1$ and $c_{\beta} = N \tau_{\beta} K + N$, the value is

$$\text{CB}(t, T, c) = NP(t, T_{\beta}) + NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i),$$

and then

$$\text{RFS}(t, T, N, K) = \text{CB}(t, T, c) - \text{FRN}(t, T, N).$$

2.3. Interest Rate Caps and Floors

Definition 2.9 (Caplets and floorlets). (i) A floorlet, depending on the notional value $N$, the floor rate $K$, the expiry time $T$, and the maturity time $S > T$, is a contract, where its holder receives $N \tau(T, S) K$ and pays $N \tau(T, S) L(T, S)$ units of currency at the same time $S$, but only if $L(T, S) < K$. 


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(ii) A caplet, depending on the notional value $N$, the cap rate $K$, the expiry time $T$, and the maturity time $S > T$, is a contract, where its holder pays $NK\tau(T, S)$ and receives $N\tau(T, S)L(T, S)$ units of currency at the same time $S$, but only if $L(T, S) > K$.

**Remark 2.10 (Caplets and floorlets).** A floorlet gives its holder an interest-rate payment for the period between $T$ and $S > T$. At maturity $S$, a fixed payment based on a fixed rate $K$ is exchanged against a floating payment based on the spot rate $L(T, S)$, resetting in $T$ and with maturity $S$. However, this is done only if the spot rate does not exceed $K$. The discounted payoff at time $t \leq T$ of a floorlet is

$$N (K - L(T, S))^+ \tau(T, S)D(t, S).$$

Similarly, the discounted payoff at time $t \leq T$ of a caplet is

$$N (L(T, S) - K)^+ \tau(T, S)D(t, S),$$

meaning that the holder of a caplet is paying interest at a rate which is at most $K$, i.e., the interest rate is capped to the fixed cap rate $K$. Hence caplets and floorlets are (call and put) options on interest rates, and they can be priced with Black's formula. This will be done in Section 8.1.

**Definition 2.11 (Caps and floors).**

(i) A floor, depending on the notional value $N$, the floor rate $K$, and the set of times $T$, is a contract, where its holder receives $N\tau(T_{i-1}, T_i)K$ and pays $N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$ units of currency at the same time $T_i$, but only if $L(T_{i-1}, T_i) < K$, for all $\alpha + 1 \leq i \leq \beta$.

(ii) A cap, depending on the notional value $N$, the cap rate $K$, and the set of times $T$, is a contract, where its holder pays $N\tau(T_{i-1}, T_i)K$ and receives $N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$ units of currency at the same time $T_i$, but only if $L(T_{i-1}, T_i) > K$, for all $\alpha + 1 \leq i \leq \beta$.

**Remark 2.12 (Caps and floors).** A floor is an RFS where each exchange payment is executed only if it has positive value. It can also be considered as a portfolio
of floorlets. The discounted payoff at time $t \leq T_\alpha$ of a floor is

$$N \sum_{i=\alpha+1}^{\beta} (K - L(T_{i-1}, T_i))^+ \tau_i D(t, T_i).$$

Similarly, a cap is a PFS where each exchange payment is executed only if it has positive value. It can also be considered as a portfolio of caplets. The discounted payoff at time $t \leq T_\alpha$ of a cap is

$$N \sum_{i=\alpha+1}^{\beta} (L(T_{i-1}, T_i) - K)^+ \tau_i D(t, T_i).$$

Caps and floors can be priced with a sum of Black’s formulas. This will be done in Section 8.1.

**Definition 2.13 (ATM).** Let $K_{\text{ATM}} = S_{a,\beta}(t)$. A cap or floor is said to be at the money, briefly ATM if $K = K_{\text{ATM}}$. A cap is called in the money, briefly ITM if $K < K_{\text{ATM}}$, while a floor is said to be ITM if $K > K_{\text{ATM}}$. A cap is called out of the money, briefly OTM if $K > K_{\text{ATM}}$, while a floor is said to be ITM if $K < K_{\text{ATM}}$.

### 2.4. Swaptions

**Definition 2.14 (Swaptions).** A swap option, briefly swaption, is an option on an IRS. The time $T_\alpha$ is called the swaption maturity. The underlying IRS length $T_\beta - T_\alpha$ is called the tenor of the swaption.

(i) A European payer swaption is a contract that gives the holder the right (but no obligation) to enter a PFS at the swaption maturity.

(ii) A European receiver swaption is a contract that gives the holder the right (but no obligation) to enter an RFS at the swaption maturity.

**Remark 2.15 (Swaption).** The value of the underlying IRS of a payer swaption at time $T_\alpha$ is

$$N \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K) \tau_i P(T_\alpha, T_i).$$

The discounted payer-swaption payoff therefore is

$$N \left( \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K) \tau_i P(T_\alpha, T_i) \right)^+ D(t, T_\alpha).$$
Thus it is not possible to decompose the payer-swaption payoff as can be done for caps. However, the value of a payer swaption is smaller than or equal to the value of the corresponding cap contract, due to the inequality

\[
\left( \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K) \tau_i P(T_\alpha, T_i) \right)^+ \leq \sum_{i=\alpha+1}^{\beta} (F(T_\alpha; T_{i-1}, T_i) - K)^+ \tau_i P(T_\alpha, T_i).
\]

Swaptions can be priced with a Black-like formula. This will be done in Section 8.2.

**Definition 2.16 (ATM).** Both payer and receiver swaption are said to be *ATM* if \( K = K_{\text{ATM}} \). A payer swaption is called *ITM* if \( K < K_{\text{ATM}} \), and a receiver swaption is said to be ITM if \( K > K_{\text{ATM}} \). A payer swaption is called *OTM* if \( K > K_{\text{ATM}} \), and a receiver swaption is said to be OTM if \( K < K_{\text{ATM}} \).