CHAPTER 3

Pricing Theory

3.1. Preliminaries from Financial Mathematics

In this section we give some crucial preliminaries from financial mathematics. The results are formulated in a nonrigorous way without stating the precise assumptions.

We start by giving the solutions to two important stochastic differential equations.

**Theorem 3.1** (Linear stochastic differential equation with deterministic diffusion coefficient). The solution of the linear stochastic differential equation

\[ dX(t) = (\alpha(t) + \beta(t)X(t))dt + \gamma(t)dB(t), \]

where \( \alpha, \beta, \) and \( \gamma \) are deterministic functions, is given by

\[ X(t) = X(s)e^{\int_s^t \beta(\tau)d\tau} + \int_s^t \alpha(u)e^{\int_u^t \beta(\tau)d\tau}du + \int_s^t \gamma(u)e^{\int_u^t \beta(\tau)d\tau}dW(u). \]

**Theorem 3.2** (Geometric Brownian motion). The solution of the lognormal linear stochastic differential equation

\[ dX(t) = \mu(t)X(t)dt + \sigma(t)X(t)dW(t), \]

where \( \mu \) and \( \sigma \) are deterministic functions, is given by the generalized geometric Brownian motion

\[ X(t) = X(s)e^{\int_s^t \mu(\tau)d\tau + \int_s^t \sigma(\tau)dW(\tau)} \exp \left\{ \int_s^t \left( \mu(\tau) - \frac{\sigma^2(\tau)}{2} \right)d\tau + \int_s^t \sigma(\tau)dW(\tau) \right\}. \]

In order to check solutions of stochastic differential equations, Itô’s formula and the stochastic Leibniz rule as given next are useful.

**Theorem 3.3** (Itô’s formula).

\[ dF(t, X(t)) = F_t(t, X(t))dt + F_x(t, X(t))dX(t) + \frac{1}{2}F_{xx}(t, X(t))(dX(t))^2, \]

where \( F(t, X(t)) \) is a deterministic function.
where \((dX(t))^2\) can be calculated by formally squaring \(dX(t)\) and using the “identities”

\[
(dt)^2 = (dt)(dW(t)) = 0 \quad \text{and} \quad (dW(t))^2 = dt.
\]

**Theorem 3.4** (Stochastic Leibniz rule).

\[
d(X(t)Y(t)) = X(t)(dY(t)) + (dX(t))Y(t) + (dX(t))(dY(t)).
\]

The next theorem helps to determine distribution, expectation, and variance of solutions to certain stochastic differential equations.

**Theorem 3.5** (Itô integral of deterministic integrand). If \(f\) is a deterministic function, then for \(0 \leq s \leq t\),

\[
I(t) = \int_s^t f(u)dW(u),
\]

conditionally on \(\mathcal{F}(s)\), is normally distributed with

\[
E(I(t)|\mathcal{F}(s)) = 0 \quad \text{and} \quad \mathbb{V}(I(t)|\mathcal{F}(s)) = \int_s^t f^2(u)du.
\]

Now we give the theorem about the change of measure.

**Theorem 3.6** (Girsanov). Consider the stochastic differential equation

\[
dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t),
\]

where \(W\) is a Brownian motion under the probability measure \(\mathbb{P}\). Let be given a new drift \(\mu^*\), define

\[
\lambda(t) = \frac{\mu^*(X(t)) - \mu(X(t))}{\sigma(X(t))},
\]

and define the probability measure \(\mathbb{P}^*\) by

\[
\frac{d\mathbb{P}^*}{d\mathbb{P}} \bigg|_{\mathcal{F}(t)} = \exp \left\{ -\frac{1}{2} \int_0^t \lambda^2(s)ds + \int_0^t \lambda(s)dW(s) \right\}.
\]

Then \(\mathbb{P}^* \sim \mathbb{P}\) and the process \(W^*\) defined by

\[
dW^*(t) = dW(t) - \lambda(t)dt
\]

is a Brownian motion under \(\mathbb{P}^*\) and we have

\[
dX(t) = \mu^*(X(t))dt + \sigma(X(t))dW^*(t).
\]

The following theorem summarizes the change of numéraire technique.
Theorem 3.7 (Change of numéraire). If $Q = Q^B$ is a risk-neutral measure, i.e., $X/B$ is a martingale under $Q^B$ for any traded asset $X$, then for each numéraire $N$, i.e., any positive nondividend-paying asset, there exists a measure $Q^N \sim Q^B$, defined by
\[
\frac{dQ^N}{dQ^B} \bigg\rvert_{\mathcal{F}(t)} = \frac{X(t)}{X(0)B(t)},
\]
under which $X/N$ is a martingale for any traded asset $X$, i.e.,
\[
\mathbb{E}^N \left( \frac{X(T)}{N(T)} \bigg\rvert \mathcal{F}(t) \right) = \frac{X(t)}{N(t)} \quad \text{for all} \quad 0 \leq t \leq T,
\]
i.e., $X/N$ has no drift under $Q^N$.

We finally also require the following stochastic representation of solutions of partial differential equations.

Theorem 3.8 (Feynmann–Kac). The solution of the partial differential equation
\[
g_t + \mu g_x + \frac{\sigma^2}{2} g_{xx} = 0
\]
with final condition $g(T, x) = h(x)$ has the stochastic representation
\[
g(t, x) = \mathbb{E}(h(X(T)) \bigg\rvert X(t) = x),
\]
where $X$ satisfies the stochastic differential equation
\[
dX(s) = \mu(X(s))ds + \sigma(X(s))dW(s)
\]
with initial condition $X(t) = x$.

3.2. Three Examples of Numéraires

Example 3.9 (The bank account as numéraire). Any traded asset $X$ has drift $r(t)X(t)$ under the risk-neutral measure $Q = Q^B$.

Example 3.10 (The zero-coupon bond as numéraire). Let $Q^S$ be the $S$-forward measure, i.e., the measure associated to the numéraire $P(t, S)$. Then $F(t; T, S)$ is a martingale under $Q^S$.

Example 3.11 (A portfolio of zero-coupon bonds as numéraire). Let $Q^{\alpha, \beta}$ be the measure associated to the numéraire $\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$. Then $S_{\alpha, \beta}(t)$ is a martingale under $Q^{\alpha, \beta}$. 
3. PRICING THEORY

3.3. Pricing Formulas

**Theorem 3.12** (Risk-neutral pricing). For any traded asset $X$, we have

\[ X(t) = \mathbb{E}(X(T)D(t, T)|\mathcal{F}(t)). \]

**Theorem 3.13** (Zero-coupon bond price). We have

\[ P(t, T) = \mathbb{E}(D(t, T)|\mathcal{F}(t)). \]

**Remark 3.14** (Zero-coupon bond price). The price of a zero-coupon bond with maturity $T$ at time $t \in [0, T]$ is equal to the conditional expectation of the discount factor from $T$ to $t$ with respect to $\mathcal{F}(t)$ under the risk-neutral measure.

**Theorem 3.15** ($T$-forward measure pricing). For any traded asset $X$, we have

\[ X(t) = P(t, T)\mathbb{E}^T(X(T)|\mathcal{F}(t)). \]

**Theorem 3.16** (Martingale property of the forward rate). We have

\[ \mathbb{E}^S(F(t; T, S)|\mathcal{F}(u)) = F(u; T, S) \quad \text{for all} \quad 0 \leq u \leq t \leq T \leq S. \]

**Remark 3.17** (Martingale property of the forward rate). Any simply-compounded forward interest rate spanning a time interval ending in $S$ is a martingale under the $S$-forward measure.

**Corollary 3.18.** We have

\[ \mathbb{E}^S(L(T, S)|\mathcal{F}(t)) = F(t; T, S) \quad \text{for all} \quad 0 \leq t \leq T \leq S. \]

**Theorem 3.19** (Expectation of short rate under the forward measure). We have

\[ \mathbb{E}^T(r(T)|\mathcal{F}(t)) = f(t, T) \quad \text{for all} \quad 0 \leq t \leq T. \]

**Remark 3.20** (Expectation of short rate under the forward measure). The expectation of any future instantaneous spot rate under the corresponding forward measure is equal to the related instantaneous forward interest rate.

**Theorem 3.21** (Pricing of European options on zero-coupon bonds). The price of a European call option with maturity $T$, strike $K$, and written on a zero-coupon
bond with maturity $S > T$ is

\[
ZBC(t, T, S, K) = E \left( e^{-\int_t^T r(u)\,du} (P(T, S) - K)^+ | \mathcal{F}(t) \right) \\
= P(t, T) E^T ((P(T, S) - K)^+ | \mathcal{F}(t))
\]

for a call and

\[
ZBP(t, T, S, K) = E \left( e^{-\int_t^T r(u)\,du} (K - P(T, S))^+ | \mathcal{F}(t) \right) \\
= P(t, T) E^T ((K - P(T, S))^+ | \mathcal{F}(t))
\]

for a put.

**Theorem 3.22** (Pricing of caplets and floorlets). The price of a caplet with notional value $N$, cap rate $K$, expiry time $T$, and maturity time $S > T$, is given by

\[
Cpl(t, T, S, N, K) = N' ZBP(t, T, S, K'),
\]

while the price of a floorlet with notional value $N$, floor rate $K$, expiry time $T$, and maturity time $S > T$, is given by

\[
Fll(t, T, S, N, K) = N' ZBC(t, T, S, K'),
\]

where

\[
N' = N(1 + \tau(T, S)K) \quad \text{and} \quad K' = \frac{1}{1 + \tau(T, S)K}.
\]

**Theorem 3.23** (Pricing of caps and floors). The price of a cap with notional value $N$, cap rate $K$, and the set of times $\mathcal{T}$, is given by

\[
Cap(t, \mathcal{T}, N, K) = \sum_{i=\alpha+1}^{\beta} N'_i ZBP(t, T_{i-1}, T_i, K'_{i}),
\]

while the price of a floor with notional value $N$, floor rate $K$, and the set of times $\mathcal{T}$, is given by

\[
Flr(t, \mathcal{T}, N, K) = \sum_{i=\alpha+1}^{\beta} N'_i ZBC(t, T_{i-1}, T_i, K'_{i}),
\]

where

\[
N'_i = N(1 + \tau_i K) \quad \text{and} \quad K'_i = \frac{1}{1 + \tau_i K} \quad \text{for} \quad \alpha + 1 \leq i \leq \beta.
\]
3.4. Two Useful Formulas

**Theorem 3.24.** If $Y$ is normally distributed with $E(Y) = \mu$ and $V(Y) = \sigma^2$, then

$$E(e^Y) = e^{\mu + \frac{\sigma^2}{2}}.$$ 

**Theorem 3.25.** Let $K > 0$. If $Y$ is lognormally distributed such that $E(\ln(Y)) = M$ and $V(\ln(Y)) = V^2$, then

$$E((Y - K)^+) = e^{M + \frac{V^2}{2}} \Phi \left( \frac{M - \ln(K) + V^2}{V} \right) - K \Phi \left( \frac{M - \ln(K)}{V} \right),$$

where $\Phi$ is the cdf of the standard normal distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$