5.1. Ho–Le Model

Definition 5.1 (Ho–Le model). In the Ho–Le model, the short rate is assumed to satisfy the stochastic differential equation
\[ dr(t) = \theta(t)dt + \sigma dW(t), \]
where \( \sigma > 0 \), \( \theta \) is deterministic, and \( W \) is a Brownian motion under the risk-neutral measure.

Theorem 5.2 (Ho–Le model). In the Ho–Le model, we have the following formulas:
\[ r(t) = r(s) + \int_s^t \theta(u)du + \sigma (W(t) - W(s)), \]
\[ E(r(t)|\mathcal{F}(s)) = r(s) + \int_s^t \theta(u)du \quad \text{and} \quad \mathbb{V}(r(t)|\mathcal{F}(s)) = \sigma^2 (t-s), \]
\[ P(t,T) = A(t,T)e^{-r(t)(T-t)}, \]
where \( A(t,T) = \exp \left\{ \frac{\sigma^2}{6} (T-t)^3 - \int_t^T (T-u)\theta(u)du \right\}, \]
\[ dP(t,T) = r(t)P(t,T)dt - \sigma(T-t)P(t,T)dW(t), \]
\[ d \frac{1}{P(t,T)} = \frac{\sigma^2(T-t)^2 - r(t)}{P(t,T)} dt + \frac{\sigma(T-t)}{P(t,T)} dW(t), \]
\[ dW^T(t) = dW(t) + \sigma(T-t)dt, \]
\[ dr(t) = [\theta(t) - \sigma^2(T-t)] dt + \sigma dW^T(t), \]
\[ f(t,T) = r(t) - \frac{\sigma^2}{2} (T-t)^2 + \int_t^T \theta(u)du \quad \text{and} \quad df(t,T) = \sigma dW^T(t), \]
\[ dF(t;T,S) = \sigma \left( F(t;T,S) + \frac{1}{\tau(T,S)} \right) (S-T)dW^S(t), \]
\[ ZBC(t,T,S,K) = P(t,S)\Phi(h) - KP(t,T)\Phi(h-\tilde{\sigma}), \]
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\[ ZBP(t, T, S, K) = KP(t, T)\Phi(-h + \tilde{\sigma}) - P(t, S)\Phi(-h), \]

where \( \tilde{\sigma} = \sigma(S - T)\sqrt{T - t} \) and \( h = \frac{1}{\tilde{\sigma}} \ln \left( \frac{P(t, S)}{P(t, T)K} \right) + \frac{\tilde{\sigma}}{2} \).

\[ \text{Cap}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} \left[ P(t, T_{i-1})\Phi(-h_i + \tilde{\sigma}_i) - (1 + \tau_i K)P(t, T_i)\Phi(-h_i) \right], \]

\[ \text{Flr}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} \left[ (1 + \tau_i K)P(t, T_i)\Phi(h_i) - P(t, T_{i-1})\Phi(h_i - \tilde{\sigma}_i) \right], \]

where \( \tilde{\sigma}_i = \sigma(T_i - T_{i-1})\sqrt{T_{i-1} - t} \) and \( h_i = \frac{1}{\tilde{\sigma}_i} \ln \left( \frac{P(t, T_i)}{P(t, T_{i-1})K} \right) + \frac{\tilde{\sigma}_i}{2} \).

**Theorem 5.3** (Calibration in the Ho–Le model). If the Ho–Le model is calibrated to a given interest rate structure \( \{f^M(0, t) : t \geq 0\} \), i.e.,

\[ f(0, t) = f^M(0, t) \quad \text{for all} \quad t \geq 0, \]

then

\[ \theta(t) = \frac{\partial f^M(0, t)}{\partial t} + \sigma^2 t \quad \text{for all} \quad t \geq 0. \]

**Theorem 5.4** (Zero-coupon bond price in the calibrated Ho–Le model). If the Ho–Le model is calibrated to a given interest rate structure \( \{f^M(0, t) : t \geq 0\} \), then

\[ P(t, T) = e^{-r(t)(T-t)} \frac{P^M(0, T)}{P^M(0, t)} \exp \left\{ (T-t)f^M(0, t) - \frac{\sigma^2}{2} t(T-t)^2 \right\}, \]

where

\[ P^M(0, t) = \exp \left\{ - \int_0^t f^M(0, u)du \right\} \quad \text{for all} \quad t \geq 0. \]

5.2. Hull–White Model (Extended Vasicek Model)

**Definition 5.5** (Short-rate dynamics in the Hull–White model). In the Hull–White model, the short rate is assumed to satisfy the stochastic differential equation

\[ dr(t) = k(\theta(t) - r(t))dt + \sigma dW(t), \]

where \( k, \sigma > 0, \theta \) is deterministic, and \( W \) is a Brownian motion under the risk-neutral measure.
Remark 5.6 (Hull–White model). The Hull–White model is also called the extended Vasicek model or the $G++$ model and can be considered, more generally, with the constants $k$ and $\sigma$ replaced by deterministic functions.

Theorem 5.7 (Short rate in the Hull–White model). Let $0 \leq s \leq t \leq T$. The short rate in the Hull–White model is given by
\[
    r(t) = r(s)e^{-k(t-s)} + k \int_s^t \theta(u)e^{-k(t-u)}du + \sigma \int_s^t e^{-k(t-u)}dW(u)
\]
and is, conditionally on $\mathcal{F}(s)$, normally distributed with
\[
    \mathbb{E}(r(t)|\mathcal{F}(s)) = r(s)e^{-k(t-s)} + k \int_s^t \theta(u)e^{-k(t-u)}du
\]
and
\[
    \mathbb{V}(r(t)|\mathcal{F}(s)) = \frac{\sigma^2}{2k} \left(1 - e^{-2k(t-s)}\right).
\]

Remark 5.8 (Short rate in the Hull–White model). As in the Vasicek model, the short rate $r(t)$ in the extended Vasicek model, for each time $t$, can be negative with positive probability, namely, with probability
\[
    \Phi\left(\frac{-r(0)e^{-kt} + k \int_0^t \theta(u)e^{-k(t-u)}du}{\sqrt{\frac{\sigma^2}{2k}(1 - e^{-2kt})}}\right),
\]
which is often “negligible in practice”. On the other hand, the short rate in the Vasicek model is mean reverting provided
\[
    \varphi^* = \lim_{t \to \infty} \left\{k \int_0^t \theta(u)e^{-k(t-u)}du\right\}
\]
exists, and then
\[
    \mathbb{E}(r(t)) \to \varphi^* \quad \text{as} \quad t \to \infty.
\]

Theorem 5.9 (Zero-coupon bond in the Hull–White model). In the Hull–White model, the price of a zero-coupon bond with maturity $T$ at time $t \in [0, T]$ is given by
\[
    P(t, T) = \bar{A}(t, T)e^{-r(t)B(t, T)},
\]
where
\[
    \bar{A}(t, T) = A(t, T) \exp\left\{-k \int_t^T \theta(u)B(u, T)du\right\}
\]
and $A$ and $B$ are as in the Vasicek model, Theorem 4.4 with $\theta = 0$. 

Theorem 5.10 (Forward rate in the Hull–White model). In the Hull–White model, the instantaneous forward interest rate with maturity $T$ is given by

$$ f(t, T) = k \int_t^T \theta(u)e^{-k(T-u)}du - \frac{\sigma^2}{2}B^2(t, T) + r(t)e^{-k(T-t)}. $$

Theorem 5.11 (Calibration in the Hull–White model). If the Hull–White model is calibrated to a given interest rate structure $\{f^M(0, t) : t \geq 0\}$, then

$$ \theta(t) = f^M(0, t) + \frac{1}{k} \frac{\partial f^M(0, t)}{\partial t} + \frac{\sigma^2}{2k^2} (1 - e^{-2kt}) \text{ for all } t \geq 0. $$

Theorem 5.12 (Zero-coupon bond in the calibrated Hull–White model). If the Hull–White model is calibrated to a given interest rate structure, then

$$ P(t, T) = e^{-r(t)B(t, T)} P^M(0, T) \exp \left\{ B(t, T)f^M(0, t) - \frac{\sigma^2}{4k} (1 - e^{-2kt}) B^2(t, T) \right\}. $$

Theorem 5.13 (Option on a zero-coupon bond in the Hull–White model). In the Hull–White model, the price of a European call option with strike $K$ and maturity $T$ and written on a zero-coupon bond with maturity $S$ at time $t \in [0, T]$ is given by

$$ ZBC(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \tilde{\sigma}), $$

where $\tilde{\sigma}$ and $h$ are as in the Vasicek model, Theorem 4.9.

$$ ZBP(t, T, S, K) = KP(t, T)\Phi(-h + \tilde{\sigma}) - P(t, S)\Phi(-h). $$

Theorem 5.14 (Caps and floors in the Hull–White model). In the Hull–White model, the price of a cap with notional value $N$, cap rate $K$, and the set of times $T$, is given by

$$ \text{Cap}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} \left[ P(t, T_{i-1})\Phi(-h_i + \tilde{\sigma}_i) - (1 + \tau_i K)P(t, T_i)\Phi(-h_i) \right], $$

while the price of a floor with notional value $N$, floor rate $K$, and the set of times $T$, is given by

$$ \text{Flr}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} \left[ (1 + \tau_i K)P(t, T_i)\Phi(h_i) - P(t, T_{i-1})\Phi(h_i - \tilde{\sigma}_i) \right], $$

where $\tilde{\sigma}_i$ and $h_i$ are as in the Vasicek model, Theorem 4.10.
5.3. Black–Karasinski Model

**Definition 5.15** (Black–Karasinski model). In the Black–Karasinski model, the short rate is given by

\[ r(t) = e^{y(t)} \quad \text{with} \quad dy(t) = k(\theta(t) - y(t))dt + \sigma dW(t), \]

where \( k, \sigma > 0 \), \( \theta \) is deterministic, and \( W \) is a Brownian motion under the risk-neutral measure.

**Remark 5.16** (Black–Karasinski model). The Black–Karasinski model is also called the extended exponential Vasicek model and can be considered, more generally, with the constants \( k \) and \( \sigma \) replaced by deterministic functions.

**Theorem 5.17** (Short rate in the Black–Karasinski model). The short rate in the Black–Karasinski model satisfies the stochastic differential equation

\[ dr(t) = \left( k\theta(t) + \sigma^2 \frac{1}{2} - k \ln(r(t)) \right) r(t)dt + \sigma r(t)dW(t). \]

Let \( 0 \leq s \leq t \leq T \). Then \( r \) is given by

\[ r(t) = \exp \left\{ \ln(r(s))e^{-k(t-s)} + k \int_s^t e^{-k(t-u)}\theta(u)du + \sigma \int_s^t e^{-k(t-u)}dW(u) \right\} \]

and is, conditionally on \( \mathcal{F}(s) \), lognormally distributed with

\[ \mathbb{E}(r(t)|\mathcal{F}(s)) = \exp \left\{ \ln(r(s))e^{-k(t-s)} + k \int_s^t e^{-k(t-u)}\theta(u)du + \frac{\sigma^2}{4k} \left(1 - e^{-2k(t-s)}\right) \right\} \]

and

\[ \mathbb{V}(r(t)|\mathcal{F}(s)) = \exp \left\{ 2 \ln(r(s))e^{-k(t-s)} + 2k \int_s^t e^{-k(t-u)}\theta(u)du \times \right\} \]

\[ \times \exp \left\{ \frac{\sigma^2}{2k} \left(1 - e^{-2k(t-s)}\right) \right\} \left[ \exp \left\{ \frac{\sigma^2}{2k} \left(1 - e^{-2k(t-s)}\right) \right\} - 1 \right]. \]

**Remark 5.18** (Short rate in the Black–Karasinski model). Since the short rate \( r \) in the Black–Karasinski model is lognormally distributed, it is always positive. A disadvantage is that \( P(t,T) \) cannot be calculated explicitly. An advantage of the Black–Karasinski model is that \( r \) is always mean reverting provided

\[ \varphi^* = \lim_{t \to \infty} \left\{ k \int_0^t \theta(u)e^{-k(t-u)}du \right\} \]
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exists, and then

\[ \mathbb{E}(r(t) | \mathcal{F}(s)) \to \exp \left( \varphi^* + \frac{\sigma^2}{4k} \right) \quad \text{as} \quad t \to \infty \]

and

\[ \mathbb{V}(r(t) | \mathcal{F}(s)) \to \exp \left( 2\varphi^* + \frac{\sigma^2}{2k} \right) \left[ \exp \left( \frac{\sigma^2}{2k} \right) - 1 \right] \quad \text{as} \quad t \to \infty. \]

5.4. Deterministic-Shift Extended Models

**Definition 5.19 (Short rate in a deterministic-shift extended model).** In a deterministic-shift extended model, the short rate is given by

\[ r(t) = x(t) + \varphi(t) \quad \text{with} \quad dx(t) = \mu(t, x(t))dt + \sigma(t, x(t))dW(t), \]

where \( \varphi, \mu, \sigma \) are deterministic functions and \( W \) is a Brownian motion under the risk-neutral measure. The stochastic differential equation for \( x \) is called the reference model, and prices of zero-coupon bonds and forward interest rates in the reference model are denoted by \( P^\text{REF}_x(T, t) \) and \( f^\text{REF}_x(T, t) \), respectively.

**Theorem 5.20 (Zero-coupon bond in a deterministic-shift extended model).** In a deterministic-shift extended model, the price of a zero-coupon bond with maturity \( T \) at time \( t \in [0, T] \) is given by

\[ P(t, T) = \exp \left( -\int_t^T \varphi(u)du \right) P^\text{REF}_r(T, t). \]

**Theorem 5.21 (Forward rate in a deterministic-shift extended model).** In a deterministic-shift extended model, the instantaneous forward interest rate with maturity \( T \) is given by

\[ f(t, T) = \varphi(T) + f^\text{REF}_r(T, t). \]

**Theorem 5.22 (Calibration in a deterministic-shift extended model).** If a deterministic-shift extended model is calibrated to a given interest rate structure \( \{ f^M(0, t) : t \geq 0 \} \), then

\[ \varphi(t) = f^M(0, t) - f^\text{REF}_r(0, t) \quad \text{for all} \quad t \geq 0. \]
5.5. Extended CIR Model

**Theorem 5.23 (Zero-coupon bond in a calibrated deterministic-shift extended model).** If a deterministic-shift extended model is calibrated to a given interest rate structure, then

\[
P(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \frac{P^{\text{REF}}_r(0, t)}{P^{r \to \varphi}_r(0, T)} P^{r \to \varphi}_r(t, T).
\]

**Theorem 5.24 (Option on a zero-coupon bond in a deterministic-shift extended model).** In a deterministic-shift extended model, the price of a European call option with strike \(K\) and maturity \(T\) and written on a zero-coupon bond with maturity \(S\) at time \(t \in [0, T]\) is given by

\[
ZBC(t, T, S, K) = \exp \left( -\int_t^S \varphi(u)du \right) ZBC^{\text{REF}}_r(t, T, S, K')
\]

where

\[
K' = K \exp \left( \int_t^T \varphi(u)du \right).
\]

5.5. Extended CIR Model

**Definition 5.25 (Short rate in the extended CIR model).** In the extended CIR model, the short rate is given by

\[
r(t) = x(t) + \varphi(t) \quad \text{with} \quad dx(t) = k(\theta - x(t))dt + \sigma \sqrt{x(t)}dW(t),
\]

where \(k, \sigma, \theta > 0\) and \(W\) is a Brownian motion under the risk-neutral measure.

**Remark 5.26 (Extended CIR model).** The extended CIR model is also called the CIR++ model and can be considered, more generally, with the constants \(k\) and \(\sigma\) replaced by deterministic functions.

**Theorem 5.27 (Zero-coupon bond in the CIR++ model).** In the CIR++ model, the price of a zero-coupon bond with maturity \(T\) at time \(t \in [0, T]\) is given by

\[
P(t, T) = \bar{A}(t, T) e^{-r(t)B(t, T)}
\]

where

\[
\bar{A}(t, T) = A(t, T) \exp \left\{ \varphi(t)B(t, T) - \int_t^T \varphi(u)du \right\}
\]

and \(A\) and \(B\) are as in the CIR model, Theorem 4.20.
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Theorem 5.28 (Forward rate in the CIR++ model). In the CIR++ model, the
instantaneous forward interest rate with maturity $T$ is given by

$$f(t, T) = \varphi(T) - \varphi(t)B_T(t, T) + k\theta B(t, T) + r(t)B_T(t, T),$$

where $B$ is as in the CIR model, Theorem 4.20.

Theorem 5.29 (Calibration in the CIR++ model). If the CIR++ model is
calibrated to a given interest rate structure $\{f^M(0, t) : t \geq 0\}$, then

$$\varphi(t) = f^M(0, t) + \varphi(0)B_T(0, T) - k\theta B(0, T) - r(0)B_T(0, T) \quad \text{for all} \quad t \geq 0,$$

where $B$ is as in the CIR model, Theorem 4.20.

Theorem 5.30 (Zero-coupon bond in the CIR++ model). If the CIR++ model
is calibrated to a given interest rate structure, then

$$P(t, T) = \frac{P^M(0, T)}{P^M(0, t)} A(0, t) A(t, T) e^{(r(0) - \varphi(0))(B(0, T) - B(0, t)) + \varphi(t)B(t, T)} e^{-r(t)B(t, T)},$$

where $A$ and $B$ are as in the CIR model, Theorem 4.20.

5.6. Extended Affine Term-Structure Models

Theorem 5.31 (Extended affine term-structure models). Assume the reference
model is an affine term-structure model, i.e.,

$$P^\text{REF}_{r}(t, T) = A(t, T)e^{-r(t)B(t, T)}.$$  

If this model is extended according to Definition 5.19 by using the deterministic
shift $\varphi$, then we have the following formulas:

$$P(t, T) = \bar{A}(t, T)e^{-r(t)B(t, T)},$$

where

$$\bar{A}(t, T) = A(t, T) \exp \left\{ \varphi(t)B(t, T) - \int_t^T \varphi(u)du \right\},$$

$$f(t, T) = \varphi(T) - \varphi(t)B_T(t, T) - \frac{A_T(t, T)}{A(t, T)} + r(t)B_T(t, T),$$

and if the extended model is calibrated to a given interest rate structure, then

$$\varphi(t) = f^M(0, t) + (\varphi(0) - r(0))B_T(0, t) + \frac{A_T(0, t)}{A(0, t)},$$

$$P(t, T) = \frac{P^M(0, T)}{P^M(0, t)} A(0, t) A(t, T) e^{(r(0) - \varphi(0))(B(0, T) - B(0, t)) + \varphi(t)B(t, T)} e^{-r(t)B(t, T)},$$