CHAPTER 7

Heath–Jarrow–Morton Framework

7.1. Heath–Jarrow–Morton Model

Definition 7.1 (Forward-rate dynamics in the HJM model). In the Heath–Jarrow–Morton model, briefly HJM model, the instantaneous forward interest rate with maturity $T$ is assumed to satisfy the stochastic differential equation

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t),$$

where $\alpha$ and $\sigma$ are adapted and $W$ is a Brownian motion under the risk-neutral measure.

Theorem 7.2 (Bond-price dynamics in the HJM model). In the HJM model, the price of a zero-coupon bond with maturity $T$ satisfies the stochastic differential equation

$$dP(t, T) = \left(r(t) + A(t, T) + \frac{1}{2}\Sigma^2(t, T)\right)P(t, T)dt + \Sigma(t, T)P(t, T)dW(t),$$

where

$$A(t, T) = -\int_t^T \alpha(t, u)du \quad \text{and} \quad \Sigma(t, T) = -\int_t^T \sigma(t, u)du.$$

Theorem 7.3 (Bond-price dynamics implying HJM model). If the price of a zero-coupon bond with maturity $T$ satisfies the stochastic differential equation

$$dP(t, T) = m(t, T)P(t, T)dt + v(t, T)P(t, T)dW(t),$$

where $m$ and $v$ are adapted, then the forward-rate dynamics are as in the HJM model with

$$\alpha(t, T) = v(t, T)v_T(t, T) - m_T(t, T) \quad \text{and} \quad \sigma(t, T) = -v_T(t, T).$$
Theorem 7.4 (Drift restriction in the HJM model). In the HJM model, we necessarily have
\[ A(t, T) = -\frac{1}{2} \Sigma^2(t, T) \quad \text{and} \quad \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) \, du. \]

Theorem 7.5 (Bond-price dynamics in the HJM model). In the HJM model, the price of a zero-coupon bond with maturity $T$ satisfies the stochastic differential equations
\[ \frac{dP(t, T)}{P(t, T)} = r(t) \, dt + \Sigma(t, T) \, dW(t) \]
and
\[ \frac{d}{P(t, T)} = \frac{\Sigma^2(t, T) - r(t)}{P(t, T)} \, dt - \frac{\Sigma(t, T)}{P(t, T)} \, dW(t). \]

Theorem 7.6 ($T$-forward measure dynamics of the forward rate in the HJM model). Under the $T$-forward measure $Q^T$, the instantaneous forward interest rate with maturity $T$ in the HJM model satisfies
\[ df(t, T) = \sigma(t, T) \, dW^T(t), \]
where the $Q^T$-Brownian motion $W^T$ is defined by
\[ dW^T(t) = dW(t) - \Sigma(t, T) \, dt. \]

Theorem 7.7 (Forward-rate dynamics in the HJM model). In the HJM model, the simply-compounded forward interest rate for the period $[T, S]$ satisfies the stochastic differential equation
\[ dF(t; T, S) = \left( F(t; T, S) + \frac{1}{\tau(T, S)} \right) (\Sigma(t, T) - \Sigma(t, S)) \, dW^S(t). \]

Theorem 7.8 (Zero-coupon bond in the HJM model). Let $0 \leq t \leq T \leq S$. In the HJM model, the price of a zero-coupon bond with maturity $S$ at time $T$ is given by
\[ P(T, S) = \frac{P(t, S)}{P(t, T)} e^Z, \]
where
\[ Z = -\frac{1}{2} \int_t^T (\Sigma^2(u, S) - \Sigma^2(u, T)) \, du + \int_t^T (\Sigma(u, S) - \Sigma(u, T)) \, dW(u) \]
\[ = -\frac{1}{2} \int_t^T (\Sigma(u, S) - \Sigma(u, T))^2 \, du + \int_t^T (\Sigma(u, S) - \Sigma(u, T)) \, dW^T(u). \]
7.2. Gaussian HJM Model

**Definition 7.9 (Gaussian HJM Model).** A Gaussian HJM model is an HJM model in which \( \sigma \) is a deterministic function.

**Theorem 7.10 (Option on a zero-coupon bond in a Gaussian HJM model).**

In a Gaussian HJM model, the price of a European call option with strike \( K \) and maturity \( T \) and written on a zero-coupon bond with maturity \( S \) at time \( t \in [0, T] \) is given by

\[
Z_{BC}(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \sigma^*),
\]

where

\[
\sigma^* = \sqrt{\int_t^T (\Sigma(u, S) - \Sigma(u, T))^2 \, du}
\]

and

\[
h = \frac{1}{\sigma^*} \ln \left( \frac{P(t, S)}{P(t, T)K} \right) + \frac{\sigma^*}{2}.
\]

The price of a corresponding put option is given by

\[
Z_{BP}(t, T, S, K) = KP(t, T)\Phi(-h + \sigma^*) - P(t, S)\Phi(-h).
\]

**Definition 7.11 (Futures price).** The futures price at time \( t \) of an asset whose value at time \( T \geq t \geq 0 \) is \( X(T) \) is given by

\[
\text{Fut}(t, T) = \mathbb{E}(X(T)|\mathcal{F}(t)).
\]

**Theorem 7.12 (Futures contract on a zero-coupon bond in a Gaussian HJM model).** In a Gaussian HJM model, the price of a futures contract with maturity \( T \) on a zero-coupon bond at time \( T \) with maturity \( S \) is given by

\[
\text{FUT}(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp \left\{ \int_t^T \Sigma(u, T) (\Sigma(u, T) - \Sigma(u, S)) \, du \right\}.
\]

7.3. Ritchken–Sankarasubramanian Model

**Definition 7.13 (HJM model with separable volatility).** An HJM model with separable volatility is an HJM model in which there exist positive functions \( \xi \) and \( \eta \) such that

\[
\sigma(t, T) = \xi(t)\eta(T).
\]
Theorem 7.14 (Zero-coupon bond in an HJM model with separable volatility). In an HJM model with separable volatility, the price of a zero-coupon bond with maturity \( T \) at time \( t \in [0, T] \) is given by

\[
P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ f(0, t)B(t, T) - \frac{1}{2} \phi(t)B^2(t, T) \right\} e^{-r(t)B(t, T)},
\]

where

\[
\phi(t) = \int_t^0 \sigma^2(u, t)du \quad \text{and} \quad B(t, T) = \frac{1}{\eta(t)} \int_t^T \eta(u)du.
\]

Theorem 7.15 (Short-rate dynamics in an HJM model with separable volatility). In an HJM model with separable volatility, the short rate satisfies the stochastic differential equation

\[
\begin{aligned}
\frac{dr(t)}{dt} &= \left\{ \frac{\partial f(0, t)}{\partial t} + \phi(t) \right\} dt + \frac{r(t) - f(0, t)}{\eta(t)} d\eta(t) \\
&\quad + \xi(t)(d\eta(t))(dW(t)) + \sigma(t, t)dW(t),
\end{aligned}
\]

where \( \phi \) is as in Theorem 7.14.

Corollary 7.16 (Short-rate dynamics in a Gaussian HJM model with separable volatility). In an HJM model with separable volatility in which \( \eta \) is deterministic, the short rate satisfies the stochastic differential equation

\[
\begin{aligned}
\frac{dr(t)}{dt} &= \left\{ \frac{\partial f(0, t)}{\partial t} - f(0, t) \frac{\eta'(t)}{\eta(t)} + \phi(t) + r(t) \frac{\eta'(t)}{\eta(t)} \right\} dt + \sigma(t, t)dW(t),
\end{aligned}
\]

where \( \phi \) is as in Theorem 7.14.

Theorem 7.17 (Option on a zero-coupon bond in a Gaussian HJM model with separable volatility). In a Gaussian HJM model with separable volatility, the price of a European call option with strike \( K \) and maturity \( T \) and written on a zero-coupon bond with maturity \( S \) at time \( t \in [0, T] \) is given by

\[
Z_{BC}(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \sigma^*),
\]

where

\[
\sigma^* = B(T, S)\sqrt{\int_t^T \sigma^2(u, T)du} \quad \text{and} \quad h = \frac{1}{\sigma^*} \ln \left( \frac{P(t, S)}{P(t, T)K} \right) + \frac{\sigma^*}{2}
\]

with \( B \) as in Theorem 7.14. The price of a corresponding put option is given by

\[
Z_{BP}(t, T, S, K) = KP(t, T)\Phi(-h + \sigma^*) - P(t, S)\Phi(-h).
\]
Theorem 7.18 (Futures contract on a zero-coupon bond in a Gaussian HJM model with separable volatility). In a Gaussian HJM model with separable volatility, the price of a futures contract with maturity $T$ on a zero-coupon bond at time $T$ with maturity $S$ is given by

$$F(t, T, S) = P(t, S) P(t, T) \exp \left\{ -B(T, S) \int_t^T \sigma(u, u) \sigma(u, T) B(u, T) du \right\}.$$ 

Definition 7.19 (Ritchken–Sankarasubramanian model). The Ritchken–Sankarasubramanian model is an HJM model with separable volatility for which there exist functions $\sigma$ and $k$ such that

$$\xi(t) = \sigma(t) \exp \left\{ \int_0^t k(u) du \right\} \quad \text{and} \quad \eta(t) = \exp \left\{ -\int_0^t k(u) du \right\}.$$ 

Theorem 7.20 (Zero-coupon bond in the Ritchken–Sankarasubramanian model). In the Ritchken–Sankarasubramanian model, the price of a zero-coupon bond with maturity $T$ at time $t \in [0, T]$ is given by

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left\{ f(0, t) B(t, T) - \frac{1}{2} \phi(t) B^2(t, T) \right\} e^{-r(t) B(t, T)},$$

where

$$\phi(t) = \int_0^t \sigma^2(u) \exp \left\{ -2 \int_u^t k(v) dv \right\} du$$

and

$$B(t, T) = \int_t^T \exp \left\{ -\int_t^s k(u) du \right\} ds.$$ 

Theorem 7.21 (Short-rate dynamics in the Ritchken–Sankarasubramanian model). In a Ritchken–Sankarasubramanian model in which $k$ is deterministic and positive, the short rate satisfies the stochastic differential equation

$$dr(t) = \left( k(t) f(0, t) + \frac{\partial f(0, t)}{\partial t} + \phi(t) - k(t) r(t) \right) dt + \sigma(t) dW(t)$$

with $\phi$ as in Theorem 7.20.

Definition 7.22 (Gaussian HJM model with exponentially damped volatility). A Gaussian HJM model with exponentially damped volatility is a Ritchken–Sankarasubramanian model in which the functions $\sigma$ and $k$ are positive constants.
Theorem 7.23 (The Gaussian HJM model with exponentially damped volatility and the Hull–White model). Suppose \( r \) is the short rate in a Gaussian HJM model with exponentially damped volatility. Then \( r \) is equal to the short rate in the corresponding calibrated Hull–White model.

Remark 7.24. Since for a Gaussian HJM model with exponentially damped volatility we have

\[
\sigma(t, T) = \sigma e^{-k(T-t)}, \quad B(t, T) = \frac{1 - e^{-k(T-t)}}{k},
\]

\[
\int_t^T \sigma^2(u, T) du = \frac{\sigma^2}{2k} \left( 1 - e^{-2k(T-t)} \right), \quad \phi(t) = \frac{\sigma^2}{2k} \left( 1 - e^{-2kt} \right),
\]

we may use Theorem 7.23 to show that

- Theorem 7.20 implies Theorem 5.12;
- Theorem 7.17 implies Theorem 5.13;
- Theorem 7.18 implies for the Hull–White model

\[
\text{FUT}(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( - \frac{\sigma^2}{2} B(T, S) B^2(t, T) \right).
\]

Definition 7.25 (Gaussian HJM model with constant volatility). A Gaussian HJM model with constant volatility is a Ritchken–Sankarasubramanian model in which \( \sigma \) is a positive constant and \( k = 0 \).

Theorem 7.26 (The Gaussian HJM model with constant volatility and the Ho–Le model). Suppose \( r \) is the short rate in a Gaussian HJM model with constant volatility. Then \( r \) is equal to the short rate in the corresponding calibrated Ho–Le model.

Remark 7.27. Since for a Gaussian HJM model with constant volatility we have

\[
\sigma(t, T) = \sigma, \quad B(t, T) = T - t, \quad \int_t^T \sigma^2(u, T) du = \sigma^2(T - t), \quad \phi(t) = \sigma^2 t,
\]

we may use Theorem 7.26 to show that

- Theorem 7.20 implies Theorem 5.4;
- Theorem 7.17 implies the formula for ZBC from Theorem 5.2;
- Theorem 7.18 implies for the Ho–Le model

\[
\text{FUT}(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( - \frac{\sigma^2}{2} (S - T)(T - t)^2 \right).
\]
7.4. Mercurio–Moraleda Model

**Definition 7.28** (Gaussian HJM model with volatility depending on time to maturity). A Gaussian HJM model with volatility depending on time to maturity is an HJM model in which there exists a deterministic function $h$ such that

$$\sigma(t, T) = h(T - t).$$

**Theorem 7.29** (Option on a zero-coupon bond in a Gaussian HJM model with volatility depending on time to maturity). In a Gaussian HJM model with volatility depending on time to maturity, the price of a European call option with strike $K$ and maturity $T$ and written on a zero-coupon bond with maturity $S$ at time $t \in [0, T]$ is given by

$$ZBC(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \sigma^*),$$

where

$$\sigma^* = \sqrt{\int_0^\tau \left(\int_u^{u+\mu} h(x)dx\right)^2 du \quad \text{with} \quad \tau = T - t \quad \text{and} \quad \mu = S - T}$$

and

$$h = \frac{1}{\sigma^*} \ln \left(\frac{P(t, S)}{P(t, T)K}\right) + \frac{\sigma^*}{2}.$$ 

The price of a corresponding put option is given by

$$ZBP(t, T, S, K) = KP(t, T)\Phi(-h + \sigma^*) - P(t, S)\Phi(-h).$$

**Theorem 7.30** (Futures contract on a zero-coupon bond in a Gaussian HJM model with volatility depending on time to maturity). In a Gaussian HJM model with volatility depending on time to maturity, the price of a futures contract with maturity $T$ on a zero-coupon bond at time $T$ with maturity $S$ is given by

$$\text{FUT}(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp \left\{ \int_0^\tau \left(\int_0^u h(x)dx\right) \left(\int_u^{u+\mu} h(x)dx\right) du \right\}$$

with $\tau$ and $\mu$ as in Theorem 7.29.

**Definition 7.31** (Mercurio–Moraleda model). The Mercurio–Moraleda model is a Gaussian HJM model with volatility depending on time to maturity for which there exist constants $\sigma, \gamma, \lambda > 0$ such that

$$h(x) = \sigma(1 + \gamma x)e^{-\frac{\lambda}{2} x}.$$
**Theorem 7.32** (Option on a zero-coupon bond in the Mercurio–Moraleda model). In the Mercurio–Moraleda model, the price of a European call option with strike $K$ and maturity $T$ and written on a zero-coupon bond with maturity $S$ at time $t \in [0, T]$ is given by

$$Z_{BC}(t, T, S, K) = P(t, S)\Phi(h) - KP(t, T)\Phi(h - \sigma^*),$$

where

$$\sigma^* = \frac{2\sigma}{\lambda^{7/2}} \sqrt{(\alpha^2\lambda^2 + 2\alpha\beta\lambda + 2\beta^2)(1 - e^{-\lambda\tau}) - \lambda\beta\tau(2\alpha\lambda + 2\beta + \beta\lambda\tau)e^{-\lambda\tau}}$$

and

$$h = \frac{1}{\sigma^*} \ln \left( \frac{P(t, S)}{P(t, T)K} \right) + \frac{\sigma^*}{2}$$

with

$$\alpha = (\lambda + 2\gamma)(1 - e^{-\frac{1}{2}\mu}) - \gamma\lambda e^{-\frac{1}{2}\mu}, \quad \beta = \gamma\lambda(1 - e^{-\frac{1}{2}\mu})$$

and $\tau$ and $\mu$ are as in Theorem 7.29. The price of a corresponding put option is given by

$$Z_{BP}(t, T, S, K) = KP(t, T)\Phi(-h + \sigma^*) - P(t, S)\Phi(-h).$$

**Theorem 7.33** (Futures contract on a zero-coupon bond in the Mercurio–Moraleda model). In the Mercurio–Moraleda model, the price of a futures contract with maturity $T$ on a zero-coupon bond at time $T$ with maturity $S$ is given by

$$FUT(t, T, S) = \frac{P(t, S)}{P(t, T)} \exp \left( \frac{4\sigma^2}{\lambda^2} z \right)$$

with

$$z = \frac{\alpha_0\lambda^2 + \alpha_0\beta\lambda + \alpha_0\beta_0\lambda + 2\beta_0\beta_0}{\lambda^3} (e^{-\lambda\tau} - 1) + \frac{\alpha_0\beta\lambda + \beta_0\alpha\lambda + 2\beta_0\tau e^{-\lambda\tau}}{\lambda^2} + \frac{\beta_0\tau e^{-\lambda\tau}}{\lambda} + \frac{2\alpha_0(\alpha\lambda + 2\beta)}{\lambda^2} \left( 1 - e^{-\frac{1}{2}\tau} \right) - \frac{2\beta_0}{\lambda} \tau e^{-\frac{1}{2}\tau}$$

where $\alpha, \beta, \tau, \mu$ are as in Theorem 7.32 and

$$\alpha_0 = \lambda + 2\gamma \quad \text{and} \quad \beta_0 = \gamma\lambda.$$