CHAPTER 8

Market Models

8.1. Lognormal Forward-LIBOR Model

REMARK 8.1. Recall from Example 3.10 that \( F(t; T, S) \) is a martingale under \( Q^S \).

DEFINITION 8.2 (Forward-rate dynamics in the LFM model). In the lognormal forward-LIBOR model, briefly LFM model, the simply-compounded forward interest rate for the period \([T, S]\) is assumed to satisfy the stochastic differential equation

\[
dF(t; T, S) = \sigma(t; T, S)F(t; T, S)dW^S(t),
\]

where \( \sigma \) is deterministic and \( W^S \) is a Brownian motion under the \( S \)-forward measure.

THEOREM 8.3 (Pricing of caplets in the LFM model). In the LFM model, the price of a caplet with nominal value \( N \), cap rate \( K \), expiry time \( T \), and maturity time \( S \) is given by

\[
Cpl(t, T, S, N, K) = NP(t, S)\tau(T, S) \text{Bl}(K, F(t; T, S), \sqrt{\int_t^T \sigma^2(u; T, S)du}),
\]

where

\[
\text{Bl}(K, F, v) = F\Phi\left(\frac{\ln \left(\frac{F}{K}\right) + \frac{v^2}{2}}{v}\right) - K\Phi\left(\frac{\ln \left(\frac{F}{K}\right) - \frac{v^2}{2}}{v}\right).
\]

THEOREM 8.4 (Pricing of caps in the LFM model). In the LFM model, the price of a cap with nominal value \( N \), cap rate \( K \), and the set of times \( T \), is given by

\[
\text{Cap}(t, T, N, K) = N \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i \text{Bl}(K, F_i(t), \sqrt{\int_t^{T_i-1} \sigma_i^2(u)du}),
\]

where for \( \alpha + 1 \leq i \leq \beta \)

\[
F_i(t) = F(t; T_{i-1}, T_i), \quad \tau_i = \tau(T_{i-1}, T_i), \quad \sigma_i(t) = \sigma(t; T_{i-1}, T_i).
\]
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Theorem 8.5 (Brownian motions in the LFM model under different forward measures). Let $S > T$. Let $W^T$ and $W^S$ be a $Q^T$-Brownian motion and a $Q^S$-Brownian motion, respectively. In the LFM model, we then have

$$dW^S(t) - dW^T(t) = \frac{\tau(T,S)F(t;T,S)\sigma(t)}{1 + \tau(T,S)F(t;T,S)}dt.$$ 

Theorem 8.6 (Forward-rate dynamics in the LFM model). Let $i, k \in \{\alpha, \ldots, \beta\}$. In the LFM model, $F_k$ satisfies the following stochastic differential equations:

(i) If $k = i$, then

$$dF_k(t) = \sigma_k(t)F_k(t)dW^T_i(t).$$

(ii) If $k > i$, then

$$dF_k(t) = \sigma_k(t)F_k(t)dW^T_i(t) + \left(\sum_{j=i+1}^{k} \tau_j \sigma_j(t)\sigma_k(t)F_j(t)F_k(t)\frac{1}{1 + \tau_j F_j(t)}\right)dt.$$ 

(iii) If $k < i$, then

$$dF_k(t) = \sigma_k(t)F_k(t)dW^T_i(t) - \left(\sum_{j=k+1}^{i} \tau_j \sigma_j(t)\sigma_k(t)F_j(t)F_k(t)\frac{1}{1 + \tau_j F_j(t)}\right)dt.$$ 

8.2. Lognormal Forward-Swap Model

Remark 8.7. Recall from Example 3.11 that $S_{\alpha, \beta}(t)$ is a martingale under $Q^{\alpha, \beta}$.

Definition 8.8 (Forward swap rate dynamics in the LSM model). In the lognormal forward-swap model, briefly LSM model, the forward swap rate for $T = \{T_\alpha, \ldots, T_\beta\}$ is assumed to satisfy the stochastic differential equation

$$dS_{\alpha, \beta}(t) = \sigma_{\alpha, \beta}(t)S_{\alpha, \beta}(t)dW^{\alpha, \beta}(t),$$

where $\sigma$ is deterministic and $W^{\alpha, \beta}$ is a Brownian motion under $Q^{\alpha, \beta}$.

Theorem 8.9 (Pricing of swaptions in the LSM model). In the LSM model, the price of a European payer swaption with swaption maturity $T = T_\alpha$ on an IRS depending on the nominal value $N$, the fixed rate $K$, and the set of times $T$ is given by

$$PS(t, T, N, K) = N \left(\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)\right)\text{Bl} \left(K, S_{\alpha, \beta}(t), \sqrt{\int_t^T \sigma_{\alpha, \beta}^2(u)du}\right),$$