## Contents

Chapter 0. Preliminaries 1
  0.1. Sets 1
  0.2. Functions 1
  0.3. Proofs 2

Chapter 1. The Real Number System 3
  1.1. The Field Axioms 3
  1.2. The Positivity Axioms 3
  1.3. The Completeness Axiom 4
  1.4. The Archimedian Property 5
  1.5. Some Inequalities and Identities 6

Chapter 2. Sequences of Real Numbers 9
  2.1. The Convergence of Sequences 9
  2.2. Monotone Sequences 10

Chapter 3. Continuous Functions 11

Chapter 4. Differentiation 13
  4.1. The Algebra of Derivatives 13
  4.2. The Mean Value Theorems 14
  4.3. Applications of the Mean Value Theorems 15

Chapter 5. Integration 17
  5.1. The Definition of the Integral 17
  5.2. The Fundamental Theorem of Calculus 18
  5.3. Applications 18
  5.4. Improper Integrals 19

Chapter 7. Infinite Series of Functions 21
  7.1. Uniform Convergence 21
  7.2. Interchanging of Limit Processes 22
0.1. Sets

Definition 0.1 (Cantor). A set is a collection of certain distinct objects which are called elements of the set.

Notation 0.2. The following notation will be used throughout this class.

1. \( x \in A \) (or \( x \notin A \)): \( x \) is an element (or is not an element) of the set \( A \);
2. \( A \subseteq B \): \( A \) is a subset of \( B \), i.e., if \( x \in A \), then \( x \in B \) (or: \( x \in A \implies x \in B \));
3. \( \emptyset \): empty set. We have \( \emptyset \subseteq A \) for all sets \( A \);
4. \( A = \{a,b,c\} \): \( A \) consists of the elements \( a, b, \) and \( c \);
5. \( A = \{x : x \text{ has the property } P\} \): \( A \) consists of all elements \( x \) that have the property \( P \);
6. \( A \cup B := \{x : x \in A \text{ or } x \in B\} \): union of \( A \) and \( B \);
7. \( A \cap B := \{x : x \in A \text{ and } x \in B\} \): intersection of \( A \) and \( B \);
8. \( A \setminus B := \{x : x \in A \text{ and } x \notin B\} \): difference of \( A \) and \( B \);
9. \( A \times B := \{(a,b) : a \in A \text{ and } b \in B\} \): Cartesian product of \( A \) and \( B \) (with \( (a,b) = (c,d) \iff a = c \text{ and } b = d \));
10. \( A^c := \{x : x \notin A\} \): complement of \( A \) (with respect to a given set \( X \));
11. \( P(A) := \{B : B \subseteq A\} \): power set of \( A \);
12. \( A = B \iff A \subseteq B \text{ and } B \subseteq A \).

0.2. Functions

Definition 0.3. Let \( X \) and \( Y \) be nonempty sets. A function (or mapping) \( f \) from \( X \) to \( Y \) is a correspondence that associates with each point \( x \in X \) in a unique way a \( y \in Y \); we write \( y = f(x) \). \( X \) is called the domain of \( f \) while \( Y \) is called the range of \( f \). We write \( f : X \to Y \).

Remark 0.4. Two functions \( f : X \to Y \) and \( g : U \to V \) are called equal (we write \( f = g \)) if \( f = g \) iff \( X = U \) and \( Y = V \) and \( f(x) = g(x) \) for all \( x \in X \).

Notation 0.5. Let \( f : X \to Y \) be a function. Then we call

(i) \( G(f) := \{(x,f(x)) : x \in X\} \subseteq X \times Y \) the graph of \( f \);
(ii) \( f(A) := \{f(x) : x \in A\} \subseteq Y \) the image of a set \( A \subseteq X \);
(iii) \( f^{-1}(B) := \{x \in X : f(x) \in B\} \subseteq X \) the inverse image of a set \( B \subseteq Y \).

Definition 0.6. Let \( f : X \to Y \) be a function. Then \( f \) is called

(i) one-to-one if \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \) always implies \( f(x_1) \neq f(x_2) \);
(ii) onto if \( f(X) = Y \) (i.e., \( \forall y \in Y \exists x \in X : f(x) = y \)).
(iii) invertible if $f$ is both one-to-one and onto, and then $f^{-1} : Y \to X$ is called the inverse function of $f$ provided $f^{-1}(y) = x$ if $x$ is the unique solution of $f(x) = y$.

**Definition 0.7.** Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the function $g \circ f : X \to Y$ with $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is called the composite function of $f$ and $g$.

**Proposition 0.8.** A function $f : X \to Y$ is invertible iff there exists a function $g : Y \to X$ with
\[
(f \circ g)(y) = y \ \forall y \in Y \quad \text{and} \quad (g \circ f)(x) = x \ \forall x \in X,
\]
and then $g$ is the inverse function of $f$, i.e., $g = f^{-1}$.

## 0.3. Proofs

Let $P$ and $Q$ be two statements.

(i) To prove the implication $P \implies Q$, i.e., the theorem with assumption $P$ and conclusion $Q$: We say “if $P$ is true, then $Q$ is true”. Or “$P$ is sufficient for $Q$”. Or “$Q$ is necessary for $P$”. The implication $P \implies Q$ is logically equivalent to the contraposition $\neg Q \implies \neg P$ (negations).

(a) **direct proof**: assume $P$ and show $Q$;

(b) **indirect proof**: do a direct proof with the contraposition;

(c) **proof by contradiction**: assume $P$ and $\neg Q$ and derive a statement that contradicts a true statement.

(ii) To prove the equivalence $P \iff Q$, show $P \implies Q$ and $Q \implies P$. Another possibility is to introduce “intermediate” statements $P_1, P_2, \ldots, P_n$ and to prove $P \iff P_1 \iff P_2 \iff \ldots \iff P_n \iff Q$. 
CHAPTER 1

The Real Number System

1.1. The Field Axioms

Definition 1.1. Let $K$ be a set with at least two elements, and let $+,\cdot : K \times K \to K$ be two functions that we call addition and multiplication. We say that $K$ is a field provided the following field axioms are satisfied:

(F1) $a + b = b + a\ \forall a, b \in K$ (commutativity of $+$);
(F2) $(a + b) + c = a + (b + c)\ \forall a, b, c \in K$ (associativity of $+$);
(F3) $\exists 0 \in K : a + 0 = 0 + a = a\ \forall a \in K$ (additive identity);
(F4) $\forall a \in K \exists b \in K : a + b = 0$ (additive inverse);
(F5) $a \cdot b = b \cdot a\ \forall a, b \in K$ (commutativity of $\cdot$);
(F6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)\ \forall a, b, c \in K$ (associativity of $\cdot$);
(F7) $\exists 1 \in K : a \cdot 1 = 1 \cdot a = a\ \forall a \in K$ (multiplicative identity);
(F8) $\forall a \in K \setminus \{0\} \exists b \in K : a \cdot b = 1$ (multiplicative inverse);
(F9) $(a + b) \cdot c = a \cdot c + b \cdot c\ \forall a, b, c \in K$ (distributive property).

Notation 1.2. $ab := a \cdot b, a^2 := a \cdot a, a + b + c := (a + b) + c, abc := (ab)c, b$ from $(F_4)$ is denoted as $-a, b$ from $(F_8)$ is denoted as $a^{-1}, a : b = \frac{a}{b} := ab^{-1}$ if $b \neq 0, a - b := a + (-b)$.

Proposition 1.3. Let $K$ be a field and $a, b \in K$. Then

(i) $\exists x \in K : a + x = b$, namely $x = b - a$;
(ii) $-(a) = a$;
(iii) $-(a + b) = -a - b$.

Proposition 1.4. Let $K$ be a field and $a, b \in K$ with $a \neq 0$. Then

(i) $\exists x \in K : ax = b$, namely $x = \frac{b}{a}$;
(ii) $(a^{-1})^{-1} = a$;
(iii) $(ab)^{-1} = b^{-1}a^{-1}$ provided $b \neq 0$.

Proposition 1.5. Let $K$ be a field and $a, b, c, d \in K$. Then

(i) $ab = 0 \iff a = 0$ or $b = 0$;
(ii) $(-1)a = -a$ and $(-a)(-b) = ab$;
(iii) $\frac{a}{2} + \frac{a}{2} = \frac{a+bc}{2}$ and $\frac{a}{2} \cdot \frac{a}{2} = \frac{ac}{2}$ provided $bd \neq 0$;
(iv) $\frac{a}{2} : \frac{a}{2} = \frac{a}{2} \cdot \frac{a}{2}$ provided $bcd \neq 0$.

Proposition 1.6. In a field we always have $1 \neq 0$.

1.2. The Positivity Axioms

Definition 1.7. Let $K$ be a field. We call $K$ ordered if there exists a set $\mathcal{P} \subseteq K$ that satisfies the positivity axioms:
1. The Real Number System

\( (P_1) \) \( a, b \in \mathcal{P} \implies a + b, ab \in \mathcal{P}; \)
\( (P_2) \) \( \forall a \in \mathcal{K} \) either \( a \in \mathcal{P} \) or \( -a \in \mathcal{P} \) or \( a = 0 \) (trichotomy).

**Notation 1.8.** We write \( a > b \) if \( a - b \in \mathcal{P} \), \( a < b \) if \( b - a \in \mathcal{P} \), \( a \geq b \) if \( a > b \) or \( a = b \), \( a \leq b \) if \( a < b \) or \( a = b \).

**Proposition 1.9.** Let \( \mathcal{K} \) be an ordered field and \( a, b, c, d \in \mathcal{K} \). Then

(i) \( a^2 > 0 \) if \( a \neq 0 \);
(ii) \( 1 > 0 \);
(iii) \( a > 0 \implies a^{-1} > 0 \);
(iv) \( a < b \) and \( b < c \implies a < c \) (transitivity);
(v) \( a < b \) and \( c > 0 \implies ac < bc \);
(vi) \( a < b \) and \( c < 0 \implies ac > bc \);
(vii) \( 0 < a < b \implies 0 < \frac{1}{b} < \frac{1}{a} \);
(viii) \( a > 0 \) and \( b > 0 \implies ab > 0 \);
(ix) \( a = b \iff a \leq b \) and \( b \leq a \);
(x) \( 0 < a < b \iff 0 < a^2 < b^2 \).

**Remark 1.10.** There is no ordered field \( \mathcal{K} \) such that \( a^2 = -1 \) for some \( a \in \mathcal{K} \).

**Proposition 1.11.** Let \( \mathcal{K} \) be an ordered field and \( a, b \in \mathcal{K} \) with \( a < b \). Then there exists \( c \in \mathcal{K} \) with \( a < c < b \), e.g., \( c := \frac{a + b}{2} \), where \( 2 := 1 + 1 \).

**Definition 1.12.** Let \( \mathcal{K} \) be an ordered field and \( T \subset \mathcal{K} \) with \( T \neq \emptyset \). An \( m \in T \) is called minimum (or maximum) of \( T \) provided \( m \leq t \) (or \( m \geq t \)) for all \( t \in T \). We write \( m = \min T \) (or \( m = \max T \)).

**Proposition 1.13.** Let \( \mathcal{K} \) be an ordered field and \( T \subset \mathcal{K} \) with \( T \neq \emptyset \). If \( \min T \) (or \( \max T \)) exists, then it is uniquely determined.

**Notation 1.14.** If \( \mathcal{K} \) is an ordered field and \( a, b \in \mathcal{K} \), then we put

\( (a, b) := \{ x \in \mathcal{K} : a < x < b \} \),
\( [a, b] := \{ x \in \mathcal{K} : a \leq x \leq b \} \),
\( (a, b] := \{ x \in \mathcal{K} : a < x \leq b \} \),

and

\( [a, b) := \{ x \in \mathcal{K} : a \leq x < b \} \),

**Example 1.15.** For \( T = (0, 1] \) we have \( \max T = 1 \) and \( \min T \) does not exist.

1.3. The Completeness Axiom

**Definition 1.16.** Let \( \mathcal{K} \) be an ordered field and \( T \subset \mathcal{K} \). We call

(i) \( s \in \mathcal{K} \) an upper (or lower) bound of \( T \) if \( t \leq s \) (or \( t \geq s \)) for all \( t \in T \);
(ii) \( T \) bounded above (or bounded below) if it has an upper (or lower) bound;
(iii) \( T \) bounded if it is bounded above and below;
(iv) \( s = \sup T \) the supremum of \( T \) (and the infimum \( \inf T \) analogously) if
   \( a \) s is an upper bound of \( T \) and
   \( b \) s \leq \tilde{s} \) for all upper bounds \( \tilde{s} \) of \( T \).
**Proposition 1.17.** Let $K$ be an ordered field and $T \subset K$. Then

(i) $m = \max T$ exists $\iff s = \sup T$ exists and $s \in T$, and then $s = m$;
(ii) $s = \sup T \iff s \geq t \forall t \in T$ and $\forall \epsilon > 0 \exists t \in T : t > s - \epsilon$.

**Definition 1.18.** (Definition of $\mathbb{R}$). (i) An ordered field is called **complete** if $\sup T \in K$ exists whenever $T \subset K$ is a nonempty set that is bounded above (completeness axiom).
(ii) An ordered field that is complete is called the **field of the real numbers**. We denote it by $\mathbb{R}$.

**Theorem 1.19.** $\forall \epsilon > 0 \exists s \geq 0 : s^2 = c$ (we denote this $s$ by $\sqrt{c}$).

**Definition 1.20.** (Definition of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$). (i) A set $M \subset \mathbb{R}$ is called **inductive** provided that

- (a) $1 \in M$ and
- (b) $x \in M \implies x + 1 \in M$.

(ii) The intersection of all inductive subsets of $\mathbb{R}$ is called the set of the **natural numbers**. We denote it by $\mathbb{N}$. Also, we put $N_0 = \mathbb{N} \cup \{0\}$.

(iii) $\mathbb{Z} := \{ a : a \in N_0 \text{ or } -a \in N_0 \}$ is called the set of **integers**.

(iv) $\mathbb{Q} := \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \setminus \{0\} \right\}$ is called the set of **rational numbers**. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of **irrational numbers**.

**Proposition 1.21.** (Properties of $\mathbb{N}$). (i) $\mathbb{N}$ is inductive;

(ii) $\mathbb{N} \subset M$ whenever $M \subset \mathbb{R}$ is inductive;

(iii) if $A \subset \mathbb{N}$ and $A$ is inductive, then $A = \mathbb{N}$.

**Remark 1.22.** We call $m$ **even** if $m = 2k$ for some $k \in \mathbb{Z}$ and $m$ **odd** if $m = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \text{ even, } q \text{ odd or } p \text{ odd, } q \neq 0 \text{ even or } p \text{ odd, } q \text{ odd} \right\}.$$  

**Theorem 1.23.** $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$, more precisely, $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

**Theorem 1.24.** Let $\emptyset \neq S \subset \mathbb{R}$ be bounded below. Then $\inf S$ exists.

**Theorem 1.25.** (Dedekind Gap Theorem). Let $\emptyset \neq S, T \subset \mathbb{R}$ and suppose $s \leq t \forall s \in S \forall t \in T$. Then $\sup S$ and $\inf T$ exist and $\sup S \leq \inf T$. Also

$$\exists \epsilon \in \mathbb{R} : s \leq c \leq t \forall s \in S \forall t \in T \iff \sup S = \inf T \iff \forall \epsilon > 0 \exists s \in S \exists t \in T : t - s < \epsilon.$$  

### 1.4. The Archimedean Property

**Theorem 1.26.** (Principle of Mathematical Induction). Let $S(n)$ be some statement for each $n \in \mathbb{N}$. If

- (i) $S(1)$ is true and
- (ii) $S(k)$ is true $\implies S(k + 1)$ is true $\forall k \in \mathbb{N}$,

then $S(n)$ is true for all $n \in \mathbb{N}$.

**Example 1.27.** $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ holds for all $n \in \mathbb{N}$.

**Proposition 1.28.** Let $m, n \in \mathbb{N}$. Then $m + n \in \mathbb{N}$ and $m \cdot n \in \mathbb{N}$.
1. THE REAL NUMBER SYSTEM

Proposition 1.29. Let \( n \in \mathbb{N} \). Then \((n,n+1) \cap \mathbb{N} = \emptyset\).

Proposition 1.30. Let \( n \in \mathbb{N} \) and \( A \subset \{1, \ldots, n\} \) be nonempty. Then \( \max A \) and \( \min A \) exist.

Theorem 1.31 (The Well-Ordering Principle). Let \( \emptyset \neq A \subset \mathbb{N} \). Then \( \min A \) exists.

Theorem 1.32 (The Archimedean Property). For all \( c \in \mathbb{R} \) there exists \( n \in \mathbb{N} \) such that \( n > c \).

Corollary 1.33. For all \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \).

Proposition 1.34. Let \( a \in \mathbb{R} \). Then \( \lfloor a \rfloor := \max\{n \in \mathbb{Z} : n \leq a\} \) exists and we have \( \lfloor a \rfloor \leq a < \lfloor a \rfloor + 1 \).

Definition 1.35. A set \( S \subset \mathbb{R} \) is called dense in \( \mathbb{R} \) provided \((a,b) \cap S \neq \emptyset\) for all \( a, b \in \mathbb{R} \) with \( a < b \).

Theorem 1.36. Both \( \mathbb{Q} \) and \( \mathbb{R} \setminus \mathbb{Q} \) are dense in \( \mathbb{R} \).

1.5. Some Inequalities and Identities

Notation 1.37. Let \( m, n \in \mathbb{Z} \) and \( a_k \in \mathbb{R} \) for \( k \in \mathbb{Z} \). We put

\[
\sum_{k=m}^{n} a_k = \begin{cases} 0 & \text{if } n < m \\ \sum_{k=m}^{n-1} a_k + a_n & \text{if } n = m \\ \sum_{k=m}^{n} a_k & \text{if } n > m \end{cases}
\]

and

\[
\prod_{k=m}^{n} a_k = \begin{cases} 1 & \text{if } n < m \\ \prod_{k=m}^{n-1} a_k a_n & \text{if } n = m \\ \prod_{k=m}^{n} a_k & \text{if } n > m. \end{cases}
\]

The following rules are clear:

(i) \( \sum_{k=m}^{n} a_k = \sum_{\nu=m}^{n} a_{\nu} \);

(ii) \( \sum_{k=m}^{n} a_k = \sum_{k=m+p}^{n+p} a_{k-p} \) for all \( p \in \mathbb{Z} \);

(iii) \( \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} a_{n+m-k} \);

(iv) \( \sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k \);

(v) \( \sum_{k=m}^{n} 1 = n - m + 1 \) if \( n \geq m \).

Example 1.38. \( \sum_{k=m}^{n} \Delta a_k \) is called a telescoping sum, where \( \Delta a_k := a_{k+1} - a_k \) is called the forward difference operator. We have \( \sum_{k=m}^{n} \Delta a_k = a_{n+1} - a_m \).

Definition 1.39. Let \( n \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{R} \). Then we define \( n! \) (read “\( n \) factorial”) and the binomial coefficient \( \binom{\alpha}{n} \) (read “\( \alpha \) choose \( n \)”) by

\[
n! := \prod_{k=1}^{n} k \quad \text{and} \quad \binom{\alpha}{n} := \frac{\prod_{k=1}^{n-1} (\alpha + 1 - k)}{n!}.
\]

Proposition 1.40. Let \( m, n \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{R} \). Then

(i) \( \binom{\alpha}{n} + \binom{\alpha}{n+1} = \binom{\alpha+1}{n+1} \);
1.5. SOME INEQUALITIES AND IDENTITIES

(ii) $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ if $m \geq n$ (and 0 if $m < n$);
(iii) $\binom{m}{n} = \binom{m}{m-n}$ if $m \geq n$.

Definition 1.41. Let $a \in \mathbb{R}$. We define $a^0 = 1$, $a^1 = a$, and $a^{n+1} = a^n a$ for each $n \in \mathbb{N}$. If $-n \in \mathbb{N}$, then we put $a^n = \left(\frac{1}{a}\right)^{-n}$.

Proposition 1.42. Let $a, b \in \mathbb{R} \setminus \{0\}$ and $p, q \in \mathbb{Z}$. Then

(i) $a^p a^q = a^{p+q}$;
(ii) $(a^p)^q = a^{pq}$;
(iii) $(ab)^p = a^p b^p$.

Theorem 1.43 (The Binomial Formula). Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Example 1.44. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$.

Theorem 1.45 (Finite Geometric Series). Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1} - 1}{a - 1} \quad \text{if } a \neq 1 \quad \text{and} \quad a^{n+1} - b^{n+1} = (a - b) \sum_{k=0}^{n} a^k b^{n-k}.$$

Theorem 1.46 (Bernoulli’s Inequality). Let $n \in \mathbb{N}$ and $x \geq -1$. Then we have

$$(1 + x)^n \geq 1 + nx.$$

Definition 1.47. Let $x \in \mathbb{R}$. Then the absolute value of $x$ is defined by

$$|x| := \max\{x, -x\}.$$

Proposition 1.48. Let $a, b \in \mathbb{R}$. Then

(i) $|a| = |-a|$;
(ii) $|a| \geq 0$; and $|a| = 0 \iff a = 0$;
(iii) $|ab| = |a||b|$;
(iv) $a = 0 \iff |a| < \varepsilon \forall \varepsilon > 0$.

Theorem 1.49 (Triangle Inequalities). If $a, b \in \mathbb{R}$, then

$$||a| - |b|| \leq |a+b| \leq |a| + |b|.$$

Remark 1.50. Define $d(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. Then

(i) $d(x, y) = d(y, x)$;
(ii) $d(x, y) \geq 0$; and $d(x, y) = 0 \iff x = y$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
Sequences of Real Numbers

2.1. The Convergence of Sequences

Definition 2.1. If $x : N \to \mathbb{R}$ is a function, then we call $x$ a sequence (of real numbers). Instead of $x(n)$ we rather write $x_n$, $n \in N$. The sequence $s$ defined by $s_n = \sum_{k=1}^{n} x_k$, $n \in N$, is also known as a series.

Example 2.2. (i) $a_n = 1 + (-1)^n$; 
(ii) $a_n = \max\{k \in N : k \leq \sqrt{n^3}\}$; 
(iii) $a_n = \sum_{k=1}^{n} \frac{1}{k}$.

Definition 2.3. A sequence $a$ is said to be convergent if
$$\exists \alpha \in \mathbb{R} \forall \epsilon > 0 \exists N \in N \forall n \geq N : |a_n - \alpha| < \epsilon.$$ We write $\alpha = \lim_{n \to \infty} a_n$ or $a_n \to \alpha$ (as $n \to \infty$). A sequence is called divergent if it is not convergent.

Proposition 2.4. Any sequence has at most one limit.

Example 2.5. (i) $a_n = \frac{2n}{4n+3}$; 
(ii) $a_n = (-1)^n$.

Proposition 2.6 (Some Limits). We have

(i) If $a_n = \alpha$ for all $n \in N$, then $\lim_{n \to \infty} a_n = \alpha$;
(ii) $\lim_{n \to \infty} \frac{1}{n} = 0$;
(iii) if $|x| < 1$, then $\lim_{n \to \infty} x^n = 0$;
(iv) if $|x| < 1$, then $\lim_{n \to \infty} \sum_{k=0}^{n} x^k = \frac{1}{1-x}$.

Definition 2.7. A sequence $a$ is called bounded (or bounded above, or bounded below) if the set $\{a_n : n \in N\}$ is bounded (or bounded above, or bounded below).

Proposition 2.8 (Necessary Conditions for Convergence). Let $a$ be a convergent sequence. Then

(i) $a$ is bounded;
(ii) $a$ satisfies the Cauchy Condition, i.e.,
$$\forall \epsilon > 0 \exists N \in N : \forall m, n \geq N |a_n - a_m| < \epsilon.$$

Remark 2.9. $a_n \to \alpha$ implies $a_{n+1} - a_n \to 0$, $a_{2n} - a_n \to 0$.

Example 2.10. (i) $a_n = (-1)^n$; 
(ii) $a_n = \sum_{k=1}^{n} \frac{1}{k}$ (the harmonic series).

Theorem 2.11. Suppose $a_n \to \alpha$ and $b_n \to \beta$ as $n \to \infty$. Then

(i) $|a_n| \to |\alpha|$;
Example 2.12. (i) \( a_n \to a, m \in \mathbb{N} \implies a_n^m \to a^m \); 
(ii) \( \frac{n^2 - 3}{2n + 3n} \to \frac{1}{2} \) as \( n \to \infty \).

Theorem 2.13. Suppose \( a_n \to a, b_n \to b, c_n \in \mathbb{R} \). Then 
(i) \( \exists K \in \mathbb{R} \forall n \in \mathbb{N} \exists N \geq n : a_n \leq K \implies \alpha \leq K \); 
(ii) \( \forall n \in \mathbb{N} \forall N \geq n : a_N \leq b_N \implies \alpha \leq \beta \); 
(iii) \( \alpha = \beta \) and \( \exists N \in \mathbb{N} \forall n \geq N : a_n \leq c_n \leq b_n \implies \lim_{n \to \infty} c_n = \alpha \).

2.2. Monotone Sequences

Definition 2.14. A sequence \( a \) is called monotonically increasing (or monotonically decreasing, strictly increasing, strictly decreasing) provided \( a_n \leq a_{n+1} (a_n \geq a_{n+1}) \), \( a_n < a_{n+1}, a_n > a_{n+1} \) holds for all \( n \in \mathbb{N} \). We write \( a_n \nearrow (\searrow, \uparrow, \downarrow) \). The sequence is called monotone if it is either one of the above.

Theorem 2.15 (The Monotone Convergence Theorem). A monotone sequence converges iff it is bounded.

Example 2.16. (i) \( s_n = \sum_{k=1}^{n} \frac{1}{k} \); 
(ii) \( a_n = \frac{1}{n} - \frac{1}{n^2} \); 
(iii) \( a_n = (1 + \frac{1}{n})^n \). We denote the limit of this sequence by \( e \).

Definition 2.17. Let \( a_n \) be a sequence and let \( n_k \) be a sequence of natural numbers that is strictly increasing. Then the sequence \( b_k \) defined by \( b_k = a_{n_k} \) for \( k \in \mathbb{N} \) is called a subsequence of the sequence \( a_k \).

Theorem 2.18. Every sequence has a monotone subsequence.

Theorem 2.19 (Bolzano–Weierstraß). Let \( a, b \in \mathbb{R} \) with \( a < b \). Every sequence in \( [a, b] \) has a convergent subsequence that has its limit in \( [a, b] \).

Proposition 2.20. Let \( a_n \) be a convergent sequence with \( \lim_{n \to \infty} a_n = \alpha \). Then every subsequence \( a_{n_k} \) of \( a_n \) converges with \( \lim_{k \to \infty} a_{n_k} = \alpha \).

Example 2.21. (i) \( a_n \to \alpha \implies a_{2n} \to \alpha, a_{n+1} \to \alpha \); 
(ii) \( (1 + \frac{1}{n})^{2n}, (1 + \frac{1}{n^2})^{n^2} \); 
(iii) \( (-1)^n (1 + \frac{1}{n}) \).

Theorem 2.22 (The Nested Interval Theorem). Let \( a_n, b_n \in \mathbb{R} \) with \( a_n < b_n \) for all \( n \in \mathbb{N} \), put \( I_n = [a_n, b_n] \), and assume \( I_{n+1} \subseteq I_n \) for all \( n \in \mathbb{N} \) and \( b_n - a_n \to 0 \) as \( n \to \infty \). Then \( \bigcap_{n \in \mathbb{N}} I_n = \{ \alpha \} \) with \( \alpha \in \mathbb{R} \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \alpha \).
CHAPTER 3

Continuous Functions

Definition 3.1. A function \( f : D \to \mathbb{R} \) is said to be continuous at \( x_0 \in D \) provided
\[
\{ x_n : n \in \mathbb{N} \} \subset D, \quad \lim_{n \to \infty} x_n = x_0 \implies \lim_{n \to \infty} f(x_n) = f(x_0).
\]
Also, \( f \) is called continuous if it is continuous at each \( x_0 \in D \).

Example 3.2. (i) \( f(x) = x^2 + 3x - 2, \ x \in \mathbb{R} \);
(ii) \( f(x) = \sqrt{x}, \ x \geq 0 \);
(iii) \( f = \chi_{[0,1]} \);
(iv) \( f = \chi_{\mathbb{Q}} \) is called the Dirichlet function.

Notation 3.3. For two functions \( f, g : D \to \mathbb{R} \) we define the sum \( f + g : D \to \mathbb{R} \) and the product \( f \cdot g : D \to \mathbb{R} \) by \((f + g)(x) = f(x) + g(x)\) and \((f \cdot g)(x) = f(x)g(x)\) for \( x \in D \). If \( g(x) \neq 0 \) for all \( x \in D \), then \( \frac{f}{g} : D \to \mathbb{R} \) is defined by \((\frac{f}{g})(x) = \frac{f(x)}{g(x)}\) for \( x \in D \).

Theorem 3.4. Let \( f, g : D \to \mathbb{R} \) be continuous functions. Then \( f+g, f \cdot g : D \to \mathbb{R} \) are continuous. If \( g(x) \neq 0 \) for all \( x \in D \), then \( \frac{f}{g} : D \to \mathbb{R} \) is continuous.

Corollary 3.5. Let \( m \in \mathbb{N}, \ c_k \in \mathbb{R} \ (0 \leq k \leq m) \), and \( p : \mathbb{R} \to \mathbb{R} \) be defined by \( p(x) = \sum_{k=0}^{m} c_k x^k \), i.e., \( p \) is a polynomial with degree \( m \) if \( c_m \neq 0 \). Then \( p \) is continuous. Also, if \( p, q \) are both polynomials and \( D = \{ x \in \mathbb{R} : q(x) \neq 0 \} \), then the rational function \( \frac{p}{q} : D \to \mathbb{R} \) is continuous.

Theorem 3.6. If \( f : D \to \mathbb{R}, \ g : U \to \mathbb{R} \) are functions with \( f(D) \subset U \) such that \( f \) is continuous at \( x_0 \in D \) and \( g \) is continuous at \( f(x_0) \in U \), then \( g \circ f : D \to \mathbb{R} \) is continuous at \( x_0 \in D \).

Example 3.7. \( \sqrt{1-x^2}, \ x \in [-1,1] \).

Theorem 3.8. Let \( f : [a, b] \to \mathbb{R} \) be continuous, where \( a, b \in \mathbb{R} \) with \( a < b \). Assume \( f(a) < 0 \) and \( f(b) > 0 \). Then \( \exists \alpha \in (a, b) : f(\alpha) = 0 \).

Theorem 3.9 (The Intermediate Value Theorem). Let \( f : [a, b] \to \mathbb{R} \) be continuous, where \( a, b \in \mathbb{R} \) with \( a < b \). If \( f(a) < c < f(b) \) or \( f(b) < c < f(a) \), then \( \exists \alpha \in (a, b) : f(\alpha) = c \).

Example 3.10. (i) \( h(x) = x^5 + x + 1, \ x \in \mathbb{R} \), has a zero in \((-2,0)\);
(ii) \( h(x) = \frac{1}{\sqrt{1+x^2}} - x^2, \ x \in \mathbb{R} \), has a zero in \((0,1)\);
(iii) if \( I \subset \mathbb{R} \) is an interval and \( f : I \to \mathbb{R} \) is continuous, then \( f(I) \) is an interval.

Theorem 3.11 (The Extreme Value Theorem). Let \( f : I = [a, b] \to \mathbb{R} \) be continuous, where \( a, b \in \mathbb{R} \) with \( a < b \). Then both \( \max f(I) \) and \( \min f(I) \) exist.
Definition 3.12. Let \( D \subset \mathbb{R} \). The function \( f : D \rightarrow \mathbb{R} \) is called strictly increasing (or strictly decreasing, increasing, decreasing) if \( f(v) > f(u) \) (or \( f(v) < f(u) \)), \( f(v) \geq f(u) \), \( f(v) \leq f(u) \) holds for all \( u, v \in D \) with \( u < v \). We write \( f \uparrow \) (\( \downarrow \), \( \nearrow \), \( \searrow \)). Also, \( f \) is called strictly monotone if it is either strictly increasing or strictly decreasing.

Theorem 3.13. Let \( f : I \rightarrow f(I) \) be continuous and strictly monotone, where \( I \) is an interval. Then \( f \) is invertible and \( f^{-1} : f(I) \rightarrow \mathbb{R} \) is continuous and strictly monotone.

Corollary 3.14. Suppose \( I \) is an interval and \( f : I \rightarrow \mathbb{R} \) is strictly monotone. Then \( f \) is continuous iff \( f(I) \) is an interval.

Theorem 3.15. Let \( x_0 \in D \subset \mathbb{R} \) and \( f : D \rightarrow \mathbb{R} \). Then \( f \) is continuous at \( x_0 \) iff \( \forall \varepsilon > 0 \exists \delta > 0 : |f(x) - f(x_0)| < \varepsilon \forall x \in D : |x - x_0| < \delta \).

Example 3.16. (i) \( f(x) = x^3 \), \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at \( x_0 = 2 \);
(ii) \( f \) from (i) is continuous on \( D = [0, 20] \);
(iii) \( f(x) = \frac{1}{x} \), \( f : (0, 1) \rightarrow \mathbb{R} \).

Definition 3.17. Let \( D \subset \mathbb{R} \) and \( f : D \rightarrow \mathbb{R} \). Then \( f \) is called uniformly continuous (on \( D \)) if \( \forall \varepsilon > 0 \exists \delta > 0 : |f(u) - f(v)| < \varepsilon \forall u, v \in D : |u - v| < \delta \).

Theorem 3.18. Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous, where \( a, b \in \mathbb{R} \) with \( a < b \). Then \( f \) is uniformly continuous.
CHAPTER 4

Differentiation

4.1. The Algebra of Derivatives

Definition 4.1. (i) An $x_0 \in \mathbb{R}$ is called a limit point of $D$ if there exists $\{x_n : n \in \mathbb{N}\} \subset D \setminus \{x_0\}$ with $\lim_{n \to \infty} x_n = x_0$.

(ii) We write $\lim_{x \to x_0, x \in D} f(x) = l$ provided $x_0$ is a limit point of $D$ and $\lim_{n \to \infty} f(x_n) = l$ whenever $\{x_n : n \in \mathbb{N}\} \subset D \setminus \{x_0\}$ with $\lim_{n \to \infty} x_n = x_0$.

Remark 4.2. (i) Let $x_0 \in D$ be a limit point of $D$. Then $f : D \to \mathbb{R}$ is continuous at $x_0$ iff $\lim_{x \to x_0} f(x) = f(x_0)$.

(ii) If $x_0$ is a limit point of $D$ and $f, g : D \to \mathbb{R}$ with $\lim_{x \to x_0} f(x) = \alpha \in \mathbb{R}$ and $\lim_{x \to x_0} g(x) = \beta \in \mathbb{R}$, then (by Theorem 2.11)

$$\lim_{x \to x_0} ((f + g)(x)) = \alpha + \beta, \quad \lim_{x \to x_0} ((fg)(x)) = \alpha \beta,$$

and (if $\beta \neq 0$)

$$\lim_{x \to x_0} ((f/g)(x)) = \alpha/\beta.$$

Definition 4.3. Let $x_0 \in (a, b) = I$. A function $f : I \to \mathbb{R}$ is called differentiable at $x_0$ provided

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, in which case we denote this limit by $f'(x_0)$. Also, $f$ is called differentiable (on $I$) if $f'(x)$ exists for all $x \in I$. In this case, $f' : I \to \mathbb{R}$ is called the derivative of $f$.

Example 4.4. (i) $f(x) = 4x - 5$;

(ii) $f(x) = mx + b$;

(iii) $f(x) = x^2$;

(iv) $f(x) = |x|$.

Proposition 4.5. Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^m$ for all $x \in \mathbb{R}$. Then $f$ is differentiable and $f'(x) = mx^{m-1}$.

Proposition 4.6. Let $x_0 \in (a, b) = I$. If $f : I \to \mathbb{R}$ is differentiable at $x_0$, then it is continuous at $x_0$.

Theorem 4.7 (Rules of Differentiation). Let $x_0 \in (a, b) = I$. 

4. DIFFERENTIATION

(i) If $f, g : I \to \mathbb{R}$ are differentiable in $x_0$, then so is $\alpha f + \beta g$ for all $\alpha, \beta \in \mathbb{R}$, $fg$, and (if $g(x_0) \neq 0$) $f/g$ with

$$
\begin{align*}
(\alpha f + \beta g)'(x_0) &= \alpha f'(x_0) + \beta g'(x_0), \\
(fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \text{Product Rule}, \\
(f/g)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} \quad \text{Quotient Rule}.
\end{align*}
$$

(ii) If $g : I \to \mathbb{R}$ is differentiable in $x_0$ and if $f : J \to \mathbb{R}$ with $J \supset g(I)$ is differentiable in $g(x_0)$, then $f \circ g : I \to \mathbb{R}$ is differentiable in $x_0$ with

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0) \quad \text{Chain Rule}.$$ 

(iii) If $f : I \to f(I)$ is continuous and strictly monotone and differentiable in $x_0$ with $f'(x_0) \neq 0$, then $f^{-1} : f(I) \to I$ is differentiable in $y_0 = f(x_0)$ with

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1})(y_0)}.$$ 

Example 4.8. (i) $f(x) = \sqrt{x}$; (ii) $f(x) = x^{p/q}$. 

4.2. The Mean Value Theorems

Definition 4.9. Let $I \subset \mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function.

(i) An $x_0 \in I$ is called a **local maximizer** (or local minimizer) of $f$, if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ (or $f(x) \leq f(x_0)$) for all $x \in I$ with $|x - x_0| < \delta$.

(ii) An $x_0 \in I$ for which $f'(x_0)$ exists is called a **critical point** of $f$ if $f'(x_0) = 0$.

Theorem 4.10. Let $f : I \to \mathbb{R}$, $x_0 \in I \subset \mathbb{R}$, and assume that $f'(x_0)$ exists. If $x_0$ is a local maximizer (or minimizer) of $f$, then it is a critical point.

Theorem 4.11 (Rolle’s Theorem). Suppose that $f : [a, b] \to \mathbb{R}$ with $a < b$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Assume $f(a) = f(b) = 0$. Then there exists a critical point of $f$ in $(a, b)$.

Theorem 4.12 (The Lagrange Mean Value Theorem). Suppose that $f : [a, b] \to \mathbb{R}$ with $a < b$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then

$$
\exists x_0 \in (a, b) : f'(x_0) = \frac{f(b) - f(a)}{b - a}.
$$

Theorem 4.13 (The Cauchy Mean Value Theorem). Suppose that $f, g : [a, b] \to \mathbb{R}$ with $a < b$ both are continuous on $[a, b]$ and differentiable on $(a, b)$. Then

$$
\exists x_0 \in (a, b) : f'(x_0) \{g(b) - g(a)\} = g'(x_0) \{f(b) - f(a)\}.
$$

Theorem 4.14 (The Darboux Mean Value Theorem). Suppose that $f : [a, b] \to \mathbb{R}$ with $a < b$ is differentiable on $[a, b]$. If $c$ is between $f'(a)$ and $f'(b)$, then there exists $x_0 \in (a, b)$ with $f'(x_0) = c$. 

4.3. Applications of the Mean Value Theorems

Theorem 4.15 (The Identity Criterion). Let \( I \subset \mathbb{R} \) be an interval and suppose that \( f : I \to \mathbb{R} \) is differentiable on \( I \). Then \( f \) is constant on \( I \) (i.e., there exists \( c \in \mathbb{R} \) such that \( f(x) = c \) for all \( x \in I \)) if and only if \( f'(x) = 0 \) for all \( x \in I \).

Theorem 4.16. Let \( I \) be an interval and \( f : I \to \mathbb{R} \) be differentiable on \( I \).

(i) If \( f'(x) > 0 \) for all \( x \in I \), then \( f \) is strictly increasing on \( I \).
(ii) If \( f'(x) < 0 \) for all \( x \in I \), then \( f \) is strictly decreasing on \( I \).

Example 4.17. (i) \( e \) and \( l \) are strictly increasing;
(ii) \( f(x) = \frac{ax+b}{ex+d} \);
(iii) \( f(x) = \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q} \\ x + x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \)

Notation 4.18. If \( I \) is an interval and \( f : I \to \mathbb{R} \) is differentiable with \( f' : I \to \mathbb{R} \), and \( f' : I \to \mathbb{R} \) is also differentiable, then we write \( f'' = (f')' = f^{(2)} \). If \( f^{(k)} \) for \( k \in \mathbb{N} \) is defined and differentiable, we put \( f^{(k+1)} = (f^{(k)})' \). Also, we put \( f^{(0)} = f \).

Theorem 4.19. Let \( I \) be an interval, \( n \in \mathbb{N} \), and suppose \( f : I \to \mathbb{R} \) has \( n \) derivatives. If \( f^{(k)}(x_0) = 0 \) for all \( 0 \leq k \leq n-1 \) for some \( x_0 \in I \), then, for each \( x \in I \setminus \{x_0\} \), there exists a point \( z \) strictly between \( x \) and \( x_0 \) with

\[
 f(x) = \frac{f^{(n)}(z)}{n!} (x-x_0)^n.
\]

Theorem 4.20. Let \( I \) be an interval and suppose \( f : I \to \mathbb{R} \) has a third derivative. Assume \( x_0 \in I \) is a critical point of \( f \).

(i) If \( f''(x_0) > 0 \), then \( x_0 \) is a local minimizer of \( f \).
(ii) If \( f''(x_0) < 0 \), then \( x_0 \) is a local maximizer of \( f \).

Theorem 4.21 (L'Hôpital’s Rules). Let \( I = [a, b) \subset \mathbb{R} \), \( a < b \), \( b \in \mathbb{R} \) or \( b = \infty \), and suppose that \( f, g : I \to \mathbb{R} \) are differentiable on \( I \) with \( g'(x) \neq 0 \) for all \( x \in I \).

Assume that \( \alpha = \lim_{x \to b^-} f(x) g'(x) \) exists. If either

\[
 \lim_{x \to b^-} \frac{f(x)}{g'(x)} = \text{finite},
\]

then \( \lim_{x \to b^-} f(x) g'(x) \) exists and is equal to \( \alpha \).

Example 4.22. (i) \( \lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}}}{x} = 1; \)
(ii) \( \lim_{x \to 0} \frac{e^{(1+x)x} - x}{x} = -\frac{1}{2}; \)
(iii) \( \lim_{x \to \infty} \frac{e^{(1+x)x}}{x} = \infty; \)
(iv) \( \lim_{x \to 0, x>0} x l(x) = 0; \)
(v) \( \lim_{x \to 0, x>0} A(x, x) = 1. \)

Definition 4.23. Let \( I \) be an open interval containing \( x_0 \) and \( n \in \mathbb{N}_0 \). Suppose that \( f : I \to \mathbb{R} \) has \( n \) derivatives. The \( n \)th Taylor polynomial for the function \( f : I \to \mathbb{R} \) at the point \( x_0 \) is defined as

\[
p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k.
\]

Example 4.24. Find \( p_3 \) for \( f(x) = 1/x \) at \( x_0 = 1 \).
Theorem 4.25 (Lagrange Remainder Theorem). Let $I$ be an open interval containing the point $x_0$ and let $n \in \mathbb{N}_0$. Suppose that $f : I \to \mathbb{R}$ has $n + 1$ derivatives. Then for each $x \in I$, there exists $z$ strictly between $x$ and $x_0$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n)}(z)}{(n+1)!} (x - x_0)^{n+1}.$$ 

Theorem 4.26. Let $I$ be an open interval containing $x_0$ and suppose $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose there are positive numbers $r$ and $M$ such that $[x_0 - r, x_0 + r] \subset I$ and $|f^{(n)}(x)| \leq M^n$ for all $x \in [x_0 - r, x_0 + r]$. Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad \text{if} \quad |x - x_0| \leq r.$$ 

Example 4.27. $e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Note also that $e \in \mathbb{R} \setminus \mathbb{Q}$. 
CHAPTER 5

Integration

5.1. The Definition of the Integral

Definition 5.1. Let \( f : [a, b] \to \mathbb{R} \) with \( a < b \) be a function. If \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \), then \( Z = \{x_0, x_1, \ldots, x_n\} \) is called a partition of the interval \([a, b]\) with gap \( |Z| = \max\{x_k - x_{k-1} : 1 \leq k \leq n\} \), and if \( \xi_k \in [x_{k-1}, x_k] \) for all \( 1 \leq k \leq n \), then we call \( \xi = (\xi_1, \xi_2, \cdots, \xi_n) \) intermediate points of the partition \( Z \).

The sum

\[
S(f, Z, \xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})
\]

is called a Riemann sum. If \( \xi \) is such that \( f(\xi_k) = \inf f([x_{k-1}, x_k]) \) for all \( 1 \leq k \leq n \) (or \( f(\xi_k) = \sup f([x_{k-1}, x_k]) \) for all \( 1 \leq k \leq n \), then we call \( L(f, Z, \xi) = S(f, Z, \xi) \) the lower Darboux sum (or \( U(f, Z) = S(f, Z, \xi) \) the upper Darboux sum).

Definition 5.2. A function \( f : [a, b] \to \mathbb{R} \) with \( a < b \) is said to be Riemann integrable if \( \lim_{n \to \infty} S(f, Z_n, \xi^n) \) exists for any sequence of partitions \( Z_n \) with \( \lim_{n \to \infty} \|Z_n\| = 0 \) and with intermediate points \( \xi^n \).

Remark 5.3. If \( f : [a, b] \to \mathbb{R} \) is Riemann integrable, then, no matter what sequences \( Z_n \) and \( \xi^n \) we take, the limit of \( S(f, Z_n, \xi^n) \) as \( n \to \infty \) is always the same. We then call this limit \( \int_a^b f(x) dx = \int_a^b f \).

Example 5.4. \( f(x) = 2x + 3 \), \( I = [0, a] \).

Proposition 5.5. Let \( a < b \) and \( I = [a, b] \).

(i) If \( f, g : I \to \mathbb{R} \) are Riemann integrable, then so is \( \alpha f + \beta g \) for all \( \alpha, \beta \in \mathbb{R} \) with

\[
\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.
\]

(ii) If \( f(x) = c \) for all \( x \in I \), then \( f \) is Riemann integrable with \( \int_a^b f = c(b - a) \).

(iii) If \( f, g : I \to \mathbb{R} \) are Riemann integrable and \( f(x) \leq g(x) \) for all \( x \in I \), then \( \int_a^b f \leq \int_a^b g \).

(iv) If \( f : I \to \mathbb{R} \) is Riemann integrable, then \( f \) is bounded on \( I \) and

\[
\inf f(I) \leq \frac{\int_a^b f}{b - a} \leq \sup f(I).
\]

(v) If \( f : I \to \mathbb{R} \) is Riemann integrable and if \( g(x) = f(x) \) for all \( x \in I \) but a finite number of points \( x \in I \), then \( g \) is Riemann integrable and \( \int_a^b f = \int_a^b g \).
(vi) If \( c \in (a,b) \) and \( f : I \to \mathbb{R} \) and \( f : [a,c] \to \mathbb{R}, f : [c,b] \to \mathbb{R} \) are Riemann integrable, then
\[
\int_a^b f = \int_a^c f + \int_c^b f.
\]

**Theorem 5.6.** If \( f : [a,b] \to \mathbb{R} \) with \( a < b \) is continuous, then it is Riemann integrable.

**Notation 5.7.** If \( a > b \), then we put \( \int_b^a f = -\int_a^b f \). We also put \( \int_a^a f = 0 \).

### 5.2. The Fundamental Theorem of Calculus

**Theorem 5.8.** (Fundamental Theorem of Calculus, First Part). Suppose \( F : [a,b] \to \mathbb{R} \) is differentiable on \( [a,b] \) and \( F' : [a,b] \to \mathbb{R} \) is Riemann integrable on \( [a,b] \). Then
\[
\int_a^b F' = F(b) - F(a).
\]

**Definition 5.9.** A function \( F : I \to \mathbb{R} \) is called an antiderivative of \( f : I \to \mathbb{R} \) if \( F \) is differentiable with \( F'(x) = f(x) \) for all \( x \in I \).

**Example 5.10.**
(i) \( \int_0^5 x^3 dx = \frac{5^4}{4} \);
(ii) \( \int_0^1 e = e(4) - 1 \);
(iii) \( \int_0^1 s = 1 \);
(iv) \( \frac{1}{\pi^2} \sum_{k=1}^n k^5 \to \frac{1}{6} \) as \( n \to \infty \).

**Theorem 5.11.** (Fundamental Theorem of Calculus, Second Part). Let \( f : I \to \mathbb{R} \) be continuous on the interval \( I \subset \mathbb{R} \) and let \( a \in I \). Then
\[
F(x) := \int_a^x f \quad \text{for each} \quad x \in I
\]
is an antiderivative of \( f \).

**Proposition 5.12.** If \( f \) is Riemann integrable on \( I \), then \( F \) defined in the FTOC (Part II) is continuous (even Lipschitz continuous) on \( I \).

**Example 5.13.**
\[
\int_1^x \frac{dt}{t}; \quad \int_0^x \frac{dt}{1+t^2}.
\]

### 5.3. Applications

**Theorem 5.14.** Suppose \( f,g : I \to \mathbb{R} \) are continuous, \( x_0 \in I \), \( y_0 \in \mathbb{R} \). Then there exists exactly one continuously differentiable function \( y \) with \( y(x_0) = y_0 \) and \( y'(x) = f(x)g(x) + g(x) \) for all \( x \in I \), namely
\[
y(x) = e(F(x)) \left\{ y_0 + \int_{x_0}^x g(t)e(-F(t))dt \right\} \quad \text{with} \quad F(x) = \int_{x_0}^x f(t)dt.
\]

**Example 5.15.** \( xy' + 2y = 4x^2 \), \( y(1) = 2 \).
Theorem 5.16 (Integration by Parts). Let \( f, g : [a, b] \to \mathbb{R} \) be continuously differentiable. Then
\[
\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.
\]

Example 5.17. \( \int_0^1 te^{t} dt = 1. \)

Theorem 5.18 (Substitution). If \( g : [\alpha, \beta] \to \mathbb{R} \) is continuously differentiable, \( f : g([\alpha, \beta]) \to \mathbb{R} \) continuous, and \( g(\alpha) = a, \ g(\beta) = b, \) then
\[
\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(g(t))g'(t)dt.
\]

Example 5.19. \( \int_0^2 e^{\sqrt{x}} dx = 2(\sqrt{2} - 1)e^{\sqrt{2}} + 2. \)

5.4. Improper Integrals

Definition 5.20. Let \( a < b \) and \( f : (a, b) \to \mathbb{R}. \)

(i) \( f \) is said to be locally integrable on \((a, b)\) if \( f \) is integrable on each closed subinterval \([c, d] \subset (a, b). \)

(ii) \( f \) is said to be improperly integrable on \((a, b)\) if \( f \) is locally integrable on \((a, b)\) and if
\[
\int_a^b f(x)dx := \lim_{c \to a^+, d \to b^-} \int_c^d f(x)dx
\]
exists and is finite. This limit is called the improper Riemann integral of \( f \) over \((a, b). \)

Example 5.21. (i) \( \int_0^1 \frac{1}{\sqrt{x}} dx = 2; \)

(ii) \( \int_1^{\infty} \frac{1}{x^2} dx = 1. \)

Theorem 5.22. If \( f, g \) are improperly integrable on \((a, b)\) and \( \alpha, \beta \in \mathbb{R}, \) then \( \alpha f + \beta g \) is improperly integrable on \((a, b), \) and
\[
\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.
\]

Theorem 5.23 (Comparison Theorem). Suppose \( f, g : (a, b) \to \mathbb{R} \) are locally integrable. If \( 0 \leq f(x) \leq g(x) \) for all \( x \in (a, b), \) and if \( g \) is improperly integrable on \((a, b), \) then \( f \) is improperly integrable on \((a, b)\) with
\[
\int_a^b f(x)dx \leq \int_a^b g(x)dx.
\]

Example 5.24. (i) \( |s(x)/\sqrt{x^3}| \) is improperly integrable on \((0, 1]; \)

(ii) \( |l(x)/\sqrt{x^5}| \) is improperly integrable on \([1, \infty). \)

Definition 5.25. Let \( a < b \) and \( f : (a, b) \to \mathbb{R}. \)

(i) \( f \) is said to be absolutely integrable on \((a, b)\) if \(|f|\) is improperly integrable on \((a, b). \)

(ii) \( f \) is said to be conditionally integrable on \((a, b)\) if \( f \) is improperly integrable but not absolutely integrable on \((a, b). \)
**Theorem 5.26.** If $f$ is absolutely integrable on $(a, b)$, then $f$ is improperly integrable on $(a, b)$, and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$ 

**Example 5.27.** $s(x)/x$ is conditionally integrable on $[1, \infty)$. 

CHAPTER 7

Infinite Series of Functions

7.1. Uniform Convergence

Example 7.1. (i) \( \lim_{x \to x_0} \lim_{n \to \infty} (1 + x/n)^n = \lim_{n \to \infty} \lim_{x \to x_0} (1 + x/n)^n; \)
(ii) \( \frac{d}{dx} \lim_{n \to \infty} (1 + x/n)^n = \lim_{n \to \infty} \frac{d}{dx} (1 + x/n)^n; \)
(iii) \( \int_0^1 \lim_{n \to \infty} (1 + x/n)^n \, dx = \lim_{n \to \infty} \int_0^1 (1 + x/n)^n \, dx; \)
(iv) \( f_n(x) = x^n; \)
(v) \( f_n(x) = s(nx)^n. \)

Definition 7.2. Let \( f_n : I \to \mathbb{R} \) be functions for each \( n \in \mathbb{N} \) and let \( f : I \to \mathbb{R}. \)
We say that the sequence \( f_n \) converges
(i) \( \text{pointwise} \) to \( f \) if \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in I; \)
(ii) \( \text{uniformly} \) to \( f \) if
\[ \forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall n \geq N \forall x \in I) |f_n(x) - f(x)| < \varepsilon. \]
The pointwise or uniform convergence of the series \( \sum_{k=0}^{\infty} g_k \) is defined as above with \( f_n = \sum_{k=0}^{n} g_k. \)

Example 7.3. Let \( f_n(x) = x^n \) on \([0, 1].\)
(i) \( f_n \) converges uniformly on \([0, 1/2]; \)
(ii) \( f_n \) does not converge uniformly on \([0, 1]. \)

Example 7.4. \( f_n(x) = 2n^2x/(1 + n^4x^4) \) is not uniformly convergent on \( \mathbb{R}. \)

Theorem 7.5 (Cauchy Criterion). A sequence of functions \( f_n : I \to \mathbb{R} \) converges
uniformly on \( I \) iff
\[ \forall \varepsilon > 0 \exists N \in \mathbb{N} : (\forall m, n \geq N \forall x \in I) |f_n(x) - f_m(x)| < \varepsilon. \]

Theorem 7.6 (Weierstrass M-Test). Suppose \( g_k : I \to \mathbb{R} \) satisfies \( |g_k(x)| \leq M_k \) for all \( x \in I \) and for all \( k \in \mathbb{N} \) such that \( \sum_{k=1}^{\infty} M_k \) is convergent. Then \( \sum_{k=1}^{\infty} g_k(x) \) is uniformly convergent.

Example 7.7. \( \sum_{k=1}^{\infty} \frac{e^{k^2x^2}}{k^2} \) is uniformly convergent on \( \mathbb{R}. \)

Proposition 7.8. Suppose \( f_n : I = [a, b] \to \mathbb{R} \) are continuous on \( I \) for all \( n \in \mathbb{N}. \)
Then \( f_n \) is uniformly convergent on \( I \) iff for each convergent sequence \( \{x_n\} \subset [a, b] \) the limit of \( f(x_n) \) as \( n \to \infty \) exists, and then
\[ \lim_{n \to \infty} f_n(x_n) = \lim_{n \to \infty} f_n(x_0) \quad \text{if} \quad \lim_{n \to \infty} x_n = x_0. \]
7. INFINITE SERIES OF FUNCTIONS

7.2. Interchanging of Limit Processes

Theorem 7.9 (Continuity of the Limit Function). Let \( f_n : I \to \mathbb{R} \) be continuous on \( I \) for all \( n \in \mathbb{N} \) and suppose that \( f_n \to f \) uniformly on \( I \). Then \( f \) is continuous on \( I \), i.e.,

\[
\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) \quad \text{for all} \quad x_0 \in I.
\]

Example 7.10. \( f(x) = \sum_{k=1}^{\infty} \frac{a(kx)}{k^2} \) is continuous on \( \mathbb{R} \).

Theorem 7.11 (Integration of the Limit Function). Let \( f_n : I = [a,b] \to \mathbb{R} \) be Riemann integrable on \( I \) for all \( n \in \mathbb{N} \) and suppose that \( f_n \to f \) uniformly on \( I \). Then \( f \) is Riemann integrable on \( I \) with

\[
\left( \int_a^b \lim_{n \to \infty} f_n(x) \, dx \right) = \int_a^b f(x) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx.
\]

Corollary 7.12. \( \sum_{k=1}^{\infty} \int_a^b f_k(x) \, dx = \int_a^b \sum_{k=1}^{\infty} f_k(x) \, dx \) if \( f_k : [a,b] \to \mathbb{R} \) are Riemann integrable for all \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} f_k(x) \) is uniformly convergent on \( [a,b] \).

Theorem 7.13 (Differentiation of the Limit Function). Let \( f_n : I = [a,b] \to \mathbb{R} \) be differentiable on \( I \) for all \( n \in \mathbb{N} \) and suppose that \( f_n' \to g \) uniformly on \( I \). Also suppose that \( \lim_{n \to \infty} f_n(x_0) \) exists for at least one \( x_0 \in I \). Then \( f_n \) converges uniformly on \( I \), say to \( f \), and \( f \) is differentiable on \( I \) with

\[
f'(x) = g(x), \quad \text{i.e.,} \quad \lim_{n \to \infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n \to \infty} f_n(x).
\]

Corollary 7.14. \( \frac{d}{dx} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x) \) if \( f_k : [a,b] \to \mathbb{R} \) are differentiable for all \( k \in \mathbb{N} \), \( \sum_{k=1}^{\infty} f_k'(x) \) is uniformly convergent on \( [a,b] \), and \( \sum_{k=1}^{\infty} f_k(x_0) \) is convergent for at least one \( x_0 \in [a,b] \).

Example 7.15. (i) \( \sum_{k=0}^{\infty} a_k (x - x_0)^k \); (ii) \( \sum_{k=1}^{\infty} \frac{a(kx)}{k^2} \); (iii) \( f_n(x) = \frac{a(n^2x)}{n} \).