The partial derivative wrt \( x_i \) of a function \( f : M \to \mathbb{R} \) with \( M \subset \mathbb{R}^n \) in the point \( a \in M \) is defined as \[
abla f(a) = \lim_{x \to a, \|x-a\| \to 0} \frac{f(x) - f(a)}{\|x-a\|}.
\]

For example, the gradient of \( f(x, y, z) = x^2e^{yz} \) is equal to \( \nabla f(a) = \). The directional derivative of \( f \) in \( a \) in direction \( v \in \mathbb{R}^n \) with \( \|v\| = 1 \) is defined as \[
\nabla f(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.
\]

Suppose \( f : G \to \mathbb{R}^m \) such that \( G \subset \mathbb{R}^n \) is open. Then \( f \) is called differentiable in \( x_0 \) if there exists an \( m \times n \)-matrix \( C \) such that \[
\nabla f(x) = C(x) \nabla x.
\]

If \( f(x) = Ax \) for a matrix \( A \), then this matrix \( C \) is given by \( \nabla f(x) = A \). If \( f(x) = \|x\|^2 \), then this matrix \( C \) is given by \( \nabla f(x) = 2x \). If \( f(x) = x^T Ax \) for a matrix \( A \), then this matrix is given by \( \nabla f(x) = 2Ax \), and the directional derivative of \( f \) in \( x_0 \) exists for any direction \( v \) with \( \nabla f(x_0) = \). Is it true that if \( f \) is continuous in \( x_0 \), then it is differentiable in \( x_0 \)? \( \nabla f(x_0) = \). The central sufficient condition for differentiability says that \( \nabla f(x_0) = \). The product rule reads \( \nabla f(x) = \). The chain rule says \( \nabla f(x) = \). We can use the chain rule to prove that under certain conditions the function \( F(x) = \int_a^{\phi(x)} f(x, t) dt \) is differentiable with \( F'(x) = \). Using this rule, we find that the derivative of \( F(x) = \int_1^x \frac{\sin(xt)}{t} dt \) equals to \( F'(x) = \). The Hessian matrix of \( f : G \to \mathbb{R} \) in \( x \in G \subset \mathbb{R}^n \) is defined as \( H_f(x) = \), and the Schwarz theorem says that \( f \in C^2(G) \) implies that the Hessian matrix is \( H_f(x) = \). For example the Hessian of \( f(x, y, z) = x^2 + xy + y^2 - yz - \cos z \) at the point \( (0, 0, 0) \) equals \( H_f(0, 0, 0) = \) and it is \( \nabla \). A necessary condition for a local minimizer is that \( \nabla \). While a sufficient condition for a local minimizer is that \( \nabla \). The
mean value theorem for real-valued functions states that $f(x_1) - f(x_0) = \ldots$

and the mean value theorem for vector-valued functions states that $\|f(x_1) - f(x_0)\| \leq \ldots$

Taylor’s theorem for real-valued functions says $f(x) = \sum_{\nu=0}^{k} \ldots$, where we define $D^k f(a, h) = \ldots$.

For example, $D^2 f(a, h) = \ldots$ and $D^3 f(a, h) = \ldots$. Taylor’s theorem with $k = 1$ states $f(x) = \ldots$.

Suppose now that $G \subset \mathbb{R}^n$ is open and convex and that $f : G \to \mathbb{R}$ satisfies $f \in C^1(G)$. Then $f$ is convex on $G$ iff $f(x) \geq \ldots$. If $f \in C^2(G)$, then $f$ is convex on $G$ iff $H_f(x) \ldots$. If $x_0$ is a critical point for a convex function $f$, then $x_0$ automatically is a \ldots. Now state the multiplier rule of Carathéodory–John: