Taylor’s theorem for real-valued functions says

\[ f(x) = \sum_{k=0}^{\infty} \nu_k. \]

Taylor’s theorem with \( k = 1 \) states

This can be used to derive a sufficient condition for a local minimizer, namely

Suppose now that \( G \subset \mathbb{R}^n \) is open and convex and that \( f : G \to \mathbb{R} \) satisfies \( f \in C^1(G) \). Then \( f \) is convex on \( G \) iff

\[ f(x) \geq \frac{1}{2} \nabla^2 f(x) \theta^2 \]

If \( f \in C^2(G) \), then \( f \) is convex on \( G \) iff

\[ H_f(x) \geq 0 \]

If \( x_0 \) is a critical point for a convex function \( f \), then \( x_0 \) automatically is a

If \( f : G \to \mathbb{R}^n \) is in \( C^1(G) \) and \( x_0 \in G \), then the assumption of the Inverse Function Theorem is

and then a local inverse function \( f^{-1} \) exists and its derivative can be calculated using the formula

\[ (f^{-1})'(y) = \frac{1}{H_{f^{-1}}(f^{-1}(y))} \]

We showed that \( f \) is an open mapping, which means that

In the proof of the Inverse Function Theorem we used the Fixed Point Theorem which says that for \( \phi : K \to K \) with closed \( K \subset \mathbb{R}^n \) such that

which can be obtained as follows:

Now, consider the function \( F(x, y, z) = (x^2 + y^2 - 2z^2 + 1) \). We want to see whether there is a function \( f \) on a neighborhood of 0 which satisfies \( f(0) = \binom{1}{1} \) and \( F(x, f(x)) \equiv 0 \) on that entire neighborhood. According to the Implicit Function Theorem, this is true because \( F \in C^1(G) \),

and furthermore, the derivative \( f' \) on that neighborhood can be calculated using the formula

\[ f'(x) = \frac{\nabla F}{|\nabla F|^2} \]

Next, about integration. We first defined the Riemann integral over compact intervals. The volume of the compact interval \( I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \) is defined as

\[ |I| = (b_1-a_1)(b_2-a_2)\ldots(b_n-a_n) \]

Given a partition
with intermediate points $\xi$, a Riemann sum is defined as

$$S(f, Z, \xi) =$$

while the lower sum is defined as

$$S_L(f, Z) =$$

and the lower Darboux integral is defined by

$$\int_I f =$$

An upper sum is defined as

$$\bar{S}(f, Z) =$$

and the upper Darboux integral is defined by

$$\int_I f =$$

We showed that both the lower Darboux and upper Darboux integral always for bounded functions $f$ and that the lower Darboux integral is always than the upper Darboux integral. If these two values are equal, then $f$ is called . Now suppose that $f : I \to \mathbb{R}$ is bounded on the interval $I$. The Riemann Criterion for $f \in R(I)$ says . The Cauchy Criterion for $f \in R(I)$ says . The Lebesgue Criterion for $f \in R(I)$ says . And a set $M \subset \mathbb{R}^n$ is said to be of measure zero if . Is $R(I) \subset C(I)$ true? We derived a formula how to calculate integrals using iterated integrals, and this formula could be used to show that the integral of $f(x, y, z) = xyz$ over $I = [0, 1]^3$ is equal to . Next, for a set $M \subset \mathbb{R}^n$, the characteristic function of $M$ is defined as , and $M$ is called Jordan measurable if $M$ is bounded and for some $I \supset M$, and in this case $|M|$ is the $n$-dimensional Jordan volume of $M$. We immediately concluded that a set $M \subset \mathbb{R}^n$ is Jordan measurable if and only if . Now state the multiplier rule of Carathéodory–John: