89. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $f : X \to \mathbb{R}$ is measurable, and $A \in \mathcal{A}$ with $\mu(A) < \infty$. Show that f is integrable on A iff $\sum_{n=1}^{\infty} n\mu(A_n) < \infty$, where $A_n = \{x \in A : n-1 \le |f(x)| < n\}$.
90. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to \overline{\mathbb{R}}$ is integrable on X. Show that for each $\varepsilon > 0$ there exists a simple function e such that $\int_X |f - e| d\mu < \varepsilon$.
91. Let $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(X)$, and μ be the counting measure. Define $f : X \to \mathbb{R}$ by $f(n) = a_n$ for a given sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$.
    (a) Show that f is integrable iff $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Calculate $\int_X f d\mu$ in this case.
    (b) Prove that if $\{a_n\}$ and $\{b_n\}$ are real sequences with $|a_n| \le b_n$ for all $n \in \mathbb{N}$ and such that $\sum_{n=1}^{\infty} b_n < \infty$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
92. Show that for $\alpha \in \mathbb{R}$ and $\alpha > 1$ we have
$$\int_0^{\infty} \frac{x^{\alpha-1}}{e^x - 1} dx = \Gamma(\alpha) \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}.$$
93. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and f is summable. Define $\nu : \mathcal{A} \to [0, \infty]$ by $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{A}$. Show that any summable function h satisfies
$$\int_A h d\nu = \int_A h f d\mu \quad \text{for all} \quad A \in \mathcal{A}.$$
94. A sequence f_n is said to be convergent to f almost everywhere, if $\lim_{n\to\infty} f_n \sim f$. Prove the following statements.
    (a) $f_n \to f$ almost everywhere on $A \in \mathcal{A}$ iff $\mu\left(\limsup_{n\to\infty} A_n(\varepsilon)\right) = 0$ for all $\varepsilon > 0$, where $A_n(\varepsilon) = \{x \in A : |f_n(x) - f(x)| \ge \varepsilon\}$.
    (b) If $\mu(A) < \infty$, then f_n tends to f almost everywhere on $A \in \mathcal{A}$ iff $\lim_{n\to\infty} \mu(\bigcup_{\nu=n}^{\infty} A_\nu(\varepsilon)) = 0$ for all $\varepsilon > 0$.
95. For f_n from #85, show that $f_n \to 0$ pointwise on X, but f_n does not converge in $L^p(X)$.
96. Prove that $L^p(I) \subset L^1(I)$ provided I is a bounded interval.
97. Suppose I is a bounded interval and $f_n \in L^p(I)$ converge uniformly on I to f. Show that then also $f \in L^p(I)$ and that $f_n \to f$ in the L^p-sense.
98. Suppose ν is a signed measure on $(X, \mathcal{A})$ and $\nu^+(A) = \nu(AP)$, $\nu^-(A) = \nu(AN)$ (with N and P from the Hahn decomposition), and $|\nu|(A) = \nu^+(A) + \nu^-(A)$. Show the following:
    (a) ν^+ and ν^- do not depend on P and N;
    (b) At least one of the measures ν^+ and ν^- is finite;
    (c) $\nu^+(A) = \sup\{\nu(B) : B \subset A, B \in \mathcal{A}\}$;
    (d) $-\nu^-(A) = \inf\{\nu(B) : B \subset A, B \in \mathcal{A}\}$;
    (e) $|\nu(A)| \le |\nu|(A)$;
    (f) $\nu^+, \nu^-, \nu \ll |\nu|$;
    (g) $\nu^+, \nu^-, \nu \ll \mu$.