21. Let $f_n(x) = x - x^n$. Find the limit function $f$ of $\{f_n\}$ on $[0, 1]$ and decide whether $f'_n \to f'$. Does $\int_0^1 f_n(x) \, dx \to \int_0^1 f(x) \, dx$ hold?

22. Let $f_n(x) = nx(1 - x)^n$. Graph $f_1$, $f_2$, $f_5$, and $f_{11}$ on $[0, 1]$. Determine the limit function $f$ of $\{f_n\}$ as $n \to \infty$. Is the limit function continuous? Is $f'_n \to f'$ true? How about the integral?

Discuss whether $\{f_n\}$ is uniformly convergent on $[0, 1]$.

23. Let $f_n(x) = nx e^{-nx}$. Is $\{f_n\}$ uniformly convergent on $(0, \infty)$?

24. Assume $\alpha_n \to 0$. Show: $f_n \to f$ uniformly if $|f_n(x) - f(x)| \leq \alpha_n$ for all $n \geq N$ and all $x \in E$.

25. Let $f_n(x) = nx/(1 + n^2x^2)$ for $x \in [0, 1]$. Show that the sequence converges uniformly on $[q, 1]$ for any $q \in (0, 1)$ but not on $[0, 1]$.

26. Prove: If $f_n \to f$ uniformly on $E$ and $g \in B(E)$, then $f_ng \to fg$ uniformly on $E$.

27. Prove: If $f_n \to f$ uniformly on $E$ and $|f_n(x)| \geq \alpha > 0$ for all $n \in \mathbb{N}$ and all $x \in E$, then $1/f_n \to 1/f$ uniformly on $E$.

28. Show: If $\sum_{k=0}^\infty a_k$ is absolutely convergent, then $\sum_{k=0}^\infty a_k \sin(kx)$ and $\sum_{k=0}^\infty a_k \cos(kx)$ converge uniformly on $\mathbb{R}$.

29. Let $F_n = \sum_{k=1}^n f_k$. Prove that $\sum_{k=1}^\infty f_k g_k$ converges uniformly provided $\{F_n g_{n+1}\}$ and $\sum_{k=1}^\infty F_k (g_k - g_{k+1})$ converge uniformly.

30. Suppose that $\sum_{k=0}^\infty f_k$ converges uniformly, that $\{g_k(x)\}$ is a monotone sequence for each $x$, and that the sequence $\{\|g_k\|_\infty\}$ is bounded. Show that $\sum_{k=0}^\infty f_k g_k$ is uniformly convergent.

31. Suppose that $\{\|\sum_{k=1}^n f_k\|_\infty\}$ is bounded, that $\{g_k(x)\}$ is monotone for each $x$, and that $g_k \to 0$ uniformly. Show that $\sum_{k=0}^\infty f_k g_k$ is uniformly convergent.

32. Suppose that $g_1 \geq g_2 \geq \ldots$ and $g_k \to 0$ uniformly. Prove that $g_1 - g_2 + g_3 - \ldots$ converges uniformly.