Problems #7, Math 315, Dr. M. Bohner.Mar 4, 2005. Due Mar 11, 2 pm.

60. For the following functions, calculate the integral between 0 and 1 once by the Riemann method and another time by the Lebesgue method (as illustrated in Example 4.1 in class):
   (a) $f(x) = x$;
   (b) $f(x) = \sqrt{x}$.
61. Show that each countable union can be written as a countable union of pairwise disjoint sets.
62. Prove that each open set in $\mathbb{R}^N$ is the countable union of closed intervals which have pairwise disjoint interiors.
63. Let $\{A_n\}$ be a sequence of sets. The set of elements that are in almost all sets A_n is denoted by $\liminf_{n\to\infty} A_n$. The set of elements that are in inifinitely many A_n is denoted by $\limsup_{n\to\infty} A_n$. If these limits are the same, this set is denoted by $\lim_{n\to\infty} A_n$. Show:
   (a) $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$;
   (b) $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$;
   (c) $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$;
   (d) $(\limsup_{n\to\infty} A_n)^C = \liminf_{n\to\infty} A_n^C$;
   (e) $(\liminf_{n\to\infty} A_n)^C = \limsup_{n\to\infty} A_n^C$;
   (f) $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$ if $A_k \subset A_{k+1}$ for all $k \in \mathbb{N}$;
   (g) $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$ if $A_k \supset A_{k+1}$ for all $k \in \mathbb{N}$.
64. Calculate the limits from the previous problem if $A_{2n} = [0, 1/2]$ and $A_{2n-1} = [0, 1]$ for $n \in \mathbb{N}$.
65. For nonempty sets A and B the expression $\mathrm{card}A = \mathrm{card}B$ ($<$) means that there is a function $f : A \to B$ which is bijective (injective but not bijective). Show that $\mathrm{card}\mathcal{P}(X) > \mathrm{card}X$ for each set $X \neq \emptyset$.