Problems #8, Math 315, Dr. M. Bohner.Mar 16, 2005. Due Mar 25, 2 pm.

66. Prove Theorem 4.18 from the lecture notes.

67. Let $\mathcal{A}$ be a σ-algebra. Show:
    (a) $\emptyset \in \mathcal{A}$;
    (b) if $A_k \in \mathcal{A}$ for all $k \in \mathbb{N}$, then $\bigcap_{k\in\mathbb{N}} A_k \in \mathcal{A}$;
    (c) if $A, B \in \mathcal{A}$, then $A - B \in \mathcal{A}$.

68. Let $X \neq \emptyset$ and Ω be a set. Suppose that $\mathcal{A}_\omega \subset \mathcal{P}(X)$ are σ-algebras for each $\omega \in \Omega$. Show that $\bigcap_{\omega\in\Omega} \mathcal{A}_\omega$ is also a σ-algebra.

69. Let $X \neq \emptyset$ and $M \subset \mathcal{P}(X)$. Show that there exists a smallest σ-algebra among all σ-algebras that contain M.

70. Let $X = \mathbb{R}$ and $M = \{(-\infty, a] : \ a \in \mathbb{R}\}$. Show that the smallest σ-algebra containing M from the previous problem contains also all of the following sets:
    (a) All intervals of the form (a, ∞);
    (b) all half-open intervals;
    (c) all open subsets of $\mathbb{R}$;
    (d) all closed subsets of $\mathbb{R}$;
    (e) all countable unions of closed subsets of $\mathbb{R}$ ("F_σ-sets");
    (f) all countable intersections of open subsets of $\mathbb{R}$ ("G_δ-sets").

71. Let $X, Y \neq \emptyset$. Prove that if $\mathcal{A}$ is a σ-algebra on X and $f : X \to Y$ is onto, then $\{B : \ f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y.

72. Let A_k, $k \in \mathbb{N}$, and A be measurable sets. Show:
    (a) If A_k is increasing to A, then $\mu(A_k) \to \mu(A)$, $k \to \infty$.
    (b) Suppose that $\mu(A_1) < \infty$. If A_k is decreasing to A, then $\mu(A_k) \to \mu(A)$, $k \to \infty$.
    (c) The second sentence of part (b) does not need to hold if the first sentence of part (b) is not satisfied.

73. Show that the Cantor set is uncountable but has Lebesgue measure zero.