64. Consider the series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

    (a) For which $x \in \mathbb{R}$ does the series converge pointwise?

    (b) Does the series converge uniformly on $[-1,1]$?

    (c) Does the series converge in the L^2 sense on $[-1,1]$?

65. Let $\phi(x) = -1-x$ if $x \in [-1,0)$ and $\phi(x) = 1-x$ if $x \in (0,1]$ and $\phi(0) = 0$.

    (a) Find the full Fourier series of ϕ in the interval $(-1,1)$.

    (b) Graph the first five partial sums of the Fourier series (use a computer if you like).

    (c) Does the Fourier series converge in the mean square sense?

    (d) Does the Fourier series converge pointwise?

    (e) Does the Fourier series converge uniformly?

66. Let f be 2π-periodic on $\mathbb{R}$ and assume $\int_{-\pi}^{\pi}|f(x)|^2\mathrm{d}x < \infty$. Show that both $\int_{-\pi}^{\pi}f(x)\cos(nx)\mathrm{d}x$ and $\int_{-\pi}^{\pi}f(x)\sin(nx)\mathrm{d}x$ converge to zero as n tends to infinity.

67. Find $\sum_{k=1}^{n}\sin(k\theta)$.

68. For $|a| < 1$, find

    (a) $\sum_{n=0}^{\infty}a^n\cos(n\theta)$;

    (b) $\sum_{n=1}^{\infty}a^n\sin(n\theta)$.

69. Let $e_n(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}$, where $x \in (-\pi,\pi)$. Let f be continuous and 2π-periodic on $\mathbb{R}$. Define $f_m = \sum_{n=-m}^{m}(f,e_n)e_n$ and $F_m = \frac{1}{m+1}\sum_{k=0}^{m}f_k$.

    (a) Establish the formula $F_m(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)K_m(y-x)\mathrm{d}x$, where $K_m(\theta) = \frac{1}{m+1}\sum_{k=0}^{m}\sum_{n=-k}^{k}\mathrm{e}^{in\theta}$ is the so-called Fejér kernel.

    (b) Show that $K_m(\theta) = \frac{1}{m+1}\frac{\sin^2\frac{(m+1)\theta}{2}}{\sin^2\frac{\theta}{2}}$ if $\theta \neq 2\pi n$ for some $n \in \mathbb{Z}$.

    (c) Establish the formula $F_m(y) - f(y) = \frac{1}{2\pi}\int_{y-\pi}^{y+\pi}[f(x)-f(y)]K_m(y-x)\mathrm{d}x$.

    (d) Draw the graph of K_m (use a computer if you like) for $m \in \{2,5,8\}$.

70. Consider the problem $u_t = ku_{xx}$, $0 < x < l$, $u(x,0) = \phi(x)$ with the boundary conditions $u_x(0,t) = u_x(l,t) = \frac{u(l,t)-u(0,t)}{l}$.

    (a) Assume that there are no negative eigenvalues and solve the problem.

    (b) Assume that limits can be taken term by term and find A, B with $\lim_{t\to\infty}u(x,t) = A+Bx$.