49. Discuss $f'' + \lambda f = 0$, $kf(0) + f'(0) = f(l) + hf'(l) = 0$.

50. Do separation of variables with $u_{tt} = c^2 u_{xx}$ $(0 < x < l)$, $u(0,t) = 0$, $u_{tt}(l,t) + ku_x(l,t) = 0$.

51. Recall the eigenvalues of the problem $f''(x) + \lambda f(x) = 0$, $f(0) = f(1) = 0$. Divide the interval $[0,1]$ into N (where $N \in \mathbb{N}$) equally long intervals. Call the endpoints $x_0, x_1, \dots x_N$ and put $f_k = f(x_k)$. Now calculate $\lim\limits_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$ and use the result to derive a discretization (E_N) of the original problem. Find the eigenvalues (compare Problem 46) of the discrete problems (E_N). Finally, determine the limit of the kth eigenvalues of (E_N) as N tends to ∞.

52. Show that $\cos(nx)$ and $\sin(mx)$ are orthogonal (in the sense of Problem 45).

53. Find the Fourier coefficients of f if f is

    (a) even;

    (b) odd.

54. Suppose the following functions f are defined on $[-\pi, \pi]$ and 2π-periodically extended on $\mathbb{R}$. Sketch f. Find the Fourier series of f. Assume it converges to f. Find the values of the infinite sums given.

    (a) $f(x) = x$ for $x \in (-\pi, \pi)$ and $f(x) = 0$ for $x \in \{-\pi, \pi\}$. Find $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$

    (b) $f(x) = |x|$. Find $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

    (c) $f(x) = |\sin x|$. Find $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots$

    (d) $f(x) = x^2$. Find $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots$

    (e) $f(x) = \cosh(\alpha x)$, $\alpha \neq 0$. Find $\frac{1}{\alpha} + \sum\limits_{n=1}^{\infty} \frac{2\alpha}{\alpha^2+n^2}$.

55. Use the previous problem to determine the value of $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

56. Find the Fourier sine series in $(0, \pi)$ of $f(x) = \cos x$.

57. Find the complex form of the Fourier series of $f(x) = \mathrm{e}^x$.