31. Let u be a solution of the wave equation $u_{tt} = c^2 u_{xx}$. Show the following:

    (a) Let $y \in \mathbb{R}$. Then v with $v(x,t) = u(x-y,t)$ solves the wave equation.

    (b) u_x, u_t, and u_{xx} solve the wave equation (provided u is often enough differentiable).

    (c) Let $a \in \mathbb{R}$. Then v with $v(x,t) = u(ax,at)$ solves the wave equation.

32. Solve the wave equation $u_{tt} = c^2 u_{xx}$, together with the initial conditions

    (a) $u(x,0) = \mathrm{e}^x$ and $u_t(x,0) = \sin x$;

    (b) $u(x,0) = \log(1+x^2)$ and $u_t(x,0) = 4+x$;

    (c) $u(x,0) = \tanh x$ and $u_t(x,0) = 0$.

33. If both ϕ and ψ are even functions of x, show that the solution of the initial value problem given in Theorem 2.2 is also even in x for all times t.

34. Use a method similar to the methods from Theorem 2.1 and Theorem 2.2 (i.e., "factor" the operator) to find the solutions to the following initial value problems:

    (a) $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = \mathrm{e}^x$;

    (b) $u_{xx} + 2u_{xt} - 3u_{tt} = 0$, $u(x,0) = \sin x$, $u_t(x,0) = x$;

    (c) $u_{xx} - u_{xt} - 2u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = x$.

35. Find the general solution of the so-called spherical wave equation $u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ by changing variables $v = ur$. Also, find the solution of the spherical wave equation that satisfies $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$, where ϕ and ψ are differentiable.

36. Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Determine the solution of the so-called Goursat problem, namely of $u_{tt} = c^2 u_{xx}$, $u(x, \frac{x}{c}) = \phi(x)$, $u(x, h(x)) = \psi(x)$.

37. Suppose u solves the equation (with a given function h and $c > 0$) $u_{tt} + 2cu_{xt} + c^2 u_{xx} = h(x-ct)$. Introduce $v = u_t + cu_x$ and calculate $v_t + cv_x$ to obtain a PDE of first order for v. Solve this PDE using the geometric method. Thus obtain a PDE of first order for u. Solve this PDE using the geometric method. Finally, solve the problem $u_{tt} + 2cu_{xt} + c^2 u_{xx} = h(x-ct)$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

38. Prove that the total energy for the wave equation $E(t) = \frac{1}{2}\int_0^l \left\{\frac{1}{c^2}u_t^2(x,t) + u_x^2(x,t)\right\} \mathrm{d}x$ is conserved when having Neumann boundary conditions.

39. Find the general solution of the nonhomogeneous wave equation $u_{tt} - c^2 u_{xx} = h(x,t)$. Then, determine the solution of this equation that satisfies the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.