31. Let \( u \) be a solution of the wave equation \( u_{tt} = c^2 u_{xx} \). Show the following:

(a) Let \( y \in \mathbb{R} \). Then \( v \) with \( v(x,t) = u(x-y,t) \) solves the wave equation.

(b) \( u_x, u_t, \) and \( u_{xx} \) solve the wave equation (provided \( u \) is often enough differentiable).

(c) Let \( a \in \mathbb{R} \). Then \( v \) with \( v(x,t) = u(ax,at) \) solves the wave equation.

32. Solve the wave equation \( u_{tt} = c^2 u_{xx} \), together with the initial conditions

(a) \( u(x,0) = e^x \) and \( u_t(x,0) = \sin x \);

(b) \( u(x,0) = \log(1 + x^2) \) and \( u_t(x,0) = 4 + x \);

(c) \( u(x,0) = \tanh x \) and \( u_t(x,0) = 0 \).

33. If both \( \phi \) and \( \psi \) are even functions of \( x \), show that the solution of the initial value problem given in Theorem 2.2 is also even in \( x \) for all times \( t \).

34. Use a method similar to the methods from Theorem 2.1 and Theorem 2.2 (i.e., “factor” the operator) to find the solutions to the following initial value problems:

(a) \( u_{xx} - 3u_{xt} - 4u_{tt} = 0 \), \( u(x,0) = x^2 \), \( u_t(x,0) = e^x \);

(b) \( u_{xx} + 2u_{xt} - 3u_{tt} = 0 \), \( u(x,0) = \sin x \), \( u_t(x,0) = x \);

(c) \( u_{xx} - u_{xt} - 2u_{tt} = 0 \), \( u(x,0) = x^2 \), \( u_t(x,0) = x \).

35. Find the general solution of the so-called spherical wave equation \( u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right) \) by changing variables \( v = ur \). Also, find the solution of the spherical wave equation that satisfies \( u(r,0) = \phi(r) \) and \( u_t(r,0) = \psi(r) \), where \( \phi \) and \( \psi \) are differentiable.

36. Let \( h : \mathbb{R} \to \mathbb{R} \) be a strictly decreasing function. Determine the solution of the so-called Goursat problem, namely of \( u_{tt} = c^2 u_{xx} \), \( u(x, \frac{2}{c}) = \phi(x) \), \( u(x, h(x)) = \psi(x) \).

37. Suppose \( u \) solves the equation (with a given function \( h \) and \( c > 0 \)) \( u_{tt} + 2cu_{xt} + c^2 u_{xx} = h(x - ct) \).

Introduce \( v = u_t + cu_x \) and calculate \( v_t + cv_x \) to obtain a PDE of first order for \( v \). Solve this PDE using the geometric method. Thus obtain a PDE of first order for \( u \). Solve this PDE using the geometric method. Finally, solve the problem \( u_{tt} + 2cu_{xt} + c^2 u_{xx} = h(x - ct) \), \( u(x,0) = \phi(x) \), \( u_t(x,0) = \psi(x) \).

38. Prove that the total energy for the wave equation \( E(t) = \frac{1}{2} \int_0^l \left\{ \frac{1}{c^2} u_t^2(x,t) + u_x^2(x,t) \right\} dx \) is conserved when having Neumann boundary conditions.

39. Find the general solution of the nonhomogeneous wave equation \( u_{tt} - c^2 u_{xx} = h(x,t) \). Then, determine the solution of this equation that satisfies the initial conditions \( u(x,0) = \phi(x) \) and \( u_t(x,0) = \psi(x) \).