40. Let u be a solution of the diffusion equation $u_t = ku_{xx}$. Show the following:

    (a) Let $y \in \mathbb{R}$. Then v with $v(x,t) = u(x-y,t)$ solves the diffusion equation.

    (b) u_x, u_t, and u_{xx} solve the diffusion equation (provided u is often enough differentiable).

    (c) Let $a > 0$. Then v with $v(x,t) = u(x\sqrt{a}, at)$ solves the diffusion equation.

41. Let $u(x,t) = 1 - x^2 - 2kt$. Show that u solves the diffusion equation and find the locations of its extreme values in $\{(x,t) :\ 0 \le x \le 1, 0 \le t \le T\}$.

42. Let $u(x,t) = -2xt - x^2$. Show that u solves the equation $u_t = xu_{xx}$ and find the locations of its extreme values in $\{(x,t) :\ -2 \le x \le 2, 0 \le t \le 1\}$. Where exactly does the proof of the maximum principle from Theorem 2.3 break down?

43. Use the energy method (as is done in the book on page 43) to show that there is at most one solution of the Dirichlet problem for the diffusion equation.

44. For this problem you may use without proving it that $\int_{-\infty}^{\infty} \mathrm{e}^{-u^2} \mathrm{d}u = \sqrt{\pi}$ holds. We let $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$ and put $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

    (a) Calculate the maximum and the two inflection points of f.

    (b) Evaluate $\int_{-\infty}^{\infty} f(x) \mathrm{d}x$.

    (c) Evaluate $\int_{-\infty}^{\infty} x f(x) \mathrm{d}x$ and denote the result by E.

    (d) Evaluate $\int_{-\infty}^{\infty} (x-E)^2 f(x) \mathrm{d}x$.

45. Solve the diffusion equation with the initial condition

    (a) $\phi(x) = \alpha$ for all $x \in \mathbb{R}$ (where $\alpha \in \mathbb{R}$);

    (b) $\phi(x) = 1$ if $|x| < l$ and zero otherwise (where $l > 0$);

    (c) $\phi(x) = 1$ for positive x and $\phi(x) = 3$ for negative x;

    (d) $\phi(x) = \mathrm{e}^{-x}$ for positive x and $\phi(x) = 0$ for negative x.