46. Let u be a solution of the diffusion equation together with $u(x,0)=\phi(x)$. Prove:

    (a) If ϕ is odd, then u is odd;

    (b) If ϕ is even, then u is even.

47. Solve the IVP $u_t - ku_{xx} + bu = 0$, $u(x,0)=\phi(x)$ (where $b>0$) by performing a change of variables $u(x,t) = \mathrm{e}^{-bt}v(x,t)$.

48. Solve the IVP $u_t - ku_{xx} + bt^2u = 0$, $u(x,0)=\phi(x)$ (where $b>0$) by performing a change of variables $u(x,t)=\mathrm{e}^{-bt^3/3}v(x,t)$.

49. Solve the IVP $u_t - ku_{xx} + bu_x = 0$, $u(x,0)=\phi(x)$ (where $b>0$) by substituting $y = x - bt$.

50. Read Chapter 3 of the book. Work on at least one problem from each of its sections.

51. Find the solution of the wave equation with Dirichlet conditions (see Theorem 4.1) and

    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$, $\psi(x)=0$;

    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$, $\psi(x) = 4\sin\frac{\pi x}{l} + 2\sin\frac{4\pi x}{l}$.

52. Find the solution of the diffusion equation with Dirichlet conditions (see Theorem 4.2) and

    (a) $\phi(x) = 3\sin\frac{\pi x}{l}$;

    (b) $\phi(x) = 3\sin\frac{\pi x}{l} - 2\sin\frac{3\pi x}{l}$.

53. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature one everywhere. Suddenly both ends are plunged into a bath of temperature zero. Write the differential equation, boundary conditions, and initial conditions. Write the formula for the temperature $u(x,t)$ at later times. In this problem, you can use the infinite series expansion $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l} = \frac{\pi}{4}$.

54. Find all eigenvalues and eigenfunctions of $f'' + \lambda f = 0$, $f(0)=f(\pi)=0$. How many zeros inside the interval $(0,\pi)$ does the nth eigenfunction of the problem have?

55. Find all eigenvalues and eigenfunctions of $f''+\lambda f = 0$, $f(-\pi)=f(\pi)$, $f'(-\pi)=f'(\pi)$. Also show that the eigenfunctions are orthogonal in the sense that $\int\limits_{-\pi}^{\pi} e_1(x)e_2(x)\mathrm{d}x = 0$ whenever e_1 and e_2 are eigenfunctions corresponding to two different eigenvalues.