56. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with $u(0,t) = u(\pi,t) = 0$. Show that the solution of this problem satisfying in addition $u(x,0) = 0$ is not unique.

57. Use the method of separation of variables and discuss the resulting eigenvalue problems for each of the following:

    (a) $u_{xx} + u_{tt} = 0$ $(0 < x < l,\ t > 0)$, $u(0,t) = u(l,t) = 0$;

    (b) $u_{xx} + u_{tt} = 0$ $(0 < x < l,\ t > 0)$, $u_x(0,t) = u_x(l,t) = 0$;

    (c) $u_{tt} = c^2 u_{xx}$ $(0 < x < l)$, $u(0,t) = 0$, $u_{tt}(l,t) + ku_x(l,t) = 0$;

    (d) $u_{tt} + a^2 u_{xxxx} = 0$ $(0 < x < l,\ t > 0)$, $u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0$.

58. Show that $\cos(nx)$ and $\sin(mx)$ are orthogonal.

59. Find the Fourier coefficients of f on $[-l,l]$ if f is

    (a) even;

    (b) odd.

60. Find the Fourier coefficients of f on $[-\pi,\pi]$ for

    (a) $f(x) = x$;

    (b) $f(x) = |x|$;

    (c) $f(x) = |\sin x|$;

    (d) $f(x) = x^2$;

    (e) $f(x) = \cosh(\alpha x)$, $\alpha \neq 0$.

61. Consider the second order difference equation $\Delta^2 f_k + \lambda f_{k+1} = 0$ (where the forward difference operator Δ is defined by $\Delta f_k = f_{k+1} - f_k$). Determine the values of λ (and of θ) for which $f_k = \cos(k\theta)$, $f_k = \sin(k\theta)$, $f_k = \cosh(k\theta)$, and $f_k = \sinh(k\theta)$ solve the equation and hence find the general solution. Then find the eigenvalues and eigenfunctions of $\Delta^2 f_k + \lambda f_{k+1} = 0$, $f_0 = f_N = 0$, where $N \in \mathbb{N}$. How many eigenvalues does this problem have? Next, recall the eigenvalues of the problem $f''(x) + \lambda f(x) = 0$, $f(0) = f(1) = 0$. Divide the interval $[0,1]$ into N (where $N \in \mathbb{N}$) equally long intervals. Call the endpoints $x_0, x_1, \ldots, x_N$ and put $f_k = f(x_k)$. Now calculate $\lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$ and use the result to derive a discretization (E_N) of the original problem. Find the eigenvalues of the discrete problems (E_N). Finally, determine the limit of the kth eigenvalues of (E_N) as N tends to ∞.