62. Use Problem 60 to find the following infinite series:
   (a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots;$
   (b) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots;$
   (c) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots;$
   (d) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots;$
   (e) $\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2+n^2}.$

63. Use the previous problem to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

64. Find the Fourier sine series in $(0,\pi)$ of $f(x) = \cos x$.

65. Find the complex form of the Fourier series of $f(x) = \mathrm{e}^x$.

66. Let V be a complex vector space. An inner product on V is a mapping $(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$: $(x,y) = \overline{(y,x)}$, $(\lambda x, y) = \lambda(x,y)$, $(x+y,z) = (x,z) + (y,z)$, and $(x,x) > 0$ if $x \neq 0$.
   (a) Prove $(x, y+z) = (x,y) + (x,z)$ and $(x, \lambda y) = \overline{\lambda}(x,y)$.
   (b) Prove $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ and $\|\lambda x\| = |\lambda|\,\|x\|$.
   (c) Show that $(x,y) = x^T\overline{y}$ is an inner product on $\mathbb{C}^n$.
   (d) Show that $(f,g) = \int_a^b f(x)\overline{g(x)}\mathrm{d}x$ is an inner product on the vector space of all continuous complex-valued functions on $[a,b]$.

67. Let V be a real vector space with inner product $(\cdot,\cdot)$. We call $x, y \in V$ orthogonal (write $x \perp y$) if $(x,y) = 0$. Also, we put $\|x\| = \sqrt{(x,x)}$. Prove the following statements. Also draw a picture for the case $V = \mathbb{R}^2$.
   (a) $(x-y) \perp (x+y)$ iff $\|x\| = \|y\|$.
   (b) $(x-z) \perp (y-z)$ iff $\|x-z\|^2 + \|y-z\|^2 = \|x-y\|^2$.
   (c) $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.
   (d) $|(x,y)| \leq \|x\|\,\|y\|$.
   (e) $\|x+y\| \leq \|x\| + \|y\|$.

68. Let V be a real vector space with inner product $(\cdot,\cdot)$. Let $\{e_i : i \in \mathbb{N}\} \subset V$ be orthonormal, i.e., (e_i, e_j) is zero if $i \neq j$ and is one if $i = j$. Let $n \in \mathbb{N}$, $x \in V$, and $\lambda_i \in \mathbb{R}$ for $i \in \mathbb{N}$. Prove:
   (a) $\left\|\sum_{i=1}^{n} \lambda_i e_i\right\|^2 = \sum_{i=1}^{n} |\lambda_i|^2;$
   (b) $\left\|x - \sum_{i=1}^{n} \lambda_i e_i\right\|^2 = \|x\|^2 + \sum_{i=1}^{n} |\lambda_i - c_i|^2 - \sum_{i=1}^{n} |c_i|^2$ with $c_i = (x, e_i)$;
   (c) $\lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2$ exists and is less than or equal to $\|x\|^2$.

69. For this problem, use the inner products defined earlier for the various cases, respectively.
   (a) Find a set of three orthonormal vectors in $\mathbb{R}^3$.
   (b) Find α such that $e_n(t) = \alpha \mathrm{e}^{\mathrm{i}nt}$, $n \in \mathbb{Z}$ are orthonormal on $[-\pi,\pi]$.
   (c) For the set of real-valued polynomials on $[-1,1]$, show that $p(x) = x$ is orthogonal to every constant function. Next, find a quadratic polynomial that is orthogonal to both p and the constant functions. Finally, find a cubic polynomial that is orthogonal to all quadratic polynomials. Hence construct an orthonormal set with three vectors.