19. Suppose $x \in l^m$ for some $m \in [1, \infty)$. Prove that
    (a) $x \in l^p$ for all $p \geq m$;
    (b) $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$.
20. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be normed linear spaces. Show the following:
    (a) For $L \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $K \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$ we have $KL \in \mathcal{B}(\mathcal{X}, \mathcal{Z})$ and $\|KL\| \leq \|K\| \, \|L\|$;
    (b) The mapping $\cdot : \mathcal{B}(\mathcal{Y}, \mathcal{Z}) \times \mathcal{B}(\mathcal{X}, \mathcal{Y}) \to \mathcal{B}(\mathcal{X}, \mathcal{Z})$ is continuous.
21. Find the dual spaces of
    (a) $\mathbb{R}^n$;
    (b) l^p $(1 < p < \infty)$;
    (c) c_0 (sequences converging to zero).
22. For each $f \in L^2[0, 1]$, define a function Vf by $(Vf)(x) = \int_0^x f(t)\mathrm{d}t$.
    (a) Prove that $V : L^2[0, 1] \to L^2[0, 1]$ is a bounded linear operator.
    (b) Find $\mathrm{Ker}V = \{0\}$.
    (c) Is V onto $\{f \in L^2[0, 1] : \ f(0) = 0\}$?
    (d) Is V onto $\{f \in C[0, 1] : \ f(0) = 0\}$?
23. Suppose $\mathcal{X}$ is a Banach space and $K \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ with $\|K\| < 1$. Show that $I - K$ is invertible and find a formula for the inverse. Show that the inverse is a bounded linear operator and give an upper bound for its norm.
24. Here are some applications of the previous problem.
    (a) Let L be an $n \times n$ Leontieff matrix (i.e., all entries are ≥ 0 and the sums of the columns are < 1). Let $y \in \mathbb{R}^n$ have entries ≥ 0. Show that $x - Lx = y$ has a solution x with entries ≥ 0.
    (b) If k is continuous on $[a, b] \times [a, b]$, we define the Fredholm operator $F : C[a, b] \to C[a, b]$ by $(Fx)(t) = \int_a^b k(t, s)x(s)\mathrm{d}s$. Let $y \in C[a, b]$. Show that $x - Fx = y$ has a unique solution $x \in C[a, b]$ provided $\max_{a\leq t\leq b} \int_a^b |k(t, s)|\mathrm{d}s < 1$ holds.
    (c) If k is continuous on $[a, b] \times [a, b]$, we define the Volterra operator $V : C[a, b] \to C[a, b]$ by $(Vx)(t) = \int_a^t k(t, s)x(s)\mathrm{d}s$. Let $y \in C[a, b]$. Show that $x - Vx = y$ has a unique solution $x \in C[a, b]$.
25. Show that all finite dimensional subspaces of a normed space are complete and therefore closed.
26. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper closed subspace. Prove the following:
    (a) $\|x + \mathcal{M}\| = d(x, \mathcal{M})$ defines a norm on $\mathcal{X}/\mathcal{M}$;
    (b) For any $\varepsilon > 0$ there exists $x \in \mathcal{X}$ with $\|x\| = 1$ and $\|x + \mathcal{M}\| \geq 1 - \varepsilon$;
    (c) The projection $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ defined by $\pi(x) = x + \mathcal{M}$ has norm 1;
    (d) If $\mathcal{X}$ is complete, then so is $\mathcal{X}/\mathcal{M}$.
27. Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. Prove the following:
    (a) If f is a complex linear functional and $u = \mathrm{Re}f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in \mathcal{X}$;
    (b) If u is a real linear functional and f is defined by $f(x) = u(x) - iu(ix)$ for all $x \in \mathcal{X}$, then f is complex linear;
    (c) If $\mathcal{X}$ is normed, then $\|f\| = \|u\|$ in both of the above two cases.