28. Prove that $\mathcal{B}(\mathcal{X},\mathcal{Y})$ is a Banach space whenever $\mathcal{Y}$ is one (i.e., prove Theorem 1.37).

29. Let $K$ be a convex set in a real normed vector space $\mathcal{X}$. The functional $h(f)=\sup\{f(x):\ x\in K\}$ defined on $\mathcal{X}^*$ is called the *support functional* of $K$.
    (a) Show that $h$ is sublinear.
    (b) Show that $h$ is convex.
    (c) Calculate $h$ if $\mathcal{X}=\mathbb{R}^n$ and $K=[-1,1]^n$.
    (d) Calculate $h$ if $\mathcal{X}=\mathbb{R}^n$ and $K$ the ball around $x_0$ with radius $k$.
    (e) If $K\subset\mathbb{R}^n$ is compact, show that $h$ is bounded and that $h(f)=f(\tilde{x})$ for some $\tilde{x}\in\partial K$.

30. Let $\mathcal{X}$ be a nontrivial linear space and let $x\in\mathcal{X}$. Put $\mathcal{M}=\langle x\rangle$, $f(\alpha x)=\alpha$ for $\alpha x\in\mathcal{M}$, and

$$\mathcal{F}=\{(\tilde{\mathcal{M}},\tilde{f}):\ \tilde{\mathcal{M}}\text{ is a subspace of }\mathcal{X},\mathcal{M}\subset\tilde{\mathcal{M}},\tilde{f}\text{ is a linear functional on }\tilde{\mathcal{M}},\tilde{f}|_{\mathcal{M}}=f\}.$$

    Partially order $\mathcal{F}$ and show that Zorn's Lemma applies to give a maximal element. Conclude that there exists a nontrivial linear functional on $\mathcal{X}$.

31. Finish the proof of the Hahn-Banach Theorem by invoking Zorn's Lemma as indicated in the lecture (i.e., finish the proof of Theorem 2.4).

32. A normed space $X$ is called *strictly convex* if $\|x\|=\|y\|=1$ for $x,y\in X$ with $x\neq y$ and $0<\lambda<1$ implies $\|\lambda x+(1-\lambda)y\|<1$.
    (a) Let $\mathcal{X}$ be a normed space and suppose that the unit ball of $\mathcal{X}^*$ is strictly convex. Show that then all Hahn-Banach extensions (in the sense of Theorem 2.7) are unique.
    (b) Show that the unit ball of $L^2(\Omega,\mu)$ (with $\|f\|=\sqrt{\int_\Omega|f|^2\mathrm{d}\mu}$) is strictly convex.

33. Let $\mathcal{M}$ be a subspace of a normed space $\mathcal{X}$ and define $\mathcal{M}^\perp=\{f\in\mathcal{X}^*:\ f(x)=0\text{ for all }x\in\mathcal{M}\}$. Prove that $x\in\overline{\mathcal{M}}$ if and only if $f(x)=0$ for all $f\in\mathcal{M}^\perp$.

34. Let $\mathcal{X}$ be a normed space and let $x\in\mathcal{X}$. Define $\hat{x}(\varphi)=\varphi(x)$ for all $\varphi\in\mathcal{X}^*$. Show the following:
    (a) $\hat{x}\in\mathcal{X}^{**}$;
    (b) $\tilde{\varphi}(x)=\|\tilde{\varphi}\|\,\|x\|$ for some $\tilde{\varphi}\in\mathcal{X}^*$;
    (c) $\|\hat{x}\|=\|x\|$.

35. Let $\mathcal{M}$ be a subspace of a normed space $\mathcal{X}$ and let $F\in\mathcal{X}^*$. Show that

$$\sup\{|F(x)|:\ x\in\mathcal{M},\|x\|\leq 1\}=\inf\{\|F+h\|:\ h\in\mathcal{M}^\perp\}=\min\{\|F+h\|:\ h\in\mathcal{M}^\perp\}.$$

36. Calculate $\inf_{c\in\mathbb{R}}\sup_{\binom{x_1}{x_2}\in\mathbb{R}^2\setminus\{0\}}\frac{|x_1+x_2-c(x_1-x_2)|}{\sqrt{x_1^2+x_2^2}}$
    (a) directly, and
    (b) by applying the previous exercise.