45. In an inner product space (and only there) we have that $\|x-y\|^2+\|x+y\|^2=2(\|x\|^2+\|y\|^2)$. Prove this Parallelogram Law. Draw a picture. Also state and prove the Pythagorean Theorem. Draw a picture. Show that $\langle x,y\rangle=\frac{1}{4}\sum_{n=0}^{3} i^n\,\|x+i^n y\|^2$.
46. Let $\mathcal{M}$ be a closed subspace of a Hilbert space. Show that $(\mathcal{M}^\perp)^\perp=\mathcal{M}$. What is $(\mathcal{M}^\perp)^\perp$ for an arbitrary subspace $\mathcal{M}$?
47. Here are some applications of the theory presented in class (all in a Hilbert space):
    (a) Let $c_1,\ldots,c_n$ be scalars and $y_1,\ldots,y_n$ be linearly independent vectors. Find the vector with minimum norm out of all vectors that satisfy $\langle x,y_i\rangle=c_i$ for all $1\le i\le n$.
    (b) Work on (a) in the explicit case of $c_1=-1$, $c_2=0$, $c_3=1$, $y_1=(1\ 2\ 0)^T$, $y_2=(2\ 0\ 1)^T$, $y_3=(0\ 1\ 2)$. Give an interpretation of this result and try to draw a picture.
    (c) The shaft angular velocity ω of a d-c motor driven from a variable current source u is governed by $\dot\omega+\omega=u(t)$, where $u(t)$ is the field current at time t. The angular position θ of the motor shaft is the time integral of ω. Assume that the motor is initially at rest. Use (a) to find the current function u of minimum energy (the energy for u is proportional to $\int_0^1 u^2(t)\mathrm{d}t$) which rotates the shaft to the new rest position $\theta=1$, $\omega=0$ within one second.
    (d) Suppose $y\in\mathbb{R}^m$ and W is an $m\times n$-matrix with linearly independent columns. Find $\hat\beta\in\mathbb{R}^n$ which minimizes $\|y-W\beta\|$ over all $\beta\in\mathbb{R}^n$.
    (e) Work on (d) in the explicit case of $y=\frac{1}{2}(1\ 4\ 7)^T$ and $W=(1\ 2\ 3)^T$. Give an interpretation of this result and draw a picture.
48. Get familiar with the Gram-Schmidt Orthogonalization Process and construct the Legendre, Hermite, and Laguerre polynomials (see Exercises 6–8 on p. 18 of the textbook).
49. Describe some of the consequences of the theorems presented in class for classical Fourier series.
50. Let $\mathcal{H}$ be a Hilbert space.
    (a) Show that the unit ball of $\mathcal{H}$ is strictly convex.
    (b) By (a) and Problem 32 (a) it follows that all Hahn-Banach extensions are unique. Prove this fact directly.
51. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.
    (a) Let $T:\mathcal{H}\to\mathcal{K}$ be a linear operator. Show that $\|Tf\|=\|f\|$ for all $f\in\mathcal{H}$ iff $\langle Tf,Tg\rangle=\langle f,g\rangle$ for all $f,g\in\mathcal{H}$.
    (b) Suppose $U:\mathcal{H}\to\mathcal{K}$ is onto, and that $\langle Uf,Ug\rangle=\langle f,g\rangle$ for all $f,g\in\mathcal{H}$. Show that U is linear.
52. The geometry of a Hilbert space is so nice that a much stronger version of the Hahn-Banach Theorem is true. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathcal{M}$ a subspace of $\mathcal{H}$. Suppose $f:\mathcal{M}\to\mathcal{K}$ is a bounded linear operator. Show the following:
    (a) f can be extended to be bounded and linear on $\overline{\mathcal{M}}$;
    (b) f has a bounded and linear extension to $\mathcal{H}$.
53. Let $\mathcal{P}$ be the collection of all polynomials.
    (a) Show that for $p\in\mathcal{P}$ and $z\in\Delta$ there exists a $C_z>0$ with $|p(z)|\le C_z\left[\int_0^{2\pi}|p(\mathrm{e}^{i\theta})|^2\frac{\mathrm{d}\theta}{2\pi}\right]^{\frac{1}{2}}$.
    (b) Show that there exists $h_z\in H^2(\partial\Delta)$ such that $p(z)=\langle p,h_z\rangle$ for all $p\in\mathcal{P}$.
    (c) Give the h_z for each $z\in\Delta$ explicitly.
    (d) Let $0\le t_0\le 1$. Show that there is no C with $p(t_0)\le C\left[\int_0^1|p(t)|^2\mathrm{d}t\right]^{\frac{1}{2}}$ for all $p\in\mathcal{P}$.