Advanced Calculus I (Math 4209)

Spring 2018 Lecture Notes

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CHAPTER 1

Preliminaries

1.1. Sets

Definition 1.1 (Cantor)**.** A *set* is a collection of certain distinct objects which are called *elements* of the set.

Notation 1.2. The following notation will be used throughout this class:

- $x \in A$ (or $x \notin A$): x is an element (or is not an element) of the set A;
- $A \subset B$: A is a subset of B, i.e., if $x \in A$, then $x \in B$ (or: $x \in A \implies x \in B$);
- $A = B \iff A \subset B$ and $B \subset A$;
- $\emptyset$: empty set. We have $\emptyset \subset A$ for all sets A;
- $A = \{a, b, c\}$: A consists of the elements a, b, and c;
- $A = \{x : x \text{ has the property } P\}$: A consists of all elements x that have the property P;
- $A \cup B := \{x : x \in A \text{ or } x \in B\}$: union of A and B;
- $A \cap B := \{x : x \in A \text{ and } x \in B\}$: intersection of A and B;
- $A \setminus B := \{x : x \in A \text{ and } x \notin B\}$: difference of A and B;
- $A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$: Cartesian product of A and B (with $(a, b) = (c, d) \iff a = c$ and $b = d$);
- $A^{\mathrm{c}} := \{x : x \notin A\}$: complement of A (with respect to a given set X);
- $\mathcal{P}(A) := \{B : B \subset A\}$: power set of A.

Notation 1.3 (Quantifiers)**.** We use the following quantifiers throughout this class:

- universal quantifier: $\forall$ "for all";
- existential quantifier: $\exists$ "there exists".
- unique existential quantifier: $\exists!$ "there exists exactly one".

Lemma 1.4. *The operations $\cup$ and $\cap$ satisfy the commutative, associative, and distributive laws.*

1.2. Functions

Definition 1.5. Let X and Y be nonempty sets. A *function* (or mapping) f from X to Y is a correspondence that associates with each point $x \in X$ in a unique way a $y \in Y$; we write $y = f(x)$. X is called the *domain* of f while Y is called the *range* of f. We write $f : X \to Y$.

Remark 1.6. Two functions $f : X \to Y$ and $g : U \to V$ are called equal (we write $f = g$) iff $X = U$ and $Y = V$ and $f(x) = g(x)$ for all $x \in X$.

Notation 1.7. Let $f : X \to Y$ be a function. Then we call

(i) $G(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$ the *graph* of f;
(ii) $f(A) := \{f(x) : x \in A\} \subset Y$ the *image* of a set $A \subset X$;
(iii) $f^{-1}(B) := \{x \in X : f(x) \in B\} \subset X$ the *inverse image* of a set $B \subset Y$.

Lemma 1.8. *Let $f : X \to Y$ be a function and $A \subset X$, $B \subset Y$. Then we have*

(i) $y \in f(A) \iff \exists x \in A : \ f(x) = y$;
(ii) $x \in A \implies f(x) \in f(A)$;
(iii) $x \in f^{-1}(B) \iff f(x) \in B$.

Definition 1.9. Let $f : X \to Y$ be a function. Then f is called

(i) *one-to-one* if $x_1, x_2 \in X$ with $x_1 \neq x_2$ always implies $f(x_1) \neq f(x_2)$;
(ii) *onto* if $f(X) = Y$;
(iii) *invertible* if f is both one-to-one and onto.

Remark 1.10. (i) $f : X \to Y$ is a function iff $\forall x \in X \ \exists! y \in Y : \ f(x) = y$;
(ii) $f : X \to Y$ is onto iff $\forall y \in Y \ \exists x \in X : \ f(x) = y$;
(iii) $f : X \to Y$ is invertible iff $\forall y \in Y \ \exists! x \in X : \ f(x) = y$.

Definition 1.11. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the function $g \circ f : X \to Z$ with $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is called the *composite* function of f and g.

Proposition 1.12. *A function $f : X \to Y$ is invertible iff there exists exactly one function $g : Y \to X$ satisfying*

$$(f \circ g)(y) = y \ \forall y \in Y \quad \textit{and} \quad (g \circ f)(x) = x \ \forall x \in X.$$

Notation 1.13. Let $f : X \to Y$. The unique function $g : Y \to X$ from Proposition 1.12 is called the inverse function of f and denoted by f^{-1}.

1.3. Proofs

Remark 1.14 (Proof Techniques)**.** Let P and Q be two statements.

(i) To prove the *implication* $P \implies Q$, i.e., the theorem with assumption P and conclusion Q: We say "if P is true, then Q is true". Or "P is *sufficient* for Q". Or "Q is *necessary* for P". The implication $P \implies Q$ is logically equivalent to the *contraposition* $\neg Q \implies \neg P$ (negations).
  (a) *direct proof*: assume P and show Q;
  (b) *indirect proof*: do a direct proof with the contraposition;
  (c) *proof by contradiction*: assume P and $\neg Q$ and derive a statement that contradicts a true statement.

(ii) To prove the *equivalence* $P \iff Q$, show $P \implies Q$ and $Q \implies P$. Another possibility is to introduce "intermediate" statements $P_1, P_2, \ldots, P_n$ and to prove $P \iff P_1 \iff P_2 \iff \ldots \iff P_n \iff Q$.

Table 1.15 (Truth Table)**.**

P	T	T	F	F
Q	T	F	T	F
$\neg P$	F	F	T	T
$\neg Q$	F	T	F	T
$P \wedge Q$	T	F	F	F
$P \vee Q$	T	T	T	F
$P \implies Q$	T	F	T	T
$Q \implies P$	T	T	F	T
$P \iff Q$	T	F	F	T

CHAPTER 2

The Real Number System

2.1. The Field Axioms

Definition 2.1. Let K be a set with at least two elements, and let $+, \cdot : K \times K \to K$ be two functions that we call *addition* and *multiplication*. We say that K is a *field* provided the following *field axioms* are satisfied:

(F_1) $a + b = b + a \; \forall a, b \in K$ (commutativity of $+$);

(F_2) $(a + b) + c = a + (b + c) \; \forall a, b, c \in K$ (associativity of $+$);

(F_3) $\exists 0 \in K : \; a + 0 = 0 + a = a \; \forall a \in K$ (additive identity);

(F_4) $\forall a \in K \; \exists b \in K : \; a + b = 0$ (additive inverse);

(F_5) $a \cdot b = b \cdot a \; \forall a, b \in K$ (commutativity of $\cdot$);

(F_6) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \; \forall a, b, c \in K$ (associativity of $\cdot$);

(F_7) $\exists 1 \in K : \; a \cdot 1 = 1 \cdot a = a \; \forall a \in K$ (multiplicative identity);

(F_8) $\forall a \in K \setminus \{0\} \; \exists b \in K : \; a \cdot b = 1$ (multiplicative inverse);

(F_9) $(a + b) \cdot c = a \cdot c + b \cdot c \; \forall a, b, c \in K$ (distributive property).

Notation 2.2. $ab := a \cdot b$, $a^2 := a \cdot a$, $a + b + c := (a + b) + c$, $abc := (ab)c$, b from (F_4) is denoted as $-a$, b from (F_8) is denoted as a^{-1}, $a : b = \frac{a}{b} := ab^{-1}$ if $b \neq 0$, $a - b := a + (-b)$.

Proposition 2.3. *Let K be a field and $a, b \in K$. Then*

(i) $\exists! x \in K : \; a + x = b$, *namely* $x = b - a$;

(ii) $-(-a) = a$;

(iii) $-(a + b) = -a - b$.

Proposition 2.4. *Let K be a field and $a, b \in K$ with $a \neq 0$. Then*

(i) $\exists! x \in K : \; ax = b$, *namely* $x = \frac{b}{a}$;

(ii) $(a^{-1})^{-1} = a$;

(iii) $(ab)^{-1} = b^{-1} a^{-1}$ *provided* $ab \neq 0$.

Proposition 2.5. *Let K be a field and $a, b, c, d \in K$. Then*

(i) $ab = 0 \iff a = 0$ *or* $b = 0$*;*

(ii) $(-1)a = -a$ *and* $(-a)(-b) = ab$*;*

(iii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ *and* $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ *provided* $bd \neq 0$*;*

(iv) $\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$ *provided* $bcd \neq 0$.

Proposition 2.6. *In a field we always have* $1 \neq 0$.

2.2. The Positivity Axioms

Definition 2.7. Let K be a field. We call K *ordered* if there exists a set $\mathcal{P} \subset K$ that satisfies the *positivity axioms*:

(P_1) $a, b \in \mathcal{P} \implies a + b, ab \in \mathcal{P}$;

(P_2) $\forall a \in K$ either $a \in \mathcal{P}$ or $-a \in \mathcal{P}$ or $a = 0$ (trichotomy).

Notation 2.8. We write $a > b$ if $a - b \in \mathcal{P}$, $a < b$ if $b - a \in \mathcal{P}$, $a \geq b$ if $a > b$ or $a = b$, $a \leq b$ if $a < b$ or $a = b$.

Proposition 2.9. *Let K be an ordered field and $a, b, c, d \in K$. Then*

(i) $a^2 > 0$ *if* $a \neq 0$*;*

(ii) $1 > 0$*;*

(iii) $a > 0 \implies a^{-1} > 0$*;*

(iv) $a < b$ *and* $b < c \implies a < c$ *(transitivity);*

(v) $a < b$ *and* $c > 0 \implies ac < bc$*;*

(vi) $a < b$ *and* $c < 0 \implies ac > bc$*;*

(vii) $0 < a < b \implies 0 < \frac{1}{b} < \frac{1}{a}$*;*

(viii) $a > 0$ *and* $b > 0 \implies ab > 0$*;* $a < 0$ *and* $b < 0 \implies ab > 0$*;* $a > 0$ *and* $b < 0 \implies ab < 0$*;*

(ix) $a = b \iff a \leq b$ *and* $b \leq a$*;*

(x) $0 < a < b \implies 0 < a^2 < b^2$.

Remark 2.10. There is no ordered field K such that $a^2 = -1$ for some $a \in K$.

Proposition 2.11. *Let K be an ordered field and $a, b \in K$ with $a < b$. Then there exists $c \in K$ with $a < c < b$, e.g., $c := \frac{a+b}{2}$, where $2 := 1 + 1$.*

Notation 2.12. If K is an ordered field and $a, b \in K$, then we put

$$(a,b) := \{x \in K : \ a < x < b\},$$
$$[a,b] := \{x \in K : \ a \leq x \leq b\},$$
$$(a,b] := \{x \in K : \ a < x \leq b\},$$

and

$$[a,b) := \{x \in K : \ a \leq x < b\}.$$

Definition 2.13. Let K be an ordered field and $T \subset K$ with $T \neq \emptyset$. An $m \in T$ is called *minimum* (or *maximum*) of T provided $m \leq t$ (or $m \geq t$) for all $t \in T$. We write $m = \min T$ (or $m = \max T$).

Proposition 2.14. *Let K be an ordered field and $T \subset K$ with $T \neq \emptyset$. If $\min T$ (or $\max T$) exists, then it is uniquely determined.*

Example 2.15. For $T = (0, 1]$ we have $\max T = 1$ and $\min T$ does not exist.

2.3. The Completeness Axiom

Definition 2.16. Let K be an ordered field and $T \subset K$. We call

(i) $s \in K$ an *upper* (or *lower*) *bound* of T if $t \leq s$ (or $t \geq s$) for all $t \in T$;
(ii) T *bounded above* (or *bounded below*) if it has an upper (or lower) bound;
(iii) T *bounded* if it is bounded above and below;
(iv) $s = \sup T$ the *supremum* of T (and the *infimum* $\inf T$ analogously) if
    (a) s is an upper bound of T and
    (b) $s \leq \tilde{s}$ for all upper bounds $\tilde{s}$ of T.

Proposition 2.17. *Let K be an ordered field and $T \subset K$. Then*

$$m = \max T \text{ exists} \iff s = \sup T \text{ exists and } s \in T,$$

and then $s = m$.

Theorem 2.18. *Let K be an ordered field and $T \subset K$. Then*

$$s = \sup T \iff \begin{cases} \forall t \in T : \ s \geq t \quad \text{and} \\ \forall \varepsilon > 0 \ \exists t \in T : \ t > s - \varepsilon. \end{cases}$$

Definition 2.19 (Definition of $\mathbb{R}$). (i) An ordered field K is called *complete* if $\sup T \in K$ exists whenever $T \subset K$ is a nonempty set that is bounded above (*completeness axiom*).

(ii) An ordered field that is complete is called the *field of the real numbers*. We denote it by $\mathbb{R}$.

Theorem 2.20. *If* $\emptyset \neq S \subset \mathbb{R}$ *is bounded below, then* $\inf S$ *exists.*

Theorem 2.21. *If* $\emptyset \neq S, T \subset \mathbb{R}$ *and* $\forall s \in S\ \forall t \in T:\ s \leq t$*, then* $\sup S \leq \inf T$*.*

Theorem 2.22. *In* $\mathbb{R}$ *we have* $\forall c \geq 0\ \exists! s \geq 0:\ s^2 = c$*.*

Notation 2.23. The s from Theorem 2.22 is denoted by $\sqrt{c}$.

2.4. The Natural Numbers

Definition 2.24 (Definition of $\mathbb{N}$). (i) A set $M \subset \mathbb{R}$ is called *inductive* if

(a) $1 \in M$ and

(b) $x \in M \implies x + 1 \in M$.

(ii) The intersection of all inductive subsets of $\mathbb{R}$ is called the set of the *natural numbers*. We denote it by $\mathbb{N}$. Also, we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proposition 2.25 (Properties of $\mathbb{N}$). (i) $\mathbb{N}$ *is inductive;*

(ii) $\mathbb{N} \subset M$ *whenever* $M \subset \mathbb{R}$ *is inductive;*

(iii) *if* $A \subset \mathbb{N}$ *and* A *is inductive, then* $A = \mathbb{N}$*;*

(iv) $\min \mathbb{N} = 1$.

Theorem 2.26 (Principle of Mathematical Induction). *Let* $S(n)$ *be some statement for each* $n \in \mathbb{N}$*. If*

(i) $S(1)$ *is true and*

(ii) $S(k)$ *is true* $\implies$ $S(k+1)$ *is true* $\forall k \in \mathbb{N}$,

then $S(n)$ *is true for all* $n \in \mathbb{N}$*.*

Example 2.27. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ holds for all $n \in \mathbb{N}$.

Proposition 2.28. *Let* $m, n \in \mathbb{N}$*. Then* $m + n \in \mathbb{N}$ *and* $m \cdot n \in \mathbb{N}$*.*

Theorem 2.29 (The Well-Ordering Principle). *Let* $\emptyset \neq A \subset \mathbb{N}$*. Then* $\min A$ *exists.*

Theorem 2.30 (The Archimedean Property)**.** $\forall c \in \mathbb{R}\ \exists n \in \mathbb{N}:\ n > c$.

Corollary 2.31. $\forall \varepsilon > 0\ \exists n \in \mathbb{N}:\ \frac{1}{n} < \varepsilon$.

Definition 2.32 (Definition of $\mathbb{Z}$ and $\mathbb{Q}$)**.** We define the following sets.

(i) $\mathbb{Z} := \{a : a \in \mathbb{N}_0 \text{ or } -a \in \mathbb{N}_0\}$ is called the set of *integers*.
(ii) $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \setminus \{0\}\right\}$ is called the set of *rational* numbers.
(iii) The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational* numbers.

Proposition 2.33. *Let $T \subset \mathbb{Z}$ be nonempty.*

(i) *If T is bounded above, then* $\max T$ *exists;*
(ii) *If T is bounded below, then* $\min T$ *exists.*

Theorem 2.34. $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$*, more precisely,* $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

2.5. Some Inequalities and Identities

Notation 2.35. Let $m, n \in \mathbb{Z}$ and $a_k \in \mathbb{R}$ for $k \in \mathbb{Z}$. We put

$$\sum_{k=m}^{n} a_k = \begin{cases} 0 & \text{if } n < m \\ a_m & \text{if } n = m \\ \sum\limits_{k=m}^{n-1} a_k + a_n & \text{if } n > m \end{cases} \quad \text{and} \quad \prod_{k=m}^{n} a_k = \begin{cases} 1 & \text{if } n < m \\ a_m & \text{if } n = m \\ \left(\prod\limits_{k=m}^{n-1} a_k\right) a_n & \text{if } n > m. \end{cases}$$

The following rules are clear:

(i) $\sum\limits_{k=m}^{n} a_k = \sum\limits_{\nu=m}^{n} a_\nu$;
(ii) $\sum\limits_{k=m}^{n} a_k = \sum\limits_{k=m+p}^{n+p} a_{k-p}$ for all $p \in \mathbb{Z}$;
(iii) $\sum\limits_{k=m}^{n} a_k = \sum\limits_{k=m}^{n} a_{n+m-k}$;
(iv) $\sum\limits_{k=m}^{n} c a_k = c \sum\limits_{k=m}^{n} a_k$;
(v) $\sum\limits_{k=m}^{n} 1 = n - m + 1$ if $n \geq m$.

Example 2.36. $\sum\limits_{k=m}^{n} \Delta a_k$ is called a *telescoping* sum, where $\Delta a_k := a_{k+1} - a_k$ is called the *forward difference operator*. We have $\sum\limits_{k=m}^{n} \Delta a_k = a_{n+1} - a_m$.

Definition 2.37. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. Then we define $n!$ (read "*n factorial*") and the *binomial coefficient* $\binom{\alpha}{n}$ (read "α choose n") by

$$n! := \prod_{k=1}^{n} k \quad \text{and} \quad \binom{\alpha}{n} := \frac{\prod_{k=1}^{n} (\alpha + 1 - k)}{n!}.$$

Proposition 2.38. *Let $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. Then*

(i) $\binom{\alpha}{n} + \binom{\alpha}{n+1} = \binom{\alpha+1}{n+1}$;
(ii) $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ *if* $m \geq n$ *(and* 0 *if* $m < n$*);*
(iii) $\binom{m}{n} = \binom{m}{m-n}$ *if* $m \geq n$.

Definition 2.39. Let $a \in \mathbb{R}$. We define $a^0 = 1$, $a^1 = a$, and $a^{n+1} = a^n a$ for each $n \in \mathbb{N}$. If $-n \in \mathbb{N}$, then we put $a^n = \left(\frac{1}{a}\right)^{-n}$.

Proposition 2.40. *Let $a, b \in \mathbb{R} \setminus \{0\}$ and $p, q \in \mathbb{Z}$. Then*

(i) $a^p a^q = a^{p+q}$;
(ii) $(a^p)^q = a^{pq}$;
(iii) $(ab)^p = a^p b^p$.

Theorem 2.41 (The Binomial Formula)**.** *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Example 2.42. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for $n \in \mathbb{N}_0$ and $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$ for $n \in \mathbb{N}$.

Theorem 2.43 (Finite Geometric Series)**.** *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1} - 1}{a - 1} \text{ if } a \neq 1 \quad \text{and} \quad a^{n+1} - b^{n+1} = (a-b) \sum_{k=0}^{n} a^k b^{n-k}.$$

Theorem 2.44 (Bernoulli's Inequality)**.** *Let $n \in \mathbb{N}_0$ and $x \geq -1$. Then we have*

$$(1+x)^n \geq 1 + nx.$$

Definition 2.45. Let $x \in \mathbb{R}$. Then the *absolute value* of x is defined by

$$|x| := \max\{x, -x\}.$$

Proposition 2.46. *Let $a, b \in \mathbb{R}$. Then*

(i) $|a| = |-a|$;

(ii) $|a| \geq 0$; *and* $|a| = 0 \iff a = 0$;

(iii) $|ab| = |a||b|$;

(iv) $a = 0 \iff |a| < \varepsilon \; \forall \varepsilon > 0$.

Theorem 2.47 (Triangle Inequalities)**.** *If* $a, b \in \mathbb{R}$, *then*

$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$

Remark 2.48. Define $d(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. Then

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, y) \geq 0$; and $d(x, y) = 0 \iff x = y$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

CHAPTER 3

Sequences of Real Numbers

3.1. The Convergence of Sequences

Definition 3.1. If $x : \mathbb{N} \to \mathbb{R}$ is a function, then we call x a *sequence* (of real numbers). Instead of $x(n)$ we rather write x_n, $n \in \mathbb{N}$. The sequence s defined by $s_n = \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}$, is also known as a *series*.

Example 3.2.
(i) $a_n = 1 + (-1)^n$;
(ii) $a_n = \max\{k \in \mathbb{N} : k \leq \sqrt{n^3}\}$;
(iii) $x_0 = 1$ and $x_{n+1} = 2x_n$ for all $n \in \mathbb{N}_0$;
(iv) $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}_0$;
(v) $a_n = \sum_{k=1}^{n} \frac{1}{k}$.

Definition 3.3. A sequence a is said to be *convergent* if

$$\exists \alpha \in \mathbb{R}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N : \ |a_n - \alpha| < \varepsilon.$$

We write $\alpha = \lim_{n\to\infty} a_n$ or $a_n \to \alpha$ (as $n \to \infty$). A sequence is called *divergent* if it is not convergent.

Example 3.4.
(i) $a_n = \frac{2n}{4n+3}$;
(ii) $a_n = (-1)^n$.

Proposition 3.5. *Any sequence has at most one limit.*

Proposition 3.6 (Some Limits)**.** *We have*

(i) *If* $a_n = \alpha$ *for all* $n \in \mathbb{N}$*, then* $\lim_{n\to\infty} a_n = \alpha$*;*
(ii) $\lim_{n\to\infty} \frac{1}{n} = 0$*;*
(iii) *if* $|x| < 1$*, then* $\lim_{n\to\infty} x^n = 0$*;*
(iv) *if* $|x| < 1$*, then* $\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \frac{1}{1-x}$.

Definition 3.7. A sequence a is called *bounded* (or bounded above, or bounded below) if the set $\{a_n : n \in \mathbb{N}\}$ is bounded (or bounded above, or bounded below).

Proposition 3.8 (Necessary Conditions for Convergence). *Let a be a convergent sequence. Then*

(i) *a is bounded;*

(ii) *a satisfies the* Cauchy Condition, *i.e.,*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} :\ \forall m, n \geq N\ |a_n - a_m| < \varepsilon.$$

Remark 3.9. *$a_n \to \alpha$ implies $a_{n+1} - a_n \to 0,\ a_{2n} - a_n \to 0$.*

Example 3.10. (i) $a_n = (-1)^n$;

(ii) $a_n = \sum_{k=1}^{n} \frac{1}{k}$ (the *harmonic* series).

Theorem 3.11. *Suppose $a_n \to \alpha$ and $b_n \to \beta$ as $n \to \infty$. Then*

(i) $|a_n| \to |\alpha|$;

(ii) $a_n + b_n \to \alpha + \beta$;

(iii) $\forall c \in \mathbb{R} :\ ca_n \to c\alpha$;

(iv) $a_n \cdot b_n \to \alpha\beta$;

(v) $\frac{a_n}{b_n} \to \frac{\alpha}{\beta}$ *if* $\beta \neq 0$.

Example 3.12. (i) $a_n \to \alpha,\ m \in \mathbb{N} \implies a_n^m \to \alpha^m$;

(ii) $\frac{n^2-3}{2n^2+3n} \to \frac{1}{2}$ as $n \to \infty$.

Theorem 3.13. *Suppose $a_n \to \alpha$, $b_n \to \beta$, $c_n \in \mathbb{R}$. Then*

(i) $\exists K \in \mathbb{R}\ \forall n \in \mathbb{N} :\ |a_n| \leq K \implies |\alpha| \leq K$;

(ii) $\forall n \in \mathbb{N} :\ a_n \leq b_n \implies \alpha \leq \beta$;

(iii) $\alpha = \beta$ *and* $\forall n \in \mathbb{N} :\ a_n \leq c_n \leq b_n \implies \lim_{n\to\infty} c_n = \alpha$.

3.2. Monotone Sequences

Definition 3.14. A sequence a is called *monotonically increasing* (or monotonically decreasing, strictly increasing, strictly decreasing) provided $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$, $a_n < a_{n+1}$, $a_n > a_{n+1}$) holds for all $n \in \mathbb{N}$. We write $a_n \nearrow (\searrow, \uparrow, \downarrow)$. The sequence is called *monotone* if it is either one of the above.

Theorem 3.15 (The Monotone Convergence Theorem)**.** *A monotone sequence converges iff it is bounded.*

Example 3.16.
(i) $a_1 = 2$ and $a_{n+1} = \frac{a_n+6}{2}$ for all $n \in \mathbb{N}$;
(ii) $s_n = \sum_{k=1}^{n} \frac{1}{k}$;
(iii) $s_n = \sum_{k=1}^{n} \frac{1}{k2^k}$;
(iv) $a_n = \left(1 + \frac{1}{n}\right)^n$. We denote the limit of this sequence by e.

Definition 3.17. Let a_n be a sequence and let n_k be a sequence of natural numbers that is strictly increasing. Then the sequence b_k defined by $b_k = a_{n_k}$ for $k \in \mathbb{N}$ is called a *subsequence* of the sequence a_n.

Theorem 3.18. *Every sequence has a monotone subsequence.*

Theorem 3.19 (Bolzano–Weierstraß)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Every sequence in $[a, b]$ has a convergent subsequence that has its limit in $[a, b]$.*

Theorem 3.20 (Cauchy)**.** *A real sequence converges iff it is a Cauchy sequence.*

Proposition 3.21. *Let a_n be a convergent sequence with $\lim_{n\to\infty} a_n = \alpha$. Then every subsequence a_{n_k} of a_n converges with $\lim_{k\to\infty} a_{n_k} = \alpha$.*

Example 3.22.
(i) $a_n \to \alpha \implies a_{2n} \to \alpha,\ a_{n+1} \to \alpha$;
(ii) $\left(1 + \frac{1}{2n}\right)^{2n}$, $\left(1 + \frac{1}{n^2}\right)^{n^2}$;
(iii) $(-1)^n \left(1 + \frac{1}{n}\right)$.

Theorem 3.23 (The Nested Interval Theorem)**.** *Let $a_n, b_n \in \mathbb{R}$ with $a_n < b_n$ for all $n \in \mathbb{N}$, put $I_n = [a_n, b_n]$, and assume $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$ and $b_n - a_n \to 0$ as $n \to \infty$. Then $\bigcap_{n\in\mathbb{N}} I_n = \{\alpha\}$ with $\alpha \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha$ exist.*

CHAPTER 4

Continuous Functions

Definition 4.1. A function $f : D \to \mathbb{R}$ is said to be *continuous* at (or in) $x_0 \in D$ provided

$$\{x_n : n \in \mathbb{N}\} \subset D, \lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0).$$

Also, f is called *continuous* if it is continuous at each $x_0 \in D$.

Example 4.2. (i) $f(x) = x^2 + 3x - 2$, $x \in \mathbb{R}$;

(ii) $f(x) = \sqrt{x}$, $x \geq 0$;

(iii) $f = \chi_{[0,1]}$;

(iv) $f = \chi_{\mathbb{Q}}$ is called the *Dirichlet function*.

Notation 4.3. For two functions $f, g : D \to \mathbb{R}$ we define the sum $f + g : D \to \mathbb{R}$ and the product $f \cdot g : D \to \mathbb{R}$ by $(f+g)(x) = f(x)+g(x)$ and $(f \cdot g)(x) = f(x)g(x)$ for $x \in D$. If $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g} : D \to \mathbb{R}$ is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for $x \in D$.

Theorem 4.4. *Let $f, g : D \to \mathbb{R}$ be continuous functions. Then $f+g, f \cdot g : D \to \mathbb{R}$ are continuous. If $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g} : D \to \mathbb{R}$ is continuous.*

Corollary 4.5. *Let $m \in \mathbb{N}$, $c_k \in \mathbb{R}$ $(0 \leq k \leq m)$, and $p : \mathbb{R} \to \mathbb{R}$ be defined by $p(x) = \sum_{k=0}^{m} c_k x^k$, i.e., p is a* polynomial *with* degree *m if $c_m \neq 0$. Then p is continuous. Also, if p, q are both polynomials and $D = \{x \in \mathbb{R} : q(x) \neq 0\}$, then the* rational *function $\frac{p}{q} : D \to \mathbb{R}$ is continuous.*

Theorem 4.6. *If $f : D \to \mathbb{R}$, $g : U \to \mathbb{R}$ are functions with $f(D) \subset U$ such that f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$, then $g \circ f : D \to \mathbb{R}$ is continuous at $x_0 \in D$.*

Example 4.7. $\sqrt{1 - x^2}$, $x \in [-1, 1]$.

Theorem 4.8. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, where* $a, b \in \mathbb{R}$ *with* $a < b$. *Assume* $f(a) < 0$ *and* $f(b) > 0$. *Then* $\exists \alpha \in (a, b) : \ f(\alpha) = 0$.

Theorem 4.9 (The Intermediate Value Theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, where* $a, b \in \mathbb{R}$ *with* $a < b$. *If* $f(a) < c < f(b)$ *or* $f(b) < c < f(a)$, *then* $\exists\, \alpha \in (a, b) :$ $f(\alpha) = c$.

Example 4.10.
(i) $h(x) = x^5 + x + 1$, $x \in \mathbb{R}$, has a *zero* in $(-2, 0)$;
(ii) $h(x) = \frac{1}{\sqrt{1+x^2}} - x^2$, $x \in \mathbb{R}$, has a zero in $(0, 1)$;
(iii) if $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

Theorem 4.11 (The Extreme Value Theorem)**.** *Let* $f : I = [a, b] \to \mathbb{R}$ *be continuous, where* $a, b \in \mathbb{R}$ *with* $a < b$. *Then both* $\max f(I)$ *and* $\min f(I)$ *exist.*

Definition 4.12. Let $D \subset \mathbb{R}$. The function $f : D \to \mathbb{R}$ is called *strictly increasing* (or strictly decreasing, increasing, decreasing) if $f(v) > f(u)$ (or $f(v) < f(u)$, $f(v) \geq f(u)$, $f(v) \leq f(u)$) holds for all $u, v \in D$ with $u < v$. We write $f \uparrow$ $(\downarrow, \nearrow, \searrow)$. Also, f is called strictly monotone if it is either strictly increasing or strictly decreasing.

Theorem 4.13. *Let* $f : I \to f(I)$ *be strictly monotone, where* I *is an interval. Then* f *is invertible and* $f^{-1} : f(I) \to I$ *is continuous and strictly monotone.*

Corollary 4.14. *Suppose* I *is an interval and* $f : I \to \mathbb{R}$ *is strictly monotone. Then* f *is continuous iff* $f(I)$ *is an interval.*

Theorem 4.15. *Let* $x_0 \in D \subset \mathbb{R}$ *and* $f : D \to \mathbb{R}$. *Then* f *is continuous at* x_0 *iff*

$$\forall \varepsilon > 0\ \exists \delta > 0\ (\forall x \in D :\ |x - x_0| < \delta) \quad |f(x) - f(x_0)| < \varepsilon.$$

Example 4.16.
(i) $f(x) = \sqrt{x}$, $f : [0, \infty) \to [0, \infty)$ is continuous at $x_0 = 4$;
(ii) $f(x) = x^3$, $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 = 2$;
(iii) f from (ii) is continuous on $D = [0, 20]$;
(iv) $f(x) = \frac{1}{x}$, $f : (0, 1) \to \mathbb{R}$.

Definition 4.17. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is called *uniformly continuous* (on D) if

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ (\forall u, v \in D :\ |u - v| < \delta) \quad |f(u) - f(v)| < \varepsilon.$$

Theorem 4.18. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, where* $a, b \in \mathbb{R}$ *with* $a < b$*. Then* f *is uniformly continuous.*

CHAPTER 5

Differentiation

5.1. Differentiation Rules

Definition 5.1. (i) An $x_0 \in \mathbb{R}$ is called a *limit point* of D if there exists $\{x_n : n \in \mathbb{N}\} \subset D \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$.

(ii) We write $\lim_{x\to x_0, x\in D} f(x) = l$ provided x_0 is a limit point of D and $\lim_{n\to\infty} f(x_n) = l$ whenever $\{x_n : n \in \mathbb{N}\} \subset D \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$.

Example 5.2. (i) $\lim_{x\to 4}(x^2 - 2x + 3) = 11$;

(ii) $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

Remark 5.3. (i) Let $x_0 \in D$ be a limit point of D. Then $f : D \to \mathbb{R}$ is continuous at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$.

(ii) If x_0 is a limit point of D and $f, g : D \to \mathbb{R}$ with $\lim_{x\to x_0} f(x) = \alpha \in \mathbb{R}$ and $\lim_{x\to x_0} g(x) = \beta \in \mathbb{R}$, then (by Theorem 3.11)

$$\lim_{x\to x_0} ((f+g)(x)) = \alpha + \beta, \quad \lim_{x\to x_0} ((fg)(x)) = \alpha\beta,$$

and (if $\beta \neq 0$)

$$\lim_{x\to x_0} ((f/g)(x)) = \alpha/\beta.$$

Definition 5.4. Let $x_0 \in (a, b) = I$. A function $f : I \to \mathbb{R}$ is called *differentiable* at (or in) x_0 provided

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, in which case we denote this limit by $f'(x_0)$. Also, f is called differentiable (on I) if $f'(x)$ exists for all $x \in I$. In this case, $f' : I \to \mathbb{R}$ is called the *derivative* of f.

Example 5.5. (i) $f(x) = 4x - 5$;

(ii) $f(x) = mx + b$;

(iii) $f(x) = x^2$;

(iv) $f(x) = |x|$.

Proposition 5.6. *Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^m$ for all $x \in \mathbb{R}$. Then f is differentiable and $f'(x) = mx^{m-1}$.*

Proposition 5.7. *Let $x_0 \in (a, b) = I$. If $f : I \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.*

Theorem 5.8 (Rules of Differentiation)**.** *Let $x_0 \in (a, b) = I$.*

(i) *If $f, g : I \to \mathbb{R}$ are differentiable in x_0, then so is $\alpha f + \beta g$ for all $\alpha, \beta \in \mathbb{R}$, fg, and (if $g(x_0) \neq 0$) f/g with*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0),$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \text{Product Rule,}$$

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} \quad \text{Quotient Rule.}$$

(ii) *If $g : I \to g(I)$ is differentiable in x_0 and if $f : J \to \mathbb{R}$ with $J \supset g(I)$ is differentiable in $g(x_0)$, then $f \circ g : I \to \mathbb{R}$ is differentiable in x_0 with*

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0) \quad \text{Chain Rule.}$$

(iii) *If $f : I \to f(I)$ is continuous and strictly monotone and differentiable in x_0 with $f'(x_0) \neq 0$, then $f^{-1} : f(I) \to I$ is differentiable in $y_0 = f(x_0)$ with*

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Example 5.9. (i) $f(x) = \sqrt[n]{x}$;

(ii) $f(x) = x^{p/q}$.

5.2. The Mean Value Theorems

Definition 5.10. Suppose $I = (a, b)$ with $a < b$ and $f : I \to \mathbb{R}$.

(i) An $x_0 \in I$ is called a *local maximizer* (or local minimizer) of f, if there exists $\delta > 0$ such that $f(x_0) \geq f(x)$ (or $f(x_0) \leq f(x)$) for all $x \in I$ with $|x - x_0| < \delta$.

(ii) An $x_0 \in I$ for which $f'(x_0)$ exists is called a *critical point* of f provided $f'(x_0) = 0$.

Theorem 5.11. *Suppose $I = (a, b)$ with $a < b$ and $f : I \to \mathbb{R}$. Assume that $x_0 \in I$ is such that $f'(x_0)$ exists. If x_0 is a local maximizer (or minimizer) of f, then it is a critical point.*

Theorem 5.12 (Rolle's Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ with $a < b$ is continuous on $[a, b]$ and differentiable on (a, b). Assume $f(a) = f(b) = 0$. Then there exists a critical point of f in (a, b).*

Theorem 5.13 (The Lagrange Mean Value Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ with $a < b$ is continuous on $[a, b]$ and differentiable on (a, b). Then*

$$\exists\, x_0 \in (a, b) : \ f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 5.14 (The Cauchy Mean Value Theorem)**.** *Suppose that $f, g : [a, b] \to \mathbb{R}$ with $a < b$ both are continuous on $[a, b]$ and differentiable on (a, b). Then*

$$\exists\, x_0 \in (a, b) : \ f'(x_0)\{g(b) - g(a)\} = g'(x_0)\{f(b) - f(a)\}.$$

Example 5.15.
- (i) $f, g : [0, 3] \to \mathbb{R}$ defined by $f(x) = 3 - x^2$ and $g(x) = \sqrt{9 - x^2}$;
- (ii) $e(x) \geq 1 + x$ for all $x \in \mathbb{R}$;
- (iii) Generalized Bernoulli inequality.

5.3. Applications of the Mean Value Theorems

Theorem 5.16 (The Identity Criterion)**.** *Let $I \subset \mathbb{R}$ be an interval and suppose that $f : I \to \mathbb{R}$ is differentiable on I. Then f is* constant *on I (i.e., there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in I$) iff $f'(x) = 0$ for all $x \in I$.*

Theorem 5.17. *Let I be an interval and $f : I \to \mathbb{R}$ be differentiable on I.*

- (i) *If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.*
- (ii) *If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*

Example 5.18.
- (i) e and l are strictly increasing;
- (ii) $f(x) = \frac{ax+b}{cx+d}$.

Theorem 5.19 (L'Hôpital's Rules)**.** *Let $I = [a, b) \subset \mathbb{R}$, $a < b$, $b \in \mathbb{R}$ or $b = \infty$, and suppose that $f, g : I \to \mathbb{R}$ are differentiable on I with $g'(x) \neq 0$ for all $x \in I$.*

Assume that $\alpha = \lim_{x\to b, x<b} \frac{f'(x)}{g'(x)}$ *exists. If either*

$$\lim_{x\to b, x<b} f(x) = \lim_{x\to b, x<b} g(x) = 0 \quad \textit{or} \quad \lim_{x\to b, x<b} g(x) = \infty,$$

then $\lim_{x\to b, x<b} \frac{f(x)}{g(x)}$ *exists and is equal to* α.

Example 5.20.

(i) $\lim_{x\to 0} \frac{l(1+x)}{x} = 1$;
(ii) $\lim_{x\to 0} \frac{l(1+x)-x}{x^2} = -\frac{1}{2}$;
(iii) $\lim_{x\to\infty} x^n e(-x) = 0$ for all $n \in \mathbb{N}$;
(iv) $\lim_{x\to 0, x>0} x l(x) = 0$;
(v) $\lim_{x\to 0, x>0} A(x,x) = 1$.

Notation 5.21. If I is an interval and $f : I \to \mathbb{R}$ is differentiable with $f' : I \to \mathbb{R}$, and $f' : I \to \mathbb{R}$ is also differentiable, then we write $f'' = (f')' = f^{(2)}$. If $f^{(k)}$ for $k \in \mathbb{N}$ is defined and differentiable, we put $f^{(k+1)} = \left(f^{(k)}\right)'$. Also, we put $f^{(0)} = f$.

Theorem 5.22. *Let* I *be an interval,* $n \in \mathbb{N}$*, and suppose* $f : I \to \mathbb{R}$ *has* n *derivatives. If* $f^{(k)}(x_0) = 0$ *for all* $0 \le k \le n-1$ *for some* $x_0 \in I$*, then, for each* $x \in I \setminus \{x_0\}$*, there exists a point* z *strictly between* x *and* x_0 *with*

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n.$$

Theorem 5.23. *Let* I *be an interval and suppose* $f : I \to \mathbb{R}$ *is such that the below derivatives exist and are continuous. Assume* $x_0 \in I$ *is a critical point of* f.

(i) *If* $f''(x_0) > 0$*, then* x_0 *is a local minimizer of* f.
(ii) *If* $f''(x_0) < 0$*, then* x_0 *is a local maximizer of* f.
(iii) *If* $f''(x_0) = 0$ *and* $f'''(x_0) \neq 0$*, then* x_0 *is neither a local minimizer nor a local maximizer of* f.
(iv) *If* $f''(x_0) = 0$ *and* $f'''(x_0) = 0$ *and* $f''''(x_0) > 0$*, then* x_0 *is a local minimizer of* f.
(v) *If* $f''(x_0) = 0$ *and* $f'''(x_0) = 0$ *and* $f''''(x_0) < 0$*, then* x_0 *is a local maximizer of* f.

Theorem 5.24 (Lagrange Remainder Theorem)**.** *Let* I *be an open interval containing the point* x_0 *and let* $n \in \mathbb{N}_0$*. Suppose that* $f : I \to \mathbb{R}$ *has* $n+1$ *derivatives. Then for each* $x \in I \setminus \{x_0\}$*, there exists* z *strictly between* x *and* x_0 *such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1}.$$

Definition 5.25. Let I be an open interval containing x_0 and $n \in \mathbb{N}_0$. Suppose that $f : I \to \mathbb{R}$ has n derivatives. The nth *Taylor polynomial* for the function $f : I \to \mathbb{R}$ at the point x_0 is defined as

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Example 5.26. Find p_3 for $f(x) = 1/x$ at $x_0 = 1$.

Theorem 5.27. *Let I be an open interval containing x_0 and suppose $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose there are positive numbers r and M such that $[x_0 - r, x_0 + r] \subset I$ and $|f^{(n)}(x)| \leq M^n$ for all $x \in [x_0 - r, x_0 + r]$. Then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{if} \quad |x - x_0| \leq r.$$

Example 5.28. $e(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Note also that $e \in \mathbb{R} \setminus \mathbb{Q}$.

CHAPTER 6

Integration

6.1. The Definition of the Integral

Definition 6.1. Let $f : [a, b] \to \mathbb{R}$ with $a < b$ be a function. If $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, then $\mathcal{Z} = \{x_0, x_1, \cdots, x_n\}$ is called a *partition* of the interval $[a, b]$ with *gap* $\|\mathcal{Z}\| = \max\{x_k - x_{k-1} : 1 \leq k \leq n\}$, and if $\xi_k \in [x_{k-1}, x_k]$ for all $1 \leq k \leq n$, then we call $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ *intermediate points* of the partition $\mathcal{Z}$. The sum

$$S(f, \mathcal{Z}, \xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

is called a *Riemann sum.* If ξ is such that $f(\xi_k) = \inf f([x_{k-1}, x_k])$ for all $1 \leq k \leq n$ (or $f(\xi_k) = \sup f([x_{k-1}, x_k])$ for all $1 \leq k \leq n$), then we call $L(f, \mathcal{Z}) = S(f, \mathcal{Z}, \xi)$ the *lower Darboux sum* (or $U(f, \mathcal{Z}) = S(f, \mathcal{Z}, \xi)$ the *upper Darboux sum*).

Definition 6.2. A function $f : [a, b] \to \mathbb{R}$ with $a < b$ is said to be *Riemann integrable* if $\lim_{n\to\infty} S(f, \mathcal{Z}_n, \xi^n)$ exists for any sequence of partitions $\mathcal{Z}_n$ with $\lim_{n\to\infty} \|\mathcal{Z}_n\| = 0$ and with intermediate points ξ^n.

Remark 6.3. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then, no matter what sequences $\mathcal{Z}_n$ and ξ^n we take, the limit of $S(f, \mathcal{Z}_n, \xi^n)$ as $n \to \infty$ is always the same. We then call this limit $\int_a^b f(x)\mathrm{d}x = \int_a^b f$.

Example 6.4. $f(x) = x$, $I = [a, b]$.

Proposition 6.5. *Let $a < b$ and $I = [a, b]$.*

(i) *If $f, g : I \to \mathbb{R}$ are Riemann integrable, then so is $\alpha f + \beta g$ for all $\alpha, \beta \in \mathbb{R}$ with*

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

(ii) *If $f(x) = c$ for all $x \in I$, then f is Riemann integrable with $\int_a^b f = c(b - a)$.*

(iii) *If* $f, g : I \to \mathbb{R}$ *are Riemann integrable and* $f(x) \leq g(x)$ *for all* $x \in I$*, then* $\int_a^b f \leq \int_a^b g$.

(iv) *If* $f : I \to \mathbb{R}$ *is Riemann integrable, then* f *is bounded on* I *and*

$$\inf f(I) \leq \frac{\int_a^b f}{b-a} \leq \sup f(I).$$

(v) *If* $f : I \to \mathbb{R}$ *is Riemann integrable and if* $g(x) = f(x)$ *for all* $x \in I$ *but a finite number of points* $x \in I$*, then* g *is Riemann integrable and* $\int_a^b f = \int_a^b g$.

(vi) *If* $c \in (a, b)$ *and* $f : I \to \mathbb{R}$ *and* $f : [a, c] \to \mathbb{R}$, $f : [c, b] \to \mathbb{R}$ *are Riemann integrable, then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 6.6. *If* $f : [a, b] \to \mathbb{R}$ *with* $a < b$ *is continuous, then it is Riemann integrable.*

Notation 6.7. If $a > b$, then we put $\int_a^b f = -\int_b^a f$. We also put $\int_a^a f = 0$.

6.2. The Fundamental Theorem of Calculus

Theorem 6.8 (Fundamental Theorem of Calculus, First Part)**.** *Suppose* $F : [a, b] \to \mathbb{R}$ *is differentiable on* $[a, b]$ *and* $F' : [a, b] \to \mathbb{R}$ *is Riemann integrable on* $[a, b]$*. Then*

$$\int_a^b F' = F(b) - F(a).$$

Definition 6.9. A function $F : I \to \mathbb{R}$ is called an *antiderivative* of $f : I \to \mathbb{R}$ if F is differentiable with $F'(x) = f(x)$ for all $x \in \mathbb{I}$.

Remark 6.10. If f possesses an antiderivative F, then any other antiderivative of f can differ from F only by a constant.

Example 6.11. (i) $\int_0^5 x^3 \mathrm{d}x = \frac{5^4}{4}$;

(ii) $\int_0^4 e = e(4) - 1$;

(iii) $\int_0^p s = 1$;

(iv) $\frac{1}{n^6} \sum_{k=1}^n k^5 \to \frac{1}{6}$ as $n \to \infty$.

Theorem 6.12 (Fundamental Theorem of Calculus, Second Part)**.** *Let* $f : I \to \mathbb{R}$ *be continuous on the interval* $I \subset \mathbb{R}$ *and let* $a \in I$*. Then*

$$F(x) := \int_a^x f \quad \textit{for each} \quad x \in I$$

is an antiderivative of f.

Remark 6.13. Continuous functions possess antiderivatives.

Proposition 6.14. *If f is Riemann integrable on I, then F defined in the FTOC (Part II) is continuous (even Lipschitz continuous) on I.*

Example 6.15. (i) $\int_1^x \frac{\mathrm{d}t}{t}$;
(ii) $\int_0^x \frac{\mathrm{d}t}{1+t^2}$.

6.3. Applications

Theorem 6.16. *Suppose $f, g : I \to \mathbb{R}$ are continuous, $x_0 \in I$, $y_0 \in \mathbb{R}$. Then there exists exactly one continuously differentiable function y with $y(x_0) = y_0$ and $y'(x) = f(x)y(x) + g(x)$ for all $x \in I$, namely*

$$y(x) = e(F(x))\left\{y_0 + \int_{x_0}^x g(t)e(-F(t))\mathrm{d}t\right\} \qquad \textit{with} \qquad F(x) = \int_{x_0}^x f(t)\mathrm{d}t.$$

Example 6.17. $xy' + 2y = 4x^2$, $y(1) = 2$.

Theorem 6.18 (Integration by Parts). *Let $f, g : [a, b] \to \mathbb{R}$ be continuously differentiable. Then*

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\mathrm{d}x.$$

Example 6.19. $\int_0^1 te(t)\mathrm{d}t = 1$.

Theorem 6.20 (Substitution). *If $g : [\alpha, \beta] \to \mathbb{R}$ is continuously differentiable, $f : g([\alpha, \beta]) \to \mathbb{R}$ continuous, then*

$$\int_a^b f(g(t))g'(t)\mathrm{d}t = \int_{g(a)}^{g(b)} f(x)\mathrm{d}x.$$

Example 6.21. $\int_0^2 e(\sqrt{x})\mathrm{d}x = 2(\sqrt{2} - 1)e(\sqrt{2}) + 2$.

6.4. Improper Integrals

Definition 6.22. Let $a < b$ and $f : (a, b) \to \mathbb{R}$.

(i) f is said to be *locally integrable* on (a, b) if f is integrable on each closed subinterval $[c, d] \subset (a, b)$.

(ii) f is said to be *improperly integrable* on (a, b) if f is locally integrable on (a, b) and if

$$\int_a^b f(x)\mathrm{d}x := \lim_{c\to a^+, d\to b^-} \int_c^d f(x)\mathrm{d}x$$

exists and is finite. This limit is called the *improper Riemann integral* of f over (a, b).

Example 6.23. (i) $\int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x = 2$;
(ii) $\int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1$.

Theorem 6.24. *If f, g are improperly integrable on (a, b) and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is improperly integrable on (a, b), and*

$$\int_a^b (\alpha f + \beta g)(x)\mathrm{d}x = \alpha \int_a^b f(x)\mathrm{d}x + \beta \int_a^b g(x)\mathrm{d}x.$$

Theorem 6.25 (Comparison Theorem)**.** *Suppose $f, g : (a, b) \to \mathbb{R}$ are locally integrable. If $0 \le f(x) \le g(x)$ for all $x \in (a, b)$, and if g is improperly integrable on (a, b), then f is improperly integrable on (a, b) with*

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x.$$

Example 6.26. (i) $|s(x)/\sqrt{x^3}|$ is improperly integrable on $(0, 1]$;
(ii) $|l(x)/\sqrt{x^5}|$ is improperly integrable on $[1, \infty)$.

Definition 6.27. Let $a < b$ and $f : (a, b) \to \mathbb{R}$.

(i) f is said to be *absolutely integrable* on (a, b) if $|f|$ is improperly integrable on (a, b).
(ii) f is said to be *conditionally integrable* on (a, b) if f is improperly integrable but not absolutely integrable on (a, b).

Theorem 6.28. *If f is locally and absolutely integrable on (a, b), then f is improperly integrable on (a, b), and*

$$\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |f(x)|\mathrm{d}x.$$

Example 6.29. $s(x)/x$ is conditionally integrable on $[1, \infty)$.

CHAPTER 7

Infinite Series of Functions

7.1. Uniform Convergence

Example 7.1. (i) $\lim_{x\to x_0}\lim_{n\to\infty}(1+x/n)^n = \lim_{n\to\infty}\lim_{x\to x_0}(1+x/n)^n$;

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\lim_{n\to\infty}(1+x/n)^n = \lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}(1+x/n)^n$;

(iii) $\int_0^1 \lim_{n\to\infty}(1+x/n)^n \mathrm{d}x = \lim_{n\to\infty}\int_0^1 (1+x/n)^n \mathrm{d}x$;

(iv) $f_n(x) = nx/(1+nx)$, $n\to\infty$, $x\to 0$;

(v) $f_n(x) = x^n$;

(vi) $f_n(x) = \frac{s(nx)}{n}$.

Definition 7.2. Let $f_n : I \to \mathbb{R}$ be functions for each $n \in \mathbb{N}$ and let $f : I \to \mathbb{R}$. We say that the sequence f_n converges

(i) *pointwise* to f if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in I$;

(ii) *uniformly* to f if

$$\forall\varepsilon > 0\ \exists N \in \mathbb{N} :\ (\forall n \geq N\ \forall x \in I)\ |f_n(x) - f(x)| < \varepsilon.$$

The pointwise or uniform convergence of the series $\sum_{k=0}^{\infty} g_k$ is defined as above with $f_n = \sum_{k=0}^{n} g_k$.

Example 7.3. Let $f_n(x) = x^n$ on $[0, 1]$.

(i) f_n converges uniformly on $[0, 1/2]$;

(ii) f_n does not converge uniformly on $[0, 1]$.

Example 7.4. $f_n(x) = 2n^2x/(1 + n^4x^4)$ is not uniformly convergent on $\mathbb{R}$.

Theorem 7.5 (Cauchy Criterion). *A sequence of functions $f_n : I \to \mathbb{R}$ converges uniformly on I iff*

$$\forall\varepsilon > 0\ \exists N \in \mathbb{N} :\ (\forall m, n \geq N\ \forall x \in I)\ |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem 7.6 (Weierstraß M-Test)**.** *Suppose $g_k : I \to \mathbb{R}$ satisfies $|g_k(x)| \le M_k$ for all $x \in I$ and for all $k \in \mathbb{N}$ such that $\sum_{k=1}^{\infty} M_k$ is convergent. Then $\sum_{k=1}^{\infty} g_k(x)$ is uniformly convergent.*

Example 7.7. $\sum_{k=1}^{\infty} \frac{c(k^2\sqrt{x})}{k^3}$ is uniformly convergent on $\mathbb{R}$.

7.2. Interchanging of Limit Processes

Theorem 7.8 (Continuity of the Limit Function)**.** *Let $f_n : I \to \mathbb{R}$ be continuous on I for all $n \in \mathbb{N}$ and suppose that $f_n \to f$ uniformly on I. Then f is continuous on I, i.e.,*

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) \quad \textit{for all} \quad x_0 \in I.$$

Example 7.9. $f(x) = \sum_{k=1}^{\infty} \frac{s(kx)}{k^2}$ is continuous on $\mathbb{R}$.

Theorem 7.10 (Integration of the Limit Function)**.** *Let $f_n : I = [a,b] \to \mathbb{R}$ be Riemann integrable on I for all $n \in \mathbb{N}$ and suppose that $f_n \to f$ uniformly on I. Then f is Riemann integrable on I with*

$$\left(\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{d}x = \right) \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x.$$

Corollary 7.11. $\sum_{k=1}^{\infty} \int_a^b f_k(x)\mathrm{d}x = \int_a^b \sum_{k=1}^{\infty} f_k(x)\mathrm{d}x$ *if $f_k : [a,b] \to \mathbb{R}$ are Riemann integrable for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} f_k(x)$ is uniformly convergent on $[a,b]$.*

Theorem 7.12 (Differentiation of the Limit Function)**.** *Let $f_n : I = [a,b] \to \mathbb{R}$ be differentiable on I for all $n \in \mathbb{N}$ and suppose that $f_n' \to g$ uniformly on I. Also suppose that $\lim_{n\to\infty} f_n(x_0)$ exists for at least one $x_0 \in I$. Then f_n converges uniformly on I, say to f, and f is differentiable on I with*

$$f'(x) = g(x), \qquad \textit{i.e.,} \qquad \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x) = \frac{\mathrm{d}}{\mathrm{d}x} \lim_{n\to\infty} f_n(x).$$

Corollary 7.13. $\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_k(x)$ *if $f_k : [a,b] \to \mathbb{R}$ are differentiable for all $k \in \mathbb{N}$, $\sum_{k=1}^{\infty} f_k'(x)$ is uniformly convergent on $[a,b]$, and $\sum_{k=1}^{\infty} f_k(x_0)$ is convergent for at least one $x_0 \in [a,b]$.*

Example 7.14. (i) $\sum_{k=0}^{\infty} a_k (x - x_0)^k$;
(ii) $\sum_{k=1}^{\infty} \frac{s(kx)}{k^3}$;
(iii) $f_n(x) = \frac{s(n^2 x)}{n}$.