51. Define $f(x) = 6x^7 - 14x^6 + 21x^4 - 14x^3 + 1$ for $x \in I$, $f : I = [-10, 10] \to \mathbb{R}$. Find $\min f(I)$ and $\max f(I)$.

52. Let $n \in \mathbb{N}$ and suppose $f, g : (a, b) \to \mathbb{R}$ both have n derivatives. Show

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \quad \text{for all} \quad x \in (a, b).$$

53. Find **all** functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

(a) $f'(x) = f(x)$ for all $x \in \mathbb{R}$, $f(0) = 1$;

(b) $f''(x) = f(x)$ for all $x \in \mathbb{R}$, $f(0) = 0$, $f'(0) = 1$;

(c) $f''(x) = f(x)$ for all $x \in \mathbb{R}$, $f(0) = 1$, $f'(0) = 0$;

54. Consider the problem $f''(x) = -f(x)$ for all $x \in \mathbb{R}$, $f(0) = 0$, $f'(0) = 1$.

(a) Show that the above problem has at most one solution. (Assume there is a solution and call it s, put $c = s'$.)

(b) Show that s is odd (i.e., $s(-x) = -s(x)\ \forall x \in \mathbb{R}$), that c is even (i.e., $c(-x) = c(x)\ \forall x \in \mathbb{R}$), and that $s^2(a) + c^2(a) = 1$, $|s(a)| \le 1$, $|c(a)| \le 1$, $s(a+b) = s(a)c(b) + s(b)c(a)$, $c(a+b) = c(a)c(b) - s(a)s(b)\ \forall a, b \in \mathbb{R}$.

(c) Show that $p = \min\{x > 0 : c(x) = 0\}$ exists. Show that both s and c have period $4p$ (i.e., $s(x+4p) = s(x)$, $c(x+4p) = c(x)\ \forall x \in \mathbb{R}$).

(d) Show that $s : [-p, p] \to [-1, 1]$ and $c : [0, 2p] \to [-1, 1]$ are invertible and calculate the derivatives of s^{-1} and c^{-1}.