Example 6.4: Let Z be any partition of $[a,b]$, $Z=\{x_0,\dots,x_n\}$, $x_0=a$, $x_n=b$. Then let $x_{k-1}\le \xi_k \le x_k$ and define $\eta_k = \xi_k - \frac{x_k+x_{k-1}}{2}$ so that $-\frac{x_k-x_{k-1}}{2}\le \eta_k \le \frac{x_k-x_{k-1}}{2}$, i.e.,

$$\xi_k = \frac{x_k+x_{k-1}}{2}+\eta_k \text{ with } |\eta_k|\le \frac{x_k-x_{k-1}}{2}\le \frac{\|Z\|}{2}\ \forall 1\le k\le n$$

Now

$$S(f,Z,\xi)=\sum_{k=1}^{n} f(\xi_k)(x_k-x_{k-1})$$

$$=\sum_{k=1}^{n}\left(\frac{x_k+x_{k-1}}{2}+\eta_k\right)(x_k-x_{k-1})$$

$$=\frac{1}{2}\sum_{k=1}^{n}(x_k^2-x_{k-1}^2)+\sum_{k=1}^{n}\eta_k(x_k-x_{k-1})$$

$$=\frac{1}{2}(x_n^2-x_0^2)+\sum_{k=1}^{n}\eta_k(x_k-x_{k-1})$$

$$=\frac{b^2-a^2}{2}+\sum_{k=1}^{n}\eta_k(x_k-x_{k-1})$$

and hence

$$\left|S(f,Z,\xi)-\frac{b^2-a^2}{2}\right|\le\sum_{k=1}^{n}|\eta_k|(x_k-x_{k-1})$$

$$\le\frac{\|Z\|}{2}\sum_{k=1}^{n}(x_k-x_{k-1})=\frac{\|Z\|}{2}(x_n-x_0)=\|Z\|\frac{b-a}{2}$$

et $\varepsilon>0$. Let $\|Z\|<\frac{2\varepsilon}{b-a}$. Then $\left|S(f,Z,\xi)-\frac{b^2-a^2}{2}\right|<\varepsilon$. So $\int_a^b f=\frac{b^2-a^2}{2}$.