

# Asymptotic Behavior of Dynamic Equations on Time Scales

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As a way to unify a discussion of many kinds of problems for equations in the continuous and discrete case (but also in order to reveal discrepancies between both cases), a theory of “time scales” was proposed and developed by Aulbach and Hilger. In our paper we investigate the asymptotic behavior of so-called dynamic equations on time scales, and such dynamic equations are differential equations in the continuous case and difference equations in the discrete case. We offer a perturbation result that leads to a time scales version of Levinson’s Fundamental Lemma. Crucial are a dichotomy condition and a growth condition on the perturbation. Also, in the case that Levinson’s result cannot be applied immediately, we suggest several preliminary transformations that might lead to a situation where Levinson’s lemma is applicable. Such transformations have been suggested by Harris and Lutz in the continuous case and by Benzaid and Lutz in the discrete case. Both those cases are covered by our theory, plus cases “in between”. Examples for such cases will also be discussed in this paper.

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## 1. INTRODUCTION

The purpose of this paper is to provide a unified treatment leading to a framework for discussing the asymptotic behavior of solutions of

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certain kinds of dynamic equations. For two major extreme cases, the so-called “continuous” case in which the equations are of the differential type, and the “discrete” case in which the equations are of the difference type, results of this type are rather well known. The differential equations case involves results on the theory of asymptotic integration, which has been well-documented in the monograph of Eastham [6] (see also the papers by Harris and Lutz [9–11]). The difference equations case involves results on the asymptotic behavior of products, and treatments can be found in the paper by Benzaid and Lutz [4] and the monographs of Agarwal [1] and Elaydi [7].

As a way to unify a discussion of many kinds of problems for equations in the continuous and discrete cases, a theory of “time scales” was proposed and developed extensively by Aulbach and Hilger (see [3] for an excellent introduction into this subject; see also [2,12,13]). Aulbach and Hilger make the important point that while in many cases proofs and methods for the two cases are analogous, it cannot and should not be just stated that results in the continuous case can automatically or necessarily be transformed or extended to the discrete case. The theory of time scales provides a mechanism, however, for treating both cases really simultaneously.

The central object, the *time scale*  $\mathbb{T}$ , is a closed subset of  $\mathbb{R}$ , and from  $\mathbb{T}$  a naturally-defined operator  $\Delta$  on the time scale determines the dynamic equations. The case  $\mathbb{T} = \mathbb{R}$  corresponds to the continuous case where  $\Delta$  is the usual derivative, and the case  $\mathbb{T} = \mathbb{N}$  (the natural numbers) corresponds to the discrete case where  $\Delta$  is the usual forward difference operator. For other choices of time scales one obtains other types of dynamic equations (e.g.,  $q$ -difference equations) which can be thought of as lying somewhere in between the continuous and discrete cases. Here, we will develop methods which lead to the asymptotic behavior of solutions of such equations on time scales. This means finding conditions on the coefficients of a dynamic equation so that solutions may be represented asymptotically in terms of known elementary functions, functions which satisfy rather simple, scalar equations, and functions which are asymptotic to constants as  $t \rightarrow \infty$  in  $\mathbb{T}$ . So basically the idea is to reduce the study of solutions of a system of dynamic equations to the one-dimensional (scalar) case. For differential equations such simple equations can be explicitly solved using exponentials and quadrature, while in the difference equation case

solutions can be constructed as elementary products. In the general case one is lead to certain scalar dynamic equations on time scales whose solutions then are fundamental and characteristic of the particular time scale.

While this leads to a single, unified approach which contains many cases in one and so represents an economical way to simultaneously approach asymptotic behavior for solutions of many kinds of equations at once, there is another motivation for doing this than just generalization. This concerns a check and balance on the types of assumptions that arise in the asymptotic theory. While the differential equation case has a long history and was developed in many stages and a corresponding theory for difference equations was developed mainly by analogy with the continuous case, it is by no means clear that the asymptotic theories are in the end the most natural. But by proving results for general time scales, we can compare them with the known results in both of the classical cases and test whether or not the results themselves are natural in the sense that they are both special cases of a theory that bridges the gap between the cases.

The asymptotic behavior of solutions of the dynamic equation

$$x^\Delta = A(t)x$$

will mean for us a representation of a (fundamental) matrix of solutions in the form

$$Y(t) = P(t)[I + E(t)]D(t),$$

where  $P(t)$  represents an explicitly known matrix of elementary functions,  $D(t)$  represents a diagonal matrix of solutions of scalar time scale equations, and  $E(t)$  represents a matrix of functions satisfying  $E(t) \rightarrow 0$  as  $t \rightarrow \infty$ . To accomplish this, we think of  $P(t)$  constructed from certain explicit linear transformations  $x = P(t)y$  of the original system into a system whose main term is diagonal with a "small" perturbation. The factor  $[I + E(t)]$  in the representation comes from two sources. One involves further, so-called preliminary transformations, whose role is to bring the system into almost diagonal form, which means that the coefficient matrix consists of an explicit diagonal matrix (possibly different from the former one) plus an absolutely summable perturbation (with respect to the time scale). The other part contributing

to  $[I + E(t)]$  arises from determining the asymptotic behavior of the almost diagonal system, and this follows from an analogue for dynamic equations on time scales of an important and fundamental result of Levinson [14] for differential equations. For Levinson's result, two kinds of assumptions are required; the first is the summability condition on the perturbation and the second is a so-called dichotomy, or separation condition on the elements of the diagonal matrix. The purpose of the dichotomy condition is to insure that solutions of the diagonal system have sufficiently different asymptotic behavior which small perturbations cannot influence. The dichotomy conditions that we find for the time scale case are natural generalizations of both Levinson's classical ones and those of Benzaid and Lutz [4] in the case of difference equations.

The paper is organized as follows. In the next Section 2, we first present (for the reader's convenience) some basic definitions and results on time scales as developed by Hilger and Aulbach. An additional result, an analogue of Hölder's inequality for time scales, is presented for later use. Next, in Section 3, we prove a theorem concerning the asymptotic behavior of bounded solutions of perturbed, dynamic equations on time scales, which plays an essential role in the proof of the analogue of Levinson's Fundamental "Lemma" in the following Section 4. Then, in Section 5, we discuss various types of preliminary transformations which are used to bring a system into a form so that Levinson's Lemma on time scales is applicable. Finally in Section 6, we first discuss the asymptotic behavior of solutions of scalar, first order dynamic equations on time scales. These are the simplest functions which are needed for the asymptotic analysis and are the natural analogues for time scales of exponential functions for the differential equation case and products in the difference equation case. Then we apply our results to a special  $q$ -difference equation.

## 2. PRELIMINARIES ON TIME SCALES

For an introduction to the theory of time scales we refer to the original work by Hilger [12], to the paper by Aulbach and Hilger [3], and to the recently appeared monograph by Kaymakçalan, Lakshmikantham, and Sivasundaram [13]. In this section we wish to give a short overview

on those parts of the theory that are important for our present treatment of asymptotic behavior of dynamic equations on time scales. By a time scale we simply mean a closed subset of  $\mathbb{R}$ . In this paper we only consider time scales  $\mathbb{T}$  that are unbounded above, i.e.,

$$\mathbb{T} \subset \mathbb{R} \text{ is closed and } \lim_{m \rightarrow \infty} t_m = \infty \text{ for some set } \{t_m: m \in \mathbb{N}\} \subset \mathbb{T}, \quad (2.1)$$

and then the (forward and backward) *jump operators*

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T}: s < t\}$$

(supplemented by  $\sup \emptyset := \inf \mathbb{T}$ ) are well-defined for each  $t \in \mathbb{T}$ . The point  $t \in \mathbb{T}$  is called *left-dense*, *left-scattered*, *right-dense*, and *right-scattered* if  $\rho(t) = t$ ,  $\rho(t) < t$ ,  $\sigma(t) = t$ , and  $\sigma(t) > t$ , respectively. The *graininess*  $\mu: \mathbb{T} \rightarrow \mathbb{R}_0^+$  is defined by  $\mu(t) = \sigma(t) - t$ . We say that a function  $f: \mathbb{T} \rightarrow \mathbb{R}$  is *differentiable* at  $t \in \mathbb{T}$  provided

$$f^\Delta(t) := \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \text{ where } s \rightarrow t, s \in \mathbb{T} \setminus \{\sigma(t)\}$$

exists. The function  $f$  is called differentiable on  $\mathbb{T}$  if  $f^\Delta(t)$  exists for all  $t \in \mathbb{T}$ . For the particular cases  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$  we refer to Example 6.1. Another example is discussed in Example 6.3. Some basic facts are collected in the following lemma. We shall write  $f^\sigma$  for  $f \circ \sigma$ .

**LEMMA 2.1** *Assume (2.1). Let  $f, g: \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}$ . Then we have:*

- (i) *If  $f$  is differentiable in  $t$ , then it is continuous in  $t$ ;*
- (ii) *If  $t$  is right-scattered and  $f$  is continuous in  $t$ , then  $f^\Delta(t) = (f(\sigma(t)) - f(t))/\mu(t)$ ;*
- (iii) *If  $f^\Delta(t)$  exists, then  $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$ ;*
- (iv) *If both  $f$  and  $g$  are differentiable in  $t$ , then so is  $fg$ , and  $(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g$  holds at  $t$ ;*
- (v) *If  $f$  is differentiable in  $t$  and invertible on  $\mathbb{T}$ , then  $f^{-1}$  is differentiable in  $t$  and  $(f^{-1})^\Delta = -(f^\sigma)^{-1} f^\Delta f^{-1}$  holds at  $t$ .*

*Proof* For (i) and (ii) see [3, Theorem 3] or [13, Theorem 1.2.2]. While (iii) follows (note that  $\mu(t) = 0$  if  $t$  is right-dense) from (i) and (ii), (iv) and (v) are from [3, Theorem 4] or [13, Theorem 1.2.3].  $\square$

A function  $f: \mathbb{T} \rightarrow \mathbb{R}$  is called *rd-continuous*, if it has a left-sided limit in each left-dense  $t \in \mathbb{T}$  and if it is continuous in each right-dense  $t \in \mathbb{T}$ . A function  $F: \mathbb{T} \rightarrow \mathbb{R}$  with  $F^\Delta(t) = f(t)$  for all  $t \in \mathbb{T}$  is said to be an *antiderivative* of  $f$  on  $\mathbb{T}$ , and in this case we define

$$\int_r^s f(\tau) \Delta\tau = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

Facts about rd-continuous functions and antiderivatives are collected in the next lemma.

**LEMMA 2.2** *Assume (2.1). Let  $f, g: \mathbb{T} \rightarrow \mathbb{R}$  be rd-continuous. Then the following hold:*

- (i)  $f$  has an antiderivative;
- (ii)  $\int_t^{\sigma(t)} f(\tau) \Delta\tau = \mu(t)f(t)$  hold for all  $t \in \mathbb{T}$ ;
- (iii) If  $|f(\tau)| \leq g(\tau)$  for all  $\tau \in [r, s] \cap \mathbb{T}$ , where  $r, s \in \mathbb{T}$  with  $r \leq s$ , then  $|\int_r^s f(\tau) \Delta\tau| \leq \int_r^s g(\tau) \Delta\tau$ .
- (iv) (*Hölder's Inequality*) Let  $p > 1$ . For all  $r, s \in \mathbb{T}$  with  $r \leq s$  we have

$$\int_r^s |f(\tau)g(\tau)| \Delta\tau \leq \left\{ \int_r^s |f(\tau)|^p \Delta\tau \right\}^{1/p} \left\{ \int_r^s |g(\tau)|^{p/(p-1)} \Delta\tau \right\}^{(p-1)/p}.$$

*Proof* While (i) is from [3, Theorem 6] or [13, Theorem 1.4.4], (ii) follows from (i) and Lemma 2.1 (iv). For (iii) we refer to [13, Theorem 1.4.3 (iii)], and, using (iii), it is easy to prove (iv) in a similar way as is done in [8, Theorem 188].  $\square$

An equation involving  $f$  and  $f^\Delta$  is called a *dynamic equation*. For initial value problems of a system of dynamic equations we will need the following result.

**LEMMA 2.3** *Assume (2.1). Let  $n \in \mathbb{N}$  and let  $A$  be an  $n \times n$ -matrix-valued rd-continuous function on  $\mathbb{T}$  such that*

$$I + \mu(t)A(t) \text{ is invertible for each } t \in \mathbb{T},$$

where  $I$  denotes the  $n \times n$ -identity-matrix. Then, for  $t_0 \in \mathbb{T}$ , there exists an  $n \times n$ -matrix-valued function  $Y$  on  $\mathbb{T}$  such that

$$Y^\Delta(t) = A(t)Y(t), \quad Y(t) \text{ is invertible for each } t \in \mathbb{T} \text{ and } Y(t_0) = I.$$

*Proof* We refer to [3, Theorem 8]. □

### 3. PERTURBED LINEAR DYNAMIC SYSTEMS

Throughout we assume that  $\mathbb{T} \subset \mathbb{R}$  is a time scale which is unbounded above. We let  $n \in \mathbb{N}$ ,  $b: \mathbb{T} \rightarrow \mathbb{R}^n$  and  $f: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  rd-continuous functions, and  $A$  an rd-continuous  $n \times n$ -matrix-valued function on  $\mathbb{T}$ . The purpose of this section is to establish a certain relationship between the system of linear dynamic equations:

$$y^\Delta = A(t)y + b(t) \tag{S_1}$$

and its perturbation

$$x^\Delta = A(t)x + b(t) + f(t, x), \tag{S_2}$$

where  $f$  will be thought of as “small” with respect to the linear part in a sense that will be made precise below. In order to treat the linear part (see Lemma 2.3 from Section 2), we first require

$$I + \mu(t)A(t) \text{ is invertible for each } t \in \mathbb{T}. \tag{V_1}$$

We fix  $t_0 \in \mathbb{T}$ . Subject to (V<sub>1</sub>), according to Lemma 2.3, there exists a unique  $n \times n$ -matrix-valued function  $Y$  with

$$Y^\Delta(t) = A(t)Y(t), \quad Y(t_0) = I, \tag{3.1}$$

and

$$Y(t) \text{ is invertible for each } t \in \mathbb{T}. \tag{3.2}$$

The main result of this section (see also [4, Theorem 1.1, 5, 15]) now reads as follows.

THEOREM 3.1 *Assume (V<sub>1</sub>),*

$$\int_{t_0}^{\infty} |f(\tau, 0)| \Delta\tau < \infty, \quad (\text{V}_2)$$

$$\left. \begin{array}{l} |f(\tau, x_1) - f(\tau, x_2)| \leq \gamma(\tau)|x_1 - x_2| \text{ for all } x_1, x_2 \in \mathbb{R}^n, \\ \tau \in \mathbb{T} \text{ with } \tau \geq t_0 \text{ and such that } \int_{t_0}^{\infty} \gamma(\tau) \Delta\tau < \infty, \end{array} \right\} \quad (\text{V}_3)$$

$$\left. \begin{array}{l} \text{there exist supplementary projections } P_1, P_2 \text{ and } K > 0 \text{ with} \\ |Y(t)P_1Y^{-1}(s)| \leq K \text{ for all } t \in \mathbb{T}, t > t_0, s \in [t_0, t) \cap \mathbb{T} \\ |Y(t)P_2Y^{-1}(s)| \leq K \text{ for all } s \in \mathbb{T}, s > t_0, t \in [t_0, s) \cap \mathbb{T}. \end{array} \right\} \quad (\text{V}_4)$$

*Then there exists a one-to-one and bicontinuous correspondence between the bounded solutions of (S<sub>1</sub>) and (S<sub>2</sub>). In addition, if*

$$\lim_{t \rightarrow \infty} \{Y(t)P_1\} = 0 \quad (\text{V}_5)$$

*holds, then the difference between corresponding solutions of (S<sub>1</sub>) and (S<sub>2</sub>) tends to zero as  $t \rightarrow \infty$ ,  $t \in \mathbb{T}$ .*

*Remark 3.2* Condition (V<sub>3</sub>) is a *growth condition* on the perturbation term, and (V<sub>4</sub>) is a so-called *dichotomy condition*. We also refer to the book by Elaydi [7, Section 7.4] for a discussion of the dichotomy condition for difference equations. In (V<sub>4</sub>),  $P_1$  and  $P_2$  are  $n \times n$ -matrices with (see [5, p. 37])

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 + P_2 = I, \quad P_1P_2 = P_2P_1 = 0. \quad (3.3)$$

For the proof of Theorem 3.1 we employ an operator  $T$  defined by ( $t \geq t_1$ )

$$\begin{aligned} (Tx)(t) &= \int_{t_1}^t Y(t)P_1Y^{-1}(\sigma(\tau))f(\tau, x(\tau)) \Delta\tau \\ &\quad - \int_t^{\infty} Y(t)P_2Y^{-1}(\sigma(\tau))f(\tau, x(\tau)) \Delta\tau, \end{aligned} \quad (3.4)$$

where  $x$  is bounded on  $[t_0, \infty) \cap \mathbb{T}$ , and where  $t_1 \in [t_0, \infty) \cap \mathbb{T}$  is picked in such a manner that (observe (V<sub>3</sub>))

$$\theta = K \int_{t_1}^{\infty} \gamma(\tau) \Delta\tau < 1 \quad (3.5)$$

holds. Note that  $T$  given by (3.4) is well-defined whenever  $x: \mathbb{T} \rightarrow \mathbb{R}^n$  is bounded on  $[t_0, \infty) \cap \mathbb{T}$  (observe (3.2), (V<sub>2</sub>)–(V<sub>4</sub>)). The following useful result on the operator  $T$  is related to the variation-of-constants formula.

**LEMMA 3.3** *Assume (V<sub>1</sub>) through (V<sub>4</sub>) and define  $T$  by (3.4). Then the relationship*

$$(Tx)^\Delta - A(Tx) = f(\cdot, x) \quad (3.6)$$

*holds on  $[t_0, \infty) \cap \mathbb{T}$ .*

*Proof* By the product rule for differentiation, Lemma 2.1 (iv), we find for  $t \in [t_0, \infty) \cap \mathbb{T}$

$$\begin{aligned} (Tx)^\Delta(t) - A(t)(Tx)(t) &= Y^\Delta(t) \int_{t_1}^t P_1 Y^{-1}(\sigma(\tau)) f(\tau, x(\tau)) \Delta\tau \\ &\quad + Y^\sigma(t) P_1 Y^{-1}(\sigma(t)) f(t, x(t)) \\ &\quad - Y^\Delta(t) \int_t^\infty P_2 Y^{-1}(\sigma(\tau)) f(\tau, x(\tau)) \Delta\tau \\ &\quad + Y^\sigma(t) P_2 Y^{-1}(\sigma(t)) f(t, x(t)) \\ &\quad - A(t) Y(t) \int_{t_1}^t P_1 Y^{-1}(\sigma(\tau)) f(\tau, x(\tau)) \Delta\tau \\ &\quad + A(t) Y(t) \int_t^\infty P_2 Y^{-1}(\sigma(\tau)) f(\tau, x(\tau)) \Delta\tau \\ &= Y(\sigma(t)) \{P_1 + P_2\} Y^{-1}(\sigma(t)) f(t, x(t)) \\ &= f(t, x(t)), \end{aligned}$$

where we applied (3.1) and (3.3). □

We finish this section by giving a proof for Theorem 3.1.

*Proof of Theorem 3.1* For a bounded function (more precisely, bounded on  $[t_1, \infty) \cap \mathbb{T}$ ) we put

$$\|x\| = \sup_{t \in [t_1, \infty) \cap \mathbb{T}} |x(t)|.$$

If  $x_1$  and  $x_2$  are two bounded functions, it is easy to prove that the estimate

$$|(Tx_1)(t) - (Tx_2)(t)| \leq \theta \|x_1 - x_2\| \quad (3.7)$$

holds for each  $t \in [t_1, \infty) \cap \mathbb{T}$ , where  $\theta$  is defined by (3.5). Hence, for such  $x_1$  and  $x_2$ , we have

$$\|Tx_1 - Tx_2\| \leq \theta \|x_1 - x_2\|. \quad (3.8)$$

Next, since

$$|(T0)(t)| \leq K \int_{t_1}^{\infty} |f(\tau, 0)| \Delta\tau$$

holds by (V<sub>2</sub>), and since (3.8) and (V<sub>3</sub>) imply

$$\|Tx\| \leq \|Tx - T0\| + \|T0\| \leq \theta \|x\| + \|T0\|,$$

we also have

$$\|Tx\| \leq \theta \|x\| + K \int_{t_1}^{\infty} |f(\tau, 0)| \Delta\tau. \quad (3.9)$$

By (3.8) and (3.9),  $T$  is a contraction and maps the set of bounded (on  $[t_1, \infty) \cap \mathbb{T}$ ) functions to itself. Hence

$$y = x - Tx \quad (3.10)$$

establishes a one-to-one correspondence between bounded (for  $t \geq t_1$ ) solutions of (S<sub>1</sub>) and (S<sub>2</sub>) (using (3.6)). Using (3.7), we have

$$(1 + \theta)^{-1} \|y_1 - y_2\| \leq \|x_1 - x_2\| \leq (1 - \theta)^{-1} \|y_1 - y_2\|$$

which shows that the correspondence is bicontinuous on  $[t_1, \infty) \cap \mathbb{T}$ . Using (V<sub>1</sub>) and Lemma 2.3, this correspondence can be extended to  $[t_0, t_1] \cap \mathbb{T}$ . Finally, given  $\varepsilon > 0$ , choose  $t_2(\varepsilon) > t_1$  so that

$$K \int_{t_2(\varepsilon)}^{\infty} |f(\tau, x(\tau))| \Delta\tau < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} |x(t) - y(t)| &= |(Tx)(t)| \\ &< |Y(t)P_1| \left| \int_{t_1}^{t_2(\varepsilon)} Y^{-1}(\sigma(\tau)) f(\tau, x(\tau)) \Delta\tau \right| + \frac{\varepsilon}{2}, \end{aligned}$$

and letting  $t \rightarrow \infty$ ,  $t \in \mathbb{T}$ , we see that  $x(t) - y(t) = o(1)$  as  $t \rightarrow \infty$ .  $\square$

#### 4. LEVINSON'S PERTURBATION LEMMA FOR TIME SCALES

In this section we wish to establish a version of Levinson's perturbation lemma (see [14] and [4, Lemma 2.1]) within our time scales setting. To do so, we will apply Theorem 3.1 from the preceding section. Throughout we assume

$$\left. \begin{aligned} \lambda_k : \mathbb{T} \rightarrow \mathbb{R}, \quad 1 \leq k \leq n, \quad \text{are rd-continuous,} \\ 1 + \mu(t)\lambda_k(t) \neq 0 \quad \text{for each } t \in \mathbb{T}, \quad 1 \leq k \leq n, \end{aligned} \right\} \quad (\text{V}_1^*)$$

and we put

$$\Lambda(t) = \text{diag}\{\lambda_1(t), \dots, \lambda_n(t)\}. \quad (4.1)$$

A fundamental role is played by the solutions of the  $n^2$  initial value problems

$$w_{ij}^{\Delta} = \frac{\lambda_j(t) - \lambda_i(t)}{1 + \mu(t)\lambda_i(t)} w_{ij}, \quad w_{ij}(t_0) = 1. \quad (\text{S}_3)$$

Because of Lemma 2.3 and

$$1 + \mu(t) \frac{\lambda_j(t) - \lambda_i(t)}{1 + \mu(t)\lambda_i(t)} = \frac{1 + \mu(t)\lambda_j(t)}{1 + \mu(t)\lambda_i(t)} \neq 0,$$

the solutions  $w_{ij}: \mathbb{T} \rightarrow \mathbb{R}$  are uniquely defined. Finally, let an rd-continuous  $n \times n$ -matrix-valued function  $R$  on  $\mathbb{T}$  be given. Now a time scale version of Levinson's perturbation lemma reads as follows.

**THEOREM 4.1** *Assume  $(V_1^*)$ ,*

$$\int_{t_0}^{\infty} \left| \frac{R(\tau)}{1 + \mu(\tau)\lambda_i(\tau)} \right| \Delta\tau < \infty, \quad 1 \leq i \leq n, \quad (V_3^*)$$

*and suppose that*

$$\left. \begin{array}{l} \text{there exist } K, m > 0 \text{ such that each pair } (i, j) \\ \text{with } i \neq j \text{ satisfies either} \\ \text{(a) } \lim_{\tau \rightarrow \infty} w_{ij}(\tau) = 0 \text{ and } \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| \geq m \\ \text{for all } t \in \mathbb{T}, t \geq t_0, s \in [t_0, t] \cap \mathbb{T} \\ \text{or} \\ \text{(b) } \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| \leq K \text{ for all } t \in \mathbb{T}, t \geq t_0, s \in [t_0, t] \cap \mathbb{T}, \end{array} \right\} (V_4^*)$$

where  $w_{ij}$  are given by  $(S_3)$ . Then the linear system  $x^\Delta = [\Lambda(t) + R(t)]x$  has a fundamental matrix  $X$  such that

$$X(t) = [I + o(1)]Y(t) \quad \text{as } t \rightarrow \infty, \quad (4.2)$$

where  $Y$  satisfies  $Y^\Delta(t) = \Lambda(t)Y(t)$  and  $Y(t_0) = I$ .

*Proof* First of all we fix  $i \in \{1, \dots, n\}$ . We put

$$\Lambda_i(t) = \frac{\Lambda(t) - \lambda_i(t)I}{1 + \mu(t)\lambda_i(t)} \quad \text{and} \quad f_i(t, x) = \frac{R(t)x}{1 + \mu(t)\lambda_i(t)},$$

and now Theorem 3.1 will be applied to

$$w_i^\Delta = \Lambda_i(t)w_i \quad (S_1^*)$$

and

$$v_i^\Delta = \Lambda_i(t)v_i + f_i(t, v_i). \quad (S_2^*)$$

Condition (V<sub>1</sub>) requires nonsingularity of

$$I + \mu(t)\Lambda_i(t) = \frac{I + \mu(t)\Lambda(t)}{I + \mu(t)\lambda_i(t)},$$

and this is of course equivalent to (V<sub>1</sub><sup>\*</sup>), since  $\Lambda(t)$  is a diagonal matrix according to (4.1). Next,  $f_i$  satisfies (V<sub>2</sub>) trivially, and also (V<sub>3</sub>) with  $\gamma = R/(1 + \mu\lambda_i)$ , and this follows from (V<sub>3</sub><sup>\*</sup>). Finally, we need to verify (V<sub>4</sub>). For this purpose, we define

$$p_j = \begin{cases} 1 & \text{if } (i, j) \text{ satisfies (a) in (V}_4^*), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let  $P_1 = \text{diag}\{p_1, \dots, p_n\}$  and  $P_2 = I - P_1$  so that (3.3) holds. We observe that

$$W_i = \text{diag}\{w_{i1}, w_{i2}, \dots, w_{in}\},$$

where  $w_{ij}$  are defined by (S<sub>3</sub>), satisfy

$$W_i^\Delta = \Lambda_i(t)W_i, \quad W_i(t_0) = I$$

(see also (3.1)). Therefore, (a) of (V<sub>4</sub><sup>\*</sup>) implies  $\lim_{t \rightarrow 0} \{W_i(t)P_1\} = 0$  and hence (V<sub>5</sub>) with  $Y = W_i$  for each  $i \in \{1, \dots, n\}$ . Next, if  $t_0 \leq s \leq t$  with  $s, t \in \mathbb{T}$ , then  $W_i(t)P_1W_i^{-1}(s)$  is a diagonal matrix with entries 0 or, if  $(i, j)$  satisfies (a) of (V<sub>4</sub><sup>\*</sup>),  $w_{ij}(t)/w_{ij}(s)$ , with  $|w_{ij}(t)/w_{ij}(s)| \leq 1/m$ . Also, if  $t_0 \leq t \leq s$  with  $s, t \in \mathbb{T}$ , then  $W_i(t)P_2W_i^{-1}(s)$  is a diagonal matrix with entries 0 or, if  $(i, j)$  satisfies (b) of (V<sub>4</sub><sup>\*</sup>),  $w_{ij}(t)/w_{ij}(s)$ , with  $|w_{ij}(t)/w_{ij}(s)| \leq K$ . Thus (V<sub>4</sub>) is satisfied with  $Y = W_i$  for each  $i \in \{1, \dots, n\}$ .

Now, by Theorem 3.1 and since  $w_i(t) = e_i$  (where  $e_i \in \mathbb{R}^n$  is the  $i$ th unit vector) is trivially a bounded solution of (S<sub>1</sub><sup>\*</sup>), there exists a bounded solution  $v_i$  of (S<sub>2</sub><sup>\*</sup>) with

$$v_i = w_i + \varepsilon_i = e_i + \varepsilon_i \quad \text{and} \quad \lim_{t \rightarrow \infty} \varepsilon_i(t) = 0, \quad 1 \leq i \leq n.$$

We let  $y_i$  be the unique solution (observe (V<sub>1</sub><sup>\*</sup>)) of

$$y^\Delta = \lambda_i(t)y, \quad y(t_0) = 1,$$

and put  $x_i = y_i v_i = (e_i + \varepsilon_i) y_i$ . Observe that  $Y = \text{diag}\{y_1, \dots, y_n\}$  satisfies (3.1). We have by Lemma 2.1 (iv)

$$\begin{aligned} x_i^\Delta(t) &= (y_i v_i)^\Delta(t) = y_i^\Delta(t) v_i(t) + y_i^\sigma(t) v_i^\Delta(t) \\ &= \lambda_i(t) y_i(t) v_i(t) + \{y_i(t) + \mu(t) y_i^\Delta(t)\} \frac{\Lambda(t) - \lambda_i(t) I + R(t)}{1 + \mu(t) \lambda_i(t)} v_i(t) \\ &= \lambda_i(t) x_i(t) + y_i(t) \{\Lambda(t) - \lambda_i(t) I + R(t)\} v_i(t) \\ &= [\Lambda(t) + R(t)] x_i(t), \end{aligned}$$

and this implies (4.2) from the statement of Theorem 4.1.  $\square$

*Remark 4.2* The proof of the above Theorem 4.1 shows that less hypotheses are required provided we just are interested in the condition

$$x_i(t) = [1 + o(1)] y_i(t)$$

for some fixed  $i \in \{1, \dots, n\}$ . More precisely, for this we only need the “ $i$ th parts” of both conditions  $(V_1^*)$  and  $(V_3^*)$ , and then each  $j \in \{1, \dots, n\}$  is required to satisfy either (a) or (b) of  $(V_4^*)$ .

*Remark 4.3* In what follows we will say that an  $n \times n$ -matrix-valued function  $\Lambda$  with

$$\begin{aligned} \Lambda(t) &= \text{diag}\{\lambda_1(t), \dots, \lambda_n(t)\} \text{ diagonal and} \\ I + \mu(t)\Lambda(t) &\text{ invertible for all } t \in \mathbb{T} \end{aligned} \tag{4.3}$$

satisfies the dichotomy condition provided  $(V_4^*)$  holds (with the notation  $(S_3)$ ). We also say that an  $n \times n$ -matrix-valued function  $R$  satisfies the growth condition with respect to  $\Lambda$  if  $(V_3^*)$  holds.

## 5. PRELIMINARY TRANSFORMATIONS

Throughout this section we consider a dynamic equation of the form

$$x^\Delta = [\Lambda(t) + V(t) + R(t)]x, \tag{5.1}$$

where  $\Lambda$ ,  $V$  and  $R$  are rd-continuous  $n \times n$ -matrix-valued functions on  $\mathbb{T}$  with (4.3) and

$$V(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad t \in \mathbb{T}. \quad (5.2)$$

Now, if either Theorem 3.1 can be directly applied to (5.1) (with  $A(t) = \Lambda(t) + V(t)$  and  $f(t, x) = R(t)x$ ) or if Theorem 4.1 could be applied (with perturbation term  $V(t) + R(t)$ ), then it follows that (5.1) has a solution  $X$  satisfying

$$X(t) = [I + o(1)]Y(t) \text{ as } t \rightarrow \infty.$$

In this section we consider situations where these results cannot be directly applied, and we look for certain types of preliminary linear transformations, whose role is to modify (5.1) to a form (with a possibly different  $\Lambda$ ) so that one of these theorems are applicable to yield an asymptotic representation for solutions. For our further investigations in this direction the following auxiliary result is useful.

**LEMMA 5.1** *Suppose  $Q$  is a differentiable  $n \times n$ -matrix-valued function on  $\mathbb{T}$  such that  $Q(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Then, for sufficiently large  $t \in \mathbb{T}$ ,*

$$P(t) = (I + Q^\sigma(t))^{-1}[(\Lambda(t) + V(t))(I + Q(t)) - Q^\Delta(t)] \quad (5.3)$$

*is well-defined. If there exist rd-continuous  $n \times n$ -matrix-valued functions  $\hat{\Lambda}$  and  $\hat{R}$  with  $P = \hat{\Lambda} + \hat{R}$  such that  $\hat{\Lambda}$  satisfies the dichotomy condition and such that both  $\hat{R}$  and  $R$  satisfy the growth condition with respect to  $\hat{\Lambda}$ , then there exists a solution  $X$  of (5.1) satisfying*

$$X(t) = [I + o(1)]Y(t), \quad t \rightarrow \infty \quad \text{where } Y^\Delta = \hat{\Lambda}(t)Y, \quad Y(t_0) = I.$$

*Proof* Our assumptions make the transformation

$$w = [I + Q(t)]^{-1}x, \quad \text{i.e., } x = [I + Q(t)]w$$

well-defined for sufficiently large  $t \in \mathbb{T}$ . This implies (use Lemma 2.1 (iv))

$$\begin{aligned} [\Lambda(t) + V(t) + R(t)][I + Q(t)]w(t) &= [\Lambda(t) + V(t) + R(t)]x(t) = x^\Delta(t) \\ &= [I + Q^\sigma(t)]w^\Delta(t) + Q^\Delta(t)w(t) \end{aligned}$$

so that (use (5.3))

$$\begin{aligned} w^\Delta(t) &= [I + Q^\sigma(t)]^{-1} \{[\Lambda(t) + V(t) + R(t)][I + Q(t)] - Q^\Delta(t)\} w(t) \\ &= [P(t) + (I + Q^\sigma(t))^{-1} R(t)(I + Q(t))] w(t). \end{aligned}$$

Now, Theorem 4.1 (with the  $R$  there substituted by  $\hat{R} + (I + Q^\sigma)^{-1} \times R(I + Q)$ ) yields (note that both  $(I + Q^\sigma(t))^{-1}$  and  $(I + Q(t))$  are bounded because of  $Q(t) \rightarrow 0$  as  $t \rightarrow \infty$ ) the existence of  $W$  with

$$W^\Delta = [\hat{\Lambda}(t) + \hat{R}(t) + (I + Q^\sigma(t))^{-1} R(t)(I + Q(t))] W$$

such that  $W(t) = [I + o(1)] Y(t)$  as  $t \rightarrow \infty$ , and hence there exists a solution  $X$  of (5.1) with

$$\begin{aligned} X(t) &= [I + Q(t)] W(t) = [I + Q(t)][I + o(1)] Y(t) \\ &= [I + o(1)] Y(t) \quad \text{as } t \rightarrow \infty, \end{aligned}$$

i.e., (4.2) holds. □

In order to apply Lemma 5.1 we now consider various options for constructing  $Q(t)$  depending upon the properties of  $\Lambda$  and  $V$ . These are motivated by analogy with the continuous case [11] and the discrete case [4].

**THEOREM 5.2** *Let  $\Lambda$ ,  $V$  and  $R$  be  $n \times n$ -matrix-valued rd-continuous functions on  $\mathbb{T}$ . Suppose that*

$$Q(t) := - \int_t^\infty V(\tau) \Delta\tau \text{ is well-defined for each } t \in [t_0, \infty) \cap \mathbb{T}.$$

*If  $\Lambda$  satisfies the dichotomy condition and if both  $\Lambda Q - Q^\sigma \Lambda + VQ$  and  $R$  satisfy the growth condition with respect to  $\Lambda$ , then there exists a solution  $X$  of (5.1) satisfying*

$$X(t) = [I + o(1)] Y(t), \quad t \rightarrow \infty \quad \text{where } Y^\Delta = \Lambda(t) Y, \quad Y(t_0) = I.$$

*Proof* First, since  $\int_{t_0}^\infty V(\tau) \Delta\tau$  converges, we have  $Q(t) \rightarrow 0$  as  $t \rightarrow \infty$ . For sufficiently large  $t \in \mathbb{T}$  we hence may define

$$\hat{R}(t) = (I + Q^\sigma(t))^{-1} [\Lambda(t) Q(t) - Q^\sigma(t) \Lambda(t) + V(t) Q(t)].$$

Then we have (observe  $Q^\Delta(t) = V(t)$  for  $t \in \mathbb{T}$  with  $t \geq t_0$ ) for  $P$  from (5.3) on  $[t_1, \infty) \cap \mathbb{T}$  for sufficiently large  $t_1 \in \mathbb{T}$

$$\begin{aligned} P &= (I + Q^\sigma)^{-1}[(\Lambda + V)(I + Q) - Q^\Delta] \\ &= (I + Q^\sigma)^{-1}[\Lambda + V + \Lambda Q + VQ - V - (I + Q^\sigma)\Lambda] + \Lambda \\ &= (I + Q^\sigma)^{-1}[\Lambda Q - Q^\sigma \Lambda + VQ] + \Lambda \\ &= \Lambda + \hat{R} \end{aligned}$$

so that our result follows from Lemma 5.1.  $\square$

**THEOREM 5.3** *Let be given  $n \times n$ -matrix-valued functions  $\Lambda$ ,  $V$  and  $R$  with (4.3) and (5.2) such that  $\Lambda(t)$  tends to an  $n \times n$ -matrix with distinct eigenvalues so that  $\Lambda(t) + V(t)$  has distinct eigenvalues  $\hat{\lambda}_1(t), \dots, \hat{\lambda}_n(t)$  for sufficiently large  $t \in \mathbb{T}$ . Put  $\hat{\Lambda}(t) = \text{diag}\{\hat{\lambda}_1(t), \dots, \hat{\lambda}_n(t)\}$ . If  $\hat{\Lambda}$  satisfies the dichotomy condition and if  $V$  is differentiable and both  $V^\Delta(I + \mu\hat{\Lambda})$  and  $R$  satisfy the growth condition with respect to  $\hat{\Lambda}$ , then there exists a solution  $X$  of (5.1) satisfying*

$$X(t) = [I + o(1)]Y(t), \quad t \rightarrow \infty \quad \text{where } Y^\Delta = \hat{\Lambda}(t)Y, \quad Y(t_0) = I.$$

*Proof* Since for sufficiently large  $t \in \mathbb{T}$  the eigenvalues of  $\Lambda(t) + V(t)$  are all distinct, for those  $t$  there exist matrices  $Q(t)$  such that

$$\hat{\Lambda}(t) = (I + Q(t))^{-1}[\Lambda(t) + V(t)](I + Q(t)), \quad (5.4)$$

and the explicit construction of these matrices  $Q(t)$  works as in [9, Remark 2.1] and it shows that  $Q(t) \rightarrow 0$  and  $Q^\Delta(t) = O(|V^\Delta(t)|)$  as  $t \rightarrow \infty$ . Next, we define (for sufficiently large  $t \in \mathbb{T}$ )

$$\hat{R}(t) = -(I + Q^\sigma(t))^{-1}Q^\Delta(t)(I + \mu(t)\hat{\Lambda}(t)).$$

With this notation we have on  $[t_1, \infty) \cap \mathbb{T}$  for sufficiently large  $t_1 \in \mathbb{T}$  (see (5.3))

$$\begin{aligned} P &= (I + Q^\sigma)^{-1}[(\Lambda + V)(I + Q) - Q^\Delta] \\ &= (I + Q^\sigma)^{-1}\{[(I + Q) - (I + Q^\sigma)](I + Q)^{-1} \\ &\quad \times (\Lambda + V)(I + Q) - Q^\Delta\} + (I + Q)^{-1}(\Lambda + V)(I + Q) \\ &= (I + Q^\sigma)^{-1}\{-\mu Q^\Delta(I + Q)^{-1}(\Lambda + V)(I + Q) - Q^\Delta\} + \hat{\Lambda} \\ &= -(I + Q^\sigma)^{-1}Q^\Delta(I + \mu\hat{\Lambda}) + \hat{\Lambda} \\ &= \hat{\Lambda} + \hat{R}, \end{aligned}$$

where we used the relation  $Q^\sigma - Q = \mu Q^\Delta$  given in Lemma 2.1 (iii). Now our conclusion follows again from Lemma 5.1.  $\square$

*Remark 5.4* The assumption that  $\lim_{t \rightarrow \infty} \Lambda(t)$  exists and has distinct eigenvalues can be weakened since the proof only requires that (5.4) holds with  $\lim_{t \rightarrow \infty} Q(t) = 0$ . If there exists a non-zero, scalar-valued function  $g$  so that  $\lim_{t \rightarrow \infty} \{g(t)[\Lambda(t) + V(t)]\}$  has distinct eigenvalues, then clearly (5.4) holds with  $\hat{\Lambda}$  replaced by  $g\hat{\Lambda}$ . This remark is useful for cases in which  $\lim_{t \rightarrow \infty} \mu(t) = \infty$  (see e.g., Example 6.3 involving a so-called  $q$ -difference equation).

**THEOREM 5.5** *Let  $\Lambda$ ,  $R$  and  $V$  be  $n \times n$ -matrix-valued rd-continuous functions on  $\mathbb{T}$ . Suppose that (4.3) holds so that the  $w_{ij}$  from (S<sub>3</sub>) are never zero, and assume that for some  $p > 1$*

$$\int_{t_0}^{\infty} \left| \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_i(\tau)} \right|^p \Delta\tau < \infty \quad \text{for all } 1 \leq i \neq j \leq n, \quad (5.5)$$

where  $v_{ij}$  are the  $ij$ th entries of  $V$ . Let  $q = p/(p-1)$  and suppose that each pair  $(i, j)$  with  $i \neq j$  satisfies either

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \right|^q \Delta\tau = 0 \quad (5.6)$$

or

$$\int_t^{\infty} \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \right|^q \Delta\tau < \infty \quad \text{for all } t \in [t_0, \infty) \cap \mathbb{T}. \quad (5.7)$$

If (5.6) holds, then we put

$$q_{ij}(t) = \int_{t_0}^t \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_j(\tau)} \Delta\tau,$$

and if (5.7) holds, then we put

$$q_{ij}(t) = - \int_t^{\infty} \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_j(\tau)} \Delta\tau.$$

Let  $Q(t)$  be the matrix that has  $q_{ij}(t)$  as its  $ij$ th entries,  $i \neq j$  and  $q_{ii}(t) \equiv 0$ . Finally, let  $\text{diag } V(t)$  be the  $n \times n$ -diagonal matrix that has the same diagonal entries as  $V(t)$  and suppose that  $\hat{\Lambda} := \Lambda + \text{diag } V$  satisfies

the dichotomy condition and that both  $VQ - Q^\sigma \text{diag } V$  and  $R$  satisfy the growth condition with respect to  $\hat{\Lambda}$ . Then there exists a solution  $X$  of (5.1) satisfying

$$X(t) = [I + o(1)]Y(t), \quad t \rightarrow \infty \quad \text{where } Y^\Delta = \hat{\Lambda}(t)Y, \quad Y(t_0) = I.$$

*Proof* First of all we will show that (5.5)–(5.7) imply  $\lim_{t \rightarrow \infty} Q(t) = 0$ . Our conclusion then follows again from Lemma 5.1, since  $P = \hat{\Lambda} + \hat{R}$  (see (5.3)) with

$$\hat{R}(t) = (I + Q^\sigma(t))^{-1} [V(t)Q(t) - Q^\sigma(t) \text{diag } V(t)] \quad (5.8)$$

(for sufficiently large  $t \in \mathbb{T}$ ), which will be shown below. Now, suppose the pair  $(i, j)$  satisfies (5.6). Then for  $t \in [t_0, \infty) \cap \mathbb{T}$ , by Hölder's inequality, Lemma 2.2 (iv),

$$\begin{aligned} |q_{ij}(t)| &\leq \int_{t_0}^t \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_j(\tau)} \right| \Delta\tau \\ &\leq \left\{ \int_{t_0}^t \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \right|^{p/(p-1)} \Delta\tau \right\}^{(p-1)/p} \left\{ \int_{t_0}^t \left| \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_i(\tau)} \right|^p \Delta\tau \right\}^{1/p} \\ &\leq \left\{ \int_{t_0}^t \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \right|^{p/(p-1)} \Delta\tau \right\}^{(p-1)/p} \left\{ \int_{t_0}^\infty \left| \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_i(\tau)} \right|^p \Delta\tau \right\}^{1/p} \\ &\rightarrow 0, \quad t \rightarrow \infty \end{aligned}$$

because of (5.5). Next, if the pair  $(i, j)$  satisfies (5.7), then again by Hölder's inequality

$$\begin{aligned} |q_{ij}(t)| &\leq \int_t^\infty \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_j(\tau)} \right| \Delta\tau \\ &\leq \left\{ \int_t^\infty \left| \frac{w_{ji}(t)}{w_{ji}(\sigma(\tau))} \right|^{p/(p-1)} \Delta\tau \right\}^{(p-1)/p} \left\{ \int_t^\infty \left| \frac{v_{ij}(\tau)}{1 + \mu(\tau)\lambda_i(\tau)} \right|^p \Delta\tau \right\}^{1/p} \\ &\rightarrow 0, \quad t \rightarrow \infty \end{aligned}$$

because of (5.5).

Now we want to show that  $Q$  is a solution of

$$Q^\Delta = V(t) - \text{diag } V(t) + \Lambda(t)Q - Q^\sigma \Lambda(t). \quad (5.9)$$

First, if the pair  $(i, j)$  satisfies (5.6), then  $q_{ij}$  is differentiable, and it follows with the product rule Lemma 2.1 (iv) that

$$\begin{aligned} q_{ij}^\Delta(t) &= w_{ji}^\Delta(t) \int_{t_0}^t \frac{v_{ij}(\tau)}{(1 + \mu(\tau)\lambda_j(\tau))w_{ji}(\sigma(\tau))} \Delta\tau \\ &\quad + \frac{w_{ji}(\sigma(t))v_{ij}(t)}{(1 + \mu(t)\lambda_j(t))w_{ji}(\sigma(t))} \\ &= \frac{\lambda_i(t) - \lambda_j(t)}{1 + \mu(t)\lambda_j(t)} w_{ji}(t) \int_{t_0}^t \frac{v_{ij}(\tau)}{(1 + \mu(\tau)\lambda_j(\tau))w_{ji}(\sigma(\tau))} \Delta\tau \\ &\quad + \frac{v_{ij}(t)}{1 + \mu(t)\lambda_j(t)} = \frac{\lambda_i(t) - \lambda_j(t)}{1 + \mu(t)\lambda_j(t)} q_{ij}(t) + \frac{v_{ij}(t)}{1 + \mu(t)\lambda_j(t)} \end{aligned}$$

holds. Hence  $q_{ij}$  is the unique solution of the initial value problem

$$q_{ij}^\Delta = \frac{\lambda_i(t) - \lambda_j(t)}{1 + \mu(t)\lambda_j(t)} q_{ij} + [1 - \delta_{ij}] \frac{v_{ij}(t)}{1 + \mu(t)\lambda_j(t)}, \quad q_{ij}(t_0) = 0,$$

where  $\delta_{ij}$  stands for the usual Kronecker symbol. This initial value problem is equivalent to (multiply the equation by  $1 + \mu(t)\lambda_j(t)$  and use Lemma 2.1 (iii))

$$q_{ij}^\Delta = [1 - \delta_{ij}]v_{ij}(t) + \lambda_i(t)q_{ij} - \lambda_j(t)q_{ij}^\sigma, \quad q_{ij}(t_0) = 0.$$

Similarly,  $q_{ij}$  satisfies this same dynamic equation (with, however, a different initial condition) provided the pair  $(i, j)$  satisfies (5.7). Thus we arrive at a proof for (5.9).

Now, for sufficiently large  $t \in \mathbb{T}$ , we define  $\hat{R}$  by (5.8). If  $t_1 \in \mathbb{T}$  is sufficiently large, we have on  $[t_1, \infty) \cap \mathbb{T}$  (see (5.3))

$$\begin{aligned}
 P &= (I + Q^\sigma)^{-1}[(\Lambda + V)(I + Q) - Q^\Delta] \\
 &= (I + Q^\sigma)^{-1}[\Lambda + V + \Lambda Q + VQ - Q^\Delta] \\
 &= (I + Q^\sigma)^{-1}[\Lambda + V + \Lambda Q + VQ - V + \text{diag } V - \Lambda Q + Q^\sigma \Lambda] \\
 &= (I + Q^\sigma)^{-1}[\Lambda + VQ + \text{diag } V + Q^\sigma \Lambda] \\
 &= (I + Q^\sigma)^{-1}[I + Q^\sigma]\Lambda + (I + Q^\sigma)\text{diag } V + VQ - Q^\sigma \text{diag } V \\
 &= \Lambda + \text{diag } V + (I + Q^\sigma)^{-1}[VQ - Q^\sigma \text{diag } V] \\
 &= \hat{\Lambda} + \hat{R},
 \end{aligned}$$

where we used (5.9). Again, an application of Lemma 5.1 finishes the proof.  $\square$

*Remark 5.6* In the continuous case the result of Theorem 5.5 with  $1 < p \leq 2$  is due to Hartman and Wintner and was generalized to the case  $p > 2$  by Harris and Lutz [11]. There, the conditions (5.6) and (5.7) follow from a rather strong separation condition on  $\text{Re}[\lambda_i(t) - \lambda_j(t)]$  (see Example 6.1), and it also follows that  $Q$  is in the same  $L^p$  class as  $V$ . In the case of an arbitrary time scale, however, it appears difficult to find general conditions on  $\Lambda$  which imply (5.6) or (5.7).

## 6. SOME EXAMPLES

*Example 6.1* Let us shortly discuss the two (well-known) cases  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{Z}$ . First, if  $\mathbb{T} = \mathbb{R}$ , then for  $t \in \mathbb{R}$

$$\begin{aligned}
 \sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t), \quad \text{and} \\
 \int_a^b f(\tau) \Delta\tau = \int_a^b f(\tau) d\tau.
 \end{aligned}$$

In this case the solution of  $w^\Delta = \lambda(t)w$ ,  $w(t_0) = 1$  is given by

$$w(t) = \exp \left\{ \int_{t_0}^t \lambda(\tau) d\tau \right\}$$

provided  $\lambda$  is continuous, so that the solutions  $w_{ij}$  of  $(S_3)$  are

$$w_{ij}(t) = \exp \left\{ \int_{t_0}^t [\lambda_j(\tau) - \lambda_i(\tau)] d\tau \right\}.$$

provided  $(V_1^*)$  holds. Thus

$$\begin{aligned} \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| &= \left| \exp \left\{ \int_t^s [\lambda_j(\tau) - \lambda_i(\tau)] d\tau \right\} \right| \\ &= \exp \left\{ \int_s^t \operatorname{Re}[\lambda_i(\tau) - \lambda_j(\tau)] d\tau \right\}, \end{aligned}$$

where  $\operatorname{Re}$  denotes the real part of a complex number. Now, if there exist  $m > 0$  and  $t^* \in \mathbb{R}$  such that for each pair  $(i, j)$  with  $i \neq j$

$$|\operatorname{Re}[\lambda_i(\tau) - \lambda_j(\tau)]| \geq m \quad \text{for all } t \in [t^*, \infty) \cap \mathbb{T},$$

then it follows that

$$\left. \begin{array}{l} \text{for each pair } (i, j) \text{ with } i \neq j \text{ either} \\ \text{(a) } \operatorname{Re}[f_i(t) - f_j(t)] \geq m \quad \text{for all } t \in [t^*, \infty) \cap \mathbb{T} \\ \text{or} \\ \text{(b) } \operatorname{Re}[f_i(t) - f_j(t)] \leq -m \quad \text{for all } t \in [t^*, \infty) \cap \mathbb{T}, \end{array} \right\} \quad (6.1)$$

where  $f_k(t) = \lambda_k(t)$ , provided  $(V_1^*)$  holds so that  $\lambda_i - \lambda_j$  are continuous functions. Thus, (6.1) (a) implies  $(V_4^*)$  (a) and (6.1) (b) implies  $(V_4^*)$  (b), and hence (6.1) is a sufficient condition for the dichotomy condition  $(V_4^*)$ .

Next, if  $\mathbb{T} = \mathbb{Z}$ , then for  $t \in \mathbb{Z}$

$$\sigma(t) = t + 1, \quad \rho(t) = t - 1, \quad \mu(t) = 1, \quad f^\Delta(t) = \Delta f(t),$$

$$\int_a^b f(\tau) \Delta\tau = \sum_{\tau=a}^{b-1} f(\tau) \quad \text{if } a < b,$$

where  $\Delta$  is the usual forward difference operator defined by  $\Delta f(t) = f(t+1) - f(t)$ . In this case the solution of  $w^\Delta = \lambda(t)w$ ,  $w(t_0) = 1$  is given

by  $(t > t_0)$

$$w(t) = \prod_{\tau=t_0}^{t-1} \{1 + \lambda(\tau)\}$$

so that the solutions  $w_{ij}$  of  $(S_3)$  are (provided  $(V_1^*)$  holds)

$$w_{ij}(t) = \prod_{\tau=t_0}^{t-1} \frac{1 + \lambda_j(\tau)}{1 + \lambda_i(\tau)}.$$

Now

$$\begin{aligned} \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| &= \prod_{\tau=s}^{t-1} \left| \frac{1 + \lambda_i(\tau)}{1 + \lambda_j(\tau)} \right| \\ &= \exp \left\{ \sum_{\tau=s}^{t-1} [\log |1 + \lambda_i(\tau)| - \log |1 + \lambda_j(\tau)|] \right\} \\ &= \exp \left\{ \int_s^t [\log |1 + \lambda_i(\tau)| - \log |1 + \lambda_j(\tau)|] \Delta\tau \right\} \\ &= \exp \left\{ \int_s^t \operatorname{Re}[f_i(\tau) - f_j(\tau)] \Delta\tau \right\}, \end{aligned}$$

this time with  $f_k(t) = \log(1 + \lambda_k(t))$ , so that again (6.1) is sufficient for the dichotomy condition  $(V_4^*)$  to hold. Alternatively we can give in this case the following sufficient condition for the dichotomy condition  $(V_4^*)$ :

$$\lim_{t \rightarrow \infty} \alpha_k(t) = \alpha_k, \quad 1 \leq k \leq n \quad \text{with } |\alpha_i| \neq |\alpha_j| \quad \text{for all } 1 \leq i < j \leq n, \quad (6.2)$$

where  $\alpha_k(t) = 1 + \lambda_k(t)$ . To see that (6.2) is really sufficient for  $(V_4^*)$  we note that for a certain index pair  $(i, j)$  with  $i \neq j$  (6.2) implies

$$(a) \quad |\alpha_i| > |\alpha_j| \quad \text{or} \quad (b) \quad |\alpha_i| < |\alpha_j|. \quad (6.3)$$

It is easy to see that (6.3) (a) implies (note  $\alpha_i \neq 0$ )  $(V_4^*)$  (a) and that (6.3) (b) is sufficient (note  $\alpha_j \neq 0$ ) for  $(V_4^*)$  (b).

In the case of a general time scale it is not easy to give an explicit representation of the solutions to  $w^\Delta = \lambda(t)w$ , even if the  $\lambda(t)$  is independent of  $t$ . However, it might be easy to find the solutions to these problems (and hence the  $w_{ij}$  from (S<sub>3</sub>)) once a particular time scale is specified (see also Example 6.3).

*Example 6.2* Let us assume that  $\mathbb{T}$  satisfies

$$\text{each } t \in \mathbb{T} \text{ is both left-scattered and right-scattered.} \quad (6.4)$$

Then it is easy to see that any set  $[t, s] \cap \mathbb{T}$ ,  $t_0 \leq t \leq s$ , contains only finitely many points, and, assuming (V<sub>1</sub><sup>\*</sup>), the crucial functions  $w_{ij}$  from (S<sub>3</sub>) may easily be checked to have both of the forms ( $t \in \mathbb{T}$ ,  $t \geq t_0$ )

$$w_{ij}(t) = \prod_{\tau \in [t_0, t) \cap \mathbb{T}} \frac{\alpha_j(\tau)}{\alpha_i(\tau)} \quad \text{with } \alpha_k(\tau) = 1 + \mu(\tau)\lambda_k(\tau) \quad (6.5)$$

and (see Lemma 2.2 (ii))

$$w_{ij}(t) = \exp \left\{ \int_{t_0}^t [f_j(\tau) - f_i(\tau)] \Delta\tau \right\} \quad \text{with } f_k(\tau) = \frac{\log[1 + \mu(\tau)\alpha_k(\tau)]}{\mu(\tau)}. \quad (6.6)$$

Hence for  $s, t \in \mathbb{T}$  with  $t_0 \leq s \leq t$ , (6.5) and (6.6) imply

$$\left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| = \prod_{\tau \in [s, t) \cap \mathbb{T}} \left| \frac{\alpha_i(\tau)}{\alpha_j(\tau)} \right| = \exp \left\{ \int_s^t \operatorname{Re}[f_i(\tau) - f_j(\tau)] \Delta\tau \right\}.$$

From this we see that both (6.1) and (6.2), using the above  $f_k$  and  $\alpha_k$  from (6.6) and (6.5), are sufficient conditions for the dichotomy condition (V<sub>4</sub><sup>\*</sup>).

*Example 6.3* ( $q$ -difference equations) Let  $q > 1$  and consider  $\mathbb{T} = \{q^k \mid k \in \mathbb{N}\} \cup \{1\}$ . Then  $\sigma(t) = qt$ ,  $\mu(t) = (q - 1)t$ ,

$$y^\Delta(t) = \frac{y^\sigma(t) - y(t)}{\mu(t)} = \frac{1}{(q - 1)t} \{y(qt) - y(t)\},$$

and (see Lemma 2.2 (ii))

$$\begin{aligned}
 \int_1^t f(\tau) \Delta \tau &= \sum_{k=0}^{s-1} \int_{q^k}^{q^{k+1}} f(\tau) \Delta \tau \\
 &= \sum_{k=0}^{s-1} \mu(q^k) f(q^k) \\
 &= (q-1) \sum_{k=1}^{s-1} q^k f(q^k)
 \end{aligned} \tag{6.7}$$

if  $t = q^s \in \mathbb{T}$ . As a special example, we consider the so-called *q-difference equation of Jacobi*

$$ty(qt) - y(t) = 1. \tag{6.8}$$

It is well known that (6.8) has the Theta-series

$$\Theta(t) = - \sum_{n=0}^{\infty} q^{n(n-1)/2} t^n$$

as a formal solution, which can be shown to diverge everywhere. Recently, C. Zhang [16] has studied the summability of this series and its asymptotic properties, especially for complex  $t$ . Since we are concerned with the behavior of solutions near  $\infty$ , the formal series

$$\hat{\Theta}(t) = \sum_{n=1}^{\infty} q^{n(n+1)/2} t^{-n}$$

is more relevant for us. It also can be seen to diverge for all  $t$ , but one would expect that it, too would be an asymptotic expansion for an actual solution of (6.8) as  $t \rightarrow \infty$ . It is possible to apply Theorem 3.1 to this equation and a perturbation  $tx(qt) - x(t) = 1 - (q/t)$ , which has solution  $x(t) = q/t$ . They can be expressed in the form of (S<sub>1</sub>), resp. (S<sub>2</sub>), with  $A(t) = (1-t)/(t^2(q-1))$ ,  $b(t) = 1/(t^2(q-1))$ , and  $f(t, x) = -q/(t^3(q-1))$ . Since  $f$  is easily seen to satisfy (V<sub>2</sub>) and (V<sub>3</sub>), it follows that (6.8) has a solution  $y$  satisfying  $y(t) = x(t) + o(1)$  as  $t \rightarrow \infty$ . This is a rather weak asymptotic statement which could be improved by modifying Theorem 3.1, but we can also achieve a better asymptotic result

in the following way: Replacing  $t$  by  $qt$  to obtain  $qty(q^2t) - y(qt) = 1$  and subtracting (6.8), we obtain

$$qty(q^2t) - (t+1)y(qt) + y(t) = 0,$$

a homogeneous *second order*  $q$ -difference equation having not only the Theta-series  $\hat{\Theta}(t)$  as a formal solution, but other solutions having quite different asymptotic behavior. We put  $x(t) = \begin{pmatrix} y(t) \\ y(qt) \end{pmatrix}$  and rewrite the above equation as

$$x^\sigma = \begin{pmatrix} y(tq) \\ \frac{1}{qt} \{(t+1)y(qt) - y(t)\} \end{pmatrix} = \tilde{A}(t)x$$

with  $\tilde{A}(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{qt} & \frac{t+1}{qt} \end{pmatrix}.$  (6.9)

Let us introduce the transformation

$$x(t) = Pz(t) \quad \text{with } P = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$$

so that (6.9) is transformed into

$$z^\sigma = A^*(t)z \quad \text{with } A^*(t) = P^{-1}\tilde{A}(t)P = \begin{pmatrix} \frac{1}{t} & \frac{q-1}{t} \\ -\frac{1}{qt} & \frac{1}{q} + \frac{1-q}{qt} \end{pmatrix}$$

or, equivalently,

$$z^\Delta = A(t)z \quad \text{with } A(t) = \frac{1}{(q-1)t} \{A^*(t) - I\}.$$

Now we will apply Theorem 5.3 (see in particular also Remark 5.4) since  $A = \Lambda + V$  with

$$\Lambda(t) = \frac{1}{(q-1)t} \begin{pmatrix} \frac{1}{t} - 1 & 0 \\ 0 & \frac{1}{q} + \frac{1-q}{qt} - 1 \end{pmatrix}$$

and

$$V(t) = \begin{pmatrix} 0 & \frac{1}{t^2} \\ -\frac{1}{q(q-1)t^2} & 0 \end{pmatrix}.$$

Note that for  $h(t) = 1/t^2$  we have

$$h^\Delta(t) = \frac{h(qt) - h(t)}{(q-1)t} = \frac{1 - q^2}{q^2(q-1)t^3} = -\frac{1+q}{q^2t^3}$$

so that

$$V^\Delta(t) = \begin{pmatrix} 0 & -\frac{1+q}{q^2t^3} \\ \frac{1+q}{q^3(q-1)t^3} & 0 \end{pmatrix}.$$

Observe that the dichotomy conditions are satisfied for  $\Lambda(t)$  (use (6.2) and see the remarks in Example 6.2) and

$$V^\Delta(t)(I + \mu(t)\hat{\Lambda}(t)) = \begin{pmatrix} 0 & -\frac{1+q}{q^2t^3} \left( \frac{1}{q} + \frac{1-q}{qt} \right) \\ \frac{1+q}{q^3(q-1)t^4} & 0 \end{pmatrix} + O(t^{-5})$$

satisfies the growth condition with respect to  $\Lambda$ . Hence Theorem 5.3 implies that  $z^\Delta = [\Lambda(t) + V(t)]z$  has a matrix solution  $Z$  satisfying

$$Z(t) = [I + o(1)]W(t) \quad \text{as } t \rightarrow \infty,$$

where  $W$  is a fundamental solution of the diagonal system

$$w^\Delta = \Lambda(t)w, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The first component  $w_1$  is a solution of the scalar equation

$$w(qt) = \frac{1}{t} w(t),$$

which has the solution

$$w_1(t) = \sqrt{t} \exp \left[ -\frac{(\ln t)^2}{2 \ln q} \right].$$

The second component  $w_2$  is a solution of

$$w(qt) = \left( \frac{1}{q} + \frac{1-q}{qt} \right) w(t), \quad (6.10)$$

and we will show that there exists a solution with the asymptotic representation

$$w_2(t) = \frac{1}{t} (1 + o(1)) \quad \text{as } t \rightarrow \infty.$$

To see this, observe that  $\tilde{w}(t) = (1/t) + (q/t^2)$  is a solution of

$$w(qt) = \left( \frac{1}{q} + \frac{1-q}{qt} + r(t) \right) w(t) \quad \text{with } r(t) = \frac{q-1}{t^2} \left( 1 + \frac{q}{t} \right)^{-1}. \quad (6.11)$$

Applying Theorem 4.1 to the Equations (6.10) and (6.11), we see that (6.10) has a solution of the form

$$w_2(t) = \left( \frac{1}{t} + \frac{q}{t^2} \right) (1 + o(1)) = \frac{1}{t} (1 + o(1)) \quad \text{as } t \rightarrow \infty.$$

Finally, we see that the system (6.9) has a vector solution of the form

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{t} (1 + o(1)) \end{pmatrix} = \begin{pmatrix} \frac{q}{t} (1 + o(1)) \\ \frac{1}{t} (1 + o(1)) \end{pmatrix}$$

as  $t \rightarrow \infty$ . Hence there exists a solution of the scalar, second order  $q$ -difference equation of the form

$$y(t) = \frac{q}{t}(1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (6.12)$$

The general solution also involves terms coming from  $w_1$ , whose behavior near  $\infty$  is of somewhat smaller order than the dominant terms. Note that  $w_1$  is also a solution of the homogeneous  $q$ -difference equation corresponding to (6.8), and it plays an important role as well in explaining why the formal Theta-series solutions diverge and how to re-sum them (see [16]). Of course, the asymptotic result (6.12) is a much weaker statement than asserting that  $\hat{\Theta}$  is an *asymptotic expansion* for an actual solution, either in the sense of Poincaré or Gevrey. However, our results could be used also in the case of nonanalytic perturbations, provided they satisfy our required conditions. Based on the asymptotic expansion results for the differential and difference equations cases and also some results such as in [16] in the  $q$ -difference case, it is likely that better asymptotic results than (6.12) can be achieved in the case of general time scales, but we choose not to do this here.

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