Asymptotic Behavior of Discretized Sturm-Liouville Eigenvalue Problems

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Abstract

We consider Sturm-Liouville eigenvalue problems of second order with arbitrary separated boundary conditions and perform a suitable discretization of them. The obtained discrete Sturm-Liouville eigenvalue problems are examined and the asymptotic behavior of their eigenvalues as the norm of the partition tends to zero is investigated.

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Abbreviated title: Discretized Sturm-Liouville Eigenvalue Problems.

1. Introduction and Main Result

Let be given $q, r \in \mathbb{R}$ with $r > 0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $(\alpha^2+\beta^2)(\gamma^2+\delta^2) \neq 0$. Then we consider the *Sturm-Liouville eigenvalue problem*

$$\text{(SL)} \qquad \left\{ \begin{array}{c} \ddot{y} + (q+\lambda r)y = 0 \text{ on } [0,1], \\ \alpha y(0) + \beta \dot{y}(0) = \gamma y(1) + \delta \dot{y}(1) = 0, \end{array} \right.$$

i.e., we deal with an autonomous Sturm-Liouville equation of second order and with separated boundary conditions. As is very well known (see e.g. [3, Chapter 0]), the problem (SL) has only real, simple, and isolated eigenvalues; moreover, there exists a smallest eigenvalue and the number of eigenvalues is countable infinite so that the eigenvalues may be arranged as

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots .$$

In Section 2 of this paper we will pick $N \in \mathbb{N} \setminus \{1\}$ and partition the interval $[0,1]$ into $N+1$ equal pieces to obtain via an appropriate discretization the *discrete Sturm-Liouville eigenvalue problem*

$$\text{(SL}^{(\text{N})}\text{)} \qquad \left\{ \begin{array}{l} \Delta^2 y_k + \frac{q+\lambda r}{(N+1)^2} y_{k+1} = 0 \text{ on } [0,N] \cap \mathbb{Z}, \\ \alpha y_0 + \beta^{(N)} \Delta y_0 = \gamma y_N + \delta^{(N)} \Delta y_N = 0, \end{array} \right.$$

where $\Delta y_k = y_{k+1} - y_k$ as usual and where $\beta^{(N)} = \beta(N+1)$ and $\delta^{(N)} = \gamma + \delta(N+1)$. Again it is well-known (see e.g. [2, Chapter 7]) that each of the discrete problems (SL$^{(\text{N})}$) has only real, simple, and finitely many eigenvalues that may be arranged as

$$\lambda_1^{(N)} < \lambda_2^{(N)} < \lambda_3^{(N)} < \ldots < \lambda_{m_N}^{(N)}.$$

Now our main result reads as follows.

Theorem : $$\lim_{N\to\infty} \lambda_m^{(N)} = \lambda_m \text{ holds for all } m \in \mathbb{N}.$$

(In the formulation of the above result we of course make some convention like $\lambda_m^{(N)} = 0$ for all $m > m_N$.)

The proof of our main result will be performed in Section 4 via an application of a theorem of *Hurwitz*. In order to apply this theorem it is necessary to see that the eigenvalues of (SL) are exactly the zeros of some function Λ, that the eigenvalues of (SL$^{(\text{N})}$) are exactly the zeros of some function $\Lambda^{(N)}$, and that

$$\lim_{N\to\infty} \Lambda^{(N)}(\lambda) = \Lambda(\lambda)$$

holds normally in $\mathbb{C}$. We determine and explicitly give $\Lambda^{(N)}$ and Λ in Section 3.

2. Discretization

Let us in this section discretize problem (SL) in order to arrive at problems ($\mathrm{SL}^{(\mathrm{N})}$). By Taylor's Theorem we have for appropriate y and with $h > 0$

$$\begin{cases} y(t+h) &= y(t) + \dot{y}(t)h + \ddot{y}(t)\frac{h^2}{2} + \dddot{y}(t)\frac{h^3}{6} + o(h^3), \\ y(t-h) &= y(t) - \dot{y}(t)h + \ddot{y}(t)\frac{h^2}{2} - \dddot{y}(t)\frac{h^3}{6} + o(h^3), \end{cases}$$

and hence

$$\begin{cases} \dot{y}(t) = \frac{1}{h}\{y(t+h) - y(t)\} + o(h) = \frac{1}{h}\{y(t) - y(t-h)\} + o(h), \\ \ddot{y}(t) = \frac{1}{h^2}\{y(t+h) - 2y(t) + y(t-h)\} + o(h). \end{cases}$$

Now we pick $N \in \mathbb{N} \setminus \{1\}$, put $h = \frac{1}{N+1}$,

$$t_k = kh = \frac{k}{N+1} \quad \text{and} \quad y_k = y(t_k), \quad 0 \le k \le N+1.$$

Thus $t_k + h = t_{k+1}$, $t_k - h = t_{k-1}$,

$$\frac{1}{h^2}\{y(t_k+h) - 2y(t_k) + y(t_k-h)\} = (N+1)^2\Delta^2 y_{k-1},$$

$\frac{1}{h}\{y(t_k+h) - y(t_k)\} = (N+1)\Delta y_k$, and $\frac{1}{h}\{y(t_k) - y(t_k-h)\} = (N+1)\Delta y_{k-1}$. Let us thus write $(N+1)^2\Delta^2 y_{k-1}$ for $\ddot{y}(t_k)$, and for $\dot{y}(t_k)$ we may write $(N+1)\Delta y_k$ or $(N+1)\Delta y_{k-1}$. Hence, instead of $\ddot{y}+(q+\lambda r)y = 0$ we will consider the discretization $(N+1)^2\Delta^2 y_{k-1} + (q+\lambda r)y_k = 0$; $\alpha y(0) + \beta\dot{y}(0)$ becomes $\alpha y(t_0) + \beta(N+1)\Delta y(t_0)$, and $\gamma y(1) + \delta\dot{y}(1)$ will be replaced by

$$\gamma y(t_{N+1}) + \delta(N+1)\Delta y(t_N) = \gamma y_N + \{\gamma + \delta(N+1)\}\Delta y_N.$$

This is how problem ($\mathrm{SL}^{(\mathrm{N})}$) arises as a discretization of problem (SL).

3. Characterization of Eigenvalues

Let us define, for each $\lambda \in \mathbb{C}$, $y(t,\lambda)$ to be the (unique) solution of the initial value problem

$$\ddot{y} + (q+\lambda r)y = 0, \qquad y(0) = \beta, \quad \dot{y}(0) = -\alpha, \tag{1}$$

while we want, again for each $\lambda \in \mathbb{C}$, $y_k^{(N)}(\lambda)$ to be the (again unique) solution of the initial value problem

$$\Delta^2 y_k + \frac{q+\lambda r}{(N+1)^2}y_{k+1} = 0, \qquad y_0 = \beta, \quad \Delta y_0 = -\frac{\alpha}{(N+1)}. \tag{2}$$

Define functions

$$\Lambda(\lambda) = \gamma y(1, \lambda) + \delta \dot{y}(1, \lambda) \quad \text{and} \quad \Lambda^{(N)}(\lambda) = \gamma y_N^{(N)}(\lambda) + \delta^{(N)} \Delta y_N^{(N)}(\lambda).$$

Now we can characterize the eigenvalues as follows.

Lemma 1. λ is an eigenvalue of (SL) [(SL$^{(\mathrm{N})}$)] iff $\Lambda(\lambda) = 0$ [$\Lambda^{(N)}(\lambda) = 0$].
Proof. The first statement is a consequence of e.g. [3, Proposition 2.2.1], while the other statement is a special case of [1, Corollary 1]. ■

Of course we are now interested in how Λ and $\Lambda^{(N)}$ look. To see this, we need to solve the initial value problems (1) and (2).

Lemma 2. For all $\lambda \in \mathbb{C}$

$$y(t, \lambda) = \beta \cos(t\sqrt{q + \lambda r}) - \alpha \frac{\sin(t\sqrt{q + \lambda r})}{\sqrt{q + \lambda r}} \tag{3}$$

and, with $\mu = \mu_N(\lambda) = 1 + \frac{\sqrt{-(q+\lambda r)(1 - \frac{q+\lambda r}{4(N+1)^2})} - \frac{q+\lambda r}{2(N+1)}}{N+1}$,

$$y_k^{(N)}(\lambda) = \frac{1}{1+\mu}\left\{\beta(\mu^k + \frac{\mu}{\mu^k}) - \alpha \frac{\mu\mu^k - \frac{\mu}{\mu^k}}{(N+1)(\mu - 1)}\right\} \tag{4}$$

where we always mean the principal value of the square root, and where (3) at $-\frac{q}{r}$ and (4) at $-\frac{q}{r}$ and $\frac{4(N+1)^2 - q}{r}$ are considered to be the corresponding limits.
Proof. Let $y(t, \lambda)$ and $y_k^{(N)}(\lambda)$ be defined as in (3) and (4). Then we have

$$\dot{y}(t, \lambda) = -\beta\sqrt{q + \lambda r} \sin(t\sqrt{q + \lambda r}) - \alpha \cos(t\sqrt{q + \lambda r}). \tag{5}$$

Hence $y(t, \lambda)$ solves $\ddot{y} + (q + \lambda r)y = 0$ and has $y(0, \lambda) \equiv \beta$ and $\dot{y}(0, \lambda) \equiv -\alpha$.

Next, we note that

$$\Delta\mu^k = \mu^k(\mu - 1), \quad \Delta\frac{1}{\mu^k} = -\frac{\mu - 1}{\mu^{k+1}}, \text{ and } \quad (\mu - 1)^2 = -\mu\frac{q + \lambda r}{(N+1)^2}$$

hold. Therefore we have

$$\Delta y_k^{(N)}(\lambda) = \frac{1}{1+\mu}\left\{\beta(\mu - 1)(\mu^k - \frac{1}{\mu^k}) - \alpha\frac{\mu\mu^k + \frac{1}{\mu^k}}{N+1}\right\} \tag{6}$$

and hence

$$\begin{aligned} \Delta^2 y_k^{(N)}(\lambda) &= \frac{1}{1+\mu}\left\{\beta(\mu - 1)^2(\mu^k + \frac{1}{\mu^{k+1}}) - \alpha\frac{\mu - 1}{N+1}(\mu\mu^k - \frac{1}{\mu^{k+1}})\right\} \\ &= -\frac{q + \lambda r}{(N+1)^2} y_{k+1}^{(N)}(\lambda). \end{aligned}$$

Of course we also have $y_0^{(N)}(\lambda) \equiv \beta$ and $\Delta y_0^{(N)}(\lambda) \equiv -\frac{\alpha}{N+1}$ so that the proof of our auxiliary result is complete. ■

4. Asymptotics

Formulas (3), (4), (5), and (6) now yield the following representations for Λ and $\Lambda^{(N)}$. We have for all $\lambda \in \mathbb{C}$

$$\Lambda(\lambda) = \{\beta\gamma - \alpha\delta\} \cos\sqrt{q+\lambda r} - \{\alpha\gamma + \beta\delta(q+\lambda r)\} \frac{\sin\sqrt{q+\lambda r}}{\sqrt{q+\lambda r}} \tag{7}$$

and, by putting $\tilde{\delta}^{(N)} = \frac{\gamma}{N+1} + \delta = \frac{\delta^{(N)}}{N+1}$,

$$\begin{aligned}\Lambda^{(N)}(\lambda) \;=\; \frac{1}{1+\mu} \; \Big\{ & \beta\gamma(\mu^N + \frac{\mu}{\mu^N}) - \alpha\tilde{\delta}^{(N)}(\mu\mu^N + \frac{1}{\mu^N}) \\ & - \frac{\alpha\gamma(\mu\mu^N - \frac{\mu}{\mu^N}) + \beta\tilde{\delta}^{(N)}(q+\lambda r)(\mu^N - \frac{1}{\mu^N})}{(N+1)(\mu-1)} \Big\}.\end{aligned} \tag{8}$$

Now we have

$$\mu_N(\lambda) \to 1, \quad (N+1)(\mu_N(\lambda) - 1) \to \sqrt{-(q+\lambda r)}, \quad \mu_N^N(\lambda) \to \mathrm{e}^{\sqrt{-(q+\lambda r)}}$$

normally in $\mathbb{C}$, i.e., uniformly on compact subsets of $\mathbb{C}$, as $N \to \infty$. We hence in view of (7) and (8) have the following crucial result.

Lemma 3. $\lim_{N\to\infty} \Lambda^{(N)}(\lambda) = \Lambda(\lambda)$ holds normally in $\mathbb{C}$.

Let us at this time recall the following well-known result of Hurwitz.

Lemma 4. Let $\Lambda^{(N)}$ be a sequence of analytic functions that converges normally in $\mathbb{C}$ to a function $\Lambda \neq 0$. Then, for any domain $D \subset \mathbb{C}$ such that the boundary of D does not contain a zero of Λ, it is possible to find a number $M = M(D) \in \mathbb{N}$ such that for $N > M$ each of the function $\Lambda^{(N)}$ has inside D the same number of zeros as Λ has inside D.

Since the eigenvalues of (SL) and (SL$^{(\mathrm{N})}$) are all real and isolated, Lemma 1 yields that the zeros of Λ and of $\Lambda^{(N)}$ are all real and isolated too. In a final result we will now show that the zeros of Λ and of $\Lambda^{(N)}$ are all simple. Of course this together with the above result of Hurwitz then easily yields our main result, the Theorem from Section 1.

Lemma 5. The zeros of Λ and of $\Lambda^{(N)}$ are all simple.
Proof. For $\lambda, \mu \in \mathbb{C}$ and $t \in [0,1]$ and $k \in [0,N] \cap \mathbb{Z}$, respectively, it is easy to check the formulas

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\{y(t,\lambda)\dot{y}(t,\mu) - y(t,\mu)\dot{y}(t,\lambda)\} & = & \{\lambda-\mu\}\, y(t,\lambda)y(t,\mu), \\ \Delta\left\{y_k^{(N)}(\lambda)\Delta y_k^{(N)}(\mu) - y_k^{(N)}(\mu)\Delta y_k^{(N)}(\lambda)\right\} & = & \{\lambda-\mu\}\, y_{k+1}^{(N)}(\lambda)y_{k+1}^{(N)}(\mu). \end{array} \right.$$

However, integrating and summing, respectively, dividing by $\lambda-\mu \neq 0$, and letting μ tend to λ, yields

$$\begin{cases} v(t,\lambda) & := & \dot{y}(t,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}y(t,\lambda) - y(t,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\dot{y}(t,\lambda) & = & \int\limits_0^t \{y(\tau,\lambda)\}^2 \,\mathrm{d}\tau, \\ v_k^{(N)}(\lambda) & := & \Delta y_k^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}y_k^{(N)}(\lambda) - y_k^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta y_k^{(N)}(\lambda) & = & \sum\limits_{\kappa=1}^{k} \left\{y_\kappa^{(N)}(\lambda)\right\}^2. \end{cases}$$

Therefore, since we don't have $y(\tau,\lambda) \equiv 0$ or $y_k^{(N)}(\lambda) \equiv 0$ (observe $N \geq 2$), $v(1,\lambda) \neq 0$ and $v_N^{(N)}(\lambda) \neq 0$ hold. Next, we have

$$\begin{cases} \dot{y}(1,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda(\lambda) - \Lambda(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\dot{y}(1,\lambda) & = & \gamma v(1,\lambda), \\ \Lambda(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}y(1,\lambda) - y(1,\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda(\lambda) & = & \delta v(1,\lambda) \end{cases}$$

and

$$\begin{cases} \Delta y_N^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda^{(N)}(\lambda) - \Lambda^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta y_N^{(N)}(\lambda) & = & \gamma v_N^{(N)}(\lambda), \\ \Lambda^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}y_N^{(N)}(\lambda) - y_N^{(N)}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda^{(N)}(\lambda) & = & \{\delta(N+1)+\gamma\}\, v_N^{(N)}(\lambda). \end{cases}$$

These last four formulas make $\Lambda(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda(\lambda) = 0$ and $\Lambda^{(N)}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda^{(N)}(\lambda) = 0$ impossible so that Λ and $\Lambda^{(N)}$ can not have double zeros. ■

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