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Asymptotic behavior of second-order dynamic equations

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Abstract

We prove several growth theorems for second-order dynamic equations on time scales. These theorems contain as special cases results for second-order differential equations, difference equations, and q-difference equations.

Keywords: Time scale; Dynamic equation; Asymptotic behavior; Boundedness

1. Introduction

In this paper we prove some results for the second-order linear dynamic equation

$$x^{\Delta\Delta} + q(t)x^{\sigma} = 0 \tag{1.1}$$

as well as corresponding results for the second-order nonlinear dynamic equation

$$(p(t)x^{\Delta})^{\Delta} + q(t)(f \circ x^{\sigma}) = 0. \tag{1.2}$$

Throughout, we assume that $p, q : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions. Our results contain corresponding results for second-order differential equations (see [2]) as well as for second-order difference equations (see [6,7]). The reason for this is that dynamic equations on time scales have been designed in order to unify continuous and discrete analysis. Furthermore, they contain other important equations, for example so-called q-difference equations. In Section 2 we will give a very brief introduction to the time scales calculus; the reader is referred to [4,5] for further study. Next, in Section 3 we present some preliminary results that are needed in the remainder of this paper. Eq. (1.1) is considered in Section 4, while Section 5 is devoted to the study of (1.2).

2. Elements of time scales calculus

In this section we present some definitions and elementary results connected to the time scales calculus. For further study we refer the reader to the monographs [4,5]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed

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subset of the real numbers $\mathbb{R}$, which is assumed throughout this paper to be unbounded above. On $\mathbb{T}$ we define the *forward* and *backward jump operators* by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\} \quad \text{for } t \in \mathbb{T}.$$

A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. Next, the *graininess* function μ is defined by $\mu(t) := \sigma(t) - t$ for $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) *derivative* $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighbourhood U of t with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

A simple useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where } f^\sigma := f \circ \sigma.$$

We will use the product rule and the quotient rule for the derivative of the product fg and the quotient f/g (if $gg^\sigma \neq 0$) of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - g^\Delta f}{gg^\sigma}.$$

For $a, b \in \mathbb{T}$ and a function $f : \mathbb{T} \to \mathbb{R}$, the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where } F^\Delta = f,$$

i.e., F is an antiderivative of f. The function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous in right-dense points and if the left-sided limits exist in left-dense points. Hilger's main existence theorem [4, Theorem 1.74] says that rd-continuous functions possess antiderivatives. If $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous and *regressive* (i.e., $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$), then another existence theorem says that the initial value problem $y^\Delta = p(t)y$, $y(t_0) = 1$ (where $t_0 \in \mathbb{T}$) possesses a unique solution $e_p(\cdot, t_0)$. The set of all rd-continuous and regressive functions is denoted by C_{rd}.

Example 1. Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta(t) = f'(t),$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\sigma(t) = t + 1, \quad \mu(t) \equiv 1, \quad f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

Another important time scale is $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, for which

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t},$$

and this time scale gives rise to so-called q-difference equations.

3. Preliminary results

The following dynamic version of Gronwall's inequality [1, Theorem 3.1] together with one of its special cases [4, Corollary 6.7] will be needed. Throughout we let $t_0 \in \mathbb{T}$ with $t_0 \geqslant 1$, and we put $\mathbb{T}_0 = \mathbb{T} \cap [t_0, \infty)$.

Lemma 1. *Suppose $u, a, b, p \in C_{\text{rd}}$ and $b(t), p(t) \geqslant 0$ for all $t \in \mathbb{T}$. Then*

$$u(t) \leqslant a(t) + p(t) \int_{t_0}^t b(\tau)u(\tau)\Delta\tau \quad \textit{for all } t \in \mathbb{T}_0$$

implies

$$u(t) \leqslant a(t) + p(t) \int_{t_0}^t a(\tau)b(\tau)e_{bp}(t, \sigma(\tau))\Delta\tau \quad \textit{for all } t \in \mathbb{T}_0.$$

Corollary 1. *Suppose* $u, b \in C_{\mathrm{rd}}$, $c \in \mathbb{R}$ *and* $b(t) \geqslant 0$ *for all* $t \in \mathbb{T}$. *Then*

$$u(t) \leqslant c + \int_{t_0}^{t} b(\tau)u(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}_0$$

implies

$$u(t) \leqslant c e_b(t, t_0) \quad \text{for all } t \in \mathbb{T}_0.$$

We also make use of the following result which is proved in [3, Remark 2].

Lemma 2. *If* $p \in C_{\mathrm{rd}}$ *such that* $p(t) \geqslant 0$ *for all* $t \in \mathbb{T}$, *then*

$$1 + \int_{t_0}^{t} p(\tau)\Delta\tau \leqslant e_p(t, t_0) \leqslant \exp\left\{\int_{t_0}^{t} p(\tau)\Delta\tau\right\} \quad \text{for all } t \in \mathbb{T}_0.$$

The following lemma is needed as well. Since it is a new result, we supply a proof.

Lemma 3. *If* $f \in C_{\mathrm{rd}}$, *then*

$$\int_{t_0}^{t} \int_{t_0}^{s} f(\tau)\Delta\tau\Delta s = \int_{t_0}^{t} [t - \sigma(\tau)] f(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}_0 \tag{3.1}$$

and

$$\int_{t_0}^{\sigma(t)} \int_{t_0}^{s} f(\tau)\Delta\tau\Delta s = \int_{t_0}^{t} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}_0. \tag{3.2}$$

Proof. We first show (3.1). Since rd-continuous functions possess antiderivatives, we can find functions $F, G, H : \mathbb{T} \to \mathbb{R}$ such that

$$F^{\Delta}(t) = f(t), \quad G^{\Delta}(t) = F(t), \quad \text{and} \quad H^{\Delta}(t) = \sigma(t) f(t).$$

Let us define a function k by

$$k(t) = \int_{t_0}^{t} \int_{t_0}^{s} f(\tau)\Delta\tau\Delta s - \int_{t_0}^{t} [t - \sigma(\tau)] f(\tau)\Delta\tau.$$

This implies

$$\begin{aligned} k(t) &= \int_{t_0}^{t} [F(s) - F(t_0)]\Delta s - t[F(t) - F(t_0)] + H(t) - H(t_0) \\ &= G(t) - G(t_0) + t_0 F(t_0) - tF(t) + H(t) - H(t_0). \end{aligned}$$

Therefore $k(t_0) = 0$ and $k^{\Delta}(t) = 0$ by the product rule, so that $k(t) \equiv 0$ for all $t \in \mathbb{T}$, i.e., (3.1) holds. Now we have

$$\begin{aligned} \int_{t_0}^{\sigma(t)} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau &= \int_{t_0}^{t} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau + \int_{t}^{\sigma(t)} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau \\ &= \int_{t_0}^{t} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau + \mu(t)[\sigma(t) - \sigma(t)] f(t) = \int_{t_0}^{t} [\sigma(t) - \sigma(\tau)] f(\tau)\Delta\tau, \end{aligned}$$

and therefore (3.2) follows from (3.1). $\square$

4. Linear dynamic equations

We first give the following result.

Theorem 1. *Any solution of* (1.1) *satisfies*

$$|x(\sigma(t))| \leqslant c\sigma(t) e_{\sigma|q|}(t, t_0) \quad \text{for all } t \in \mathbb{T}_0, \tag{4.1}$$

where $c = |x(t_0) - x^{\Delta}(t_0)| + |x^{\Delta}(t_0)|$.

Proof. Suppose x solves (1.1). We integrate (1.1) between t_0 and t to obtain

$$x^{\Delta}(t) = x^{\Delta}(t_0) - \int_{t_0}^{t} q(\tau)x^{\sigma}(\tau)\Delta\tau. \tag{4.2}$$

Now, integrating (4.2) between t_0 and $\sigma(t)$ provides

$$x(\sigma(t)) = x(t_0) + x^{\Delta}(t_0)[\sigma(t) - t_0] - \int_{t_0}^{\sigma(t)}\int_{t_0}^{s} q(\tau)x^{\sigma}(\tau)\Delta\tau\Delta s = c_1 + c_2\sigma(t) - \int_{t_0}^{t}[\sigma(t) - \sigma(\tau)]q(\tau)x^{\sigma}(\tau)\Delta\tau,$$

where we used (3.2) from Lemma 3 and put $c_1 = x(t_0) - x^{\Delta}(t_0)$ and $c_2 = x^{\Delta}(t_0)$. Next, if $\sigma(t) \geqslant 1$, then

$$|x(\sigma(t))| \leqslant |c_1| + |c_2\sigma(t)| + \int_{t_0}^{t}[\sigma(t) - \sigma(\tau)]|q(\tau)||x^{\sigma}(\tau)|\Delta\tau \leqslant c\sigma(t) + \sigma(t)\int_{t_0}^{t}|q(\tau)||x^{\sigma}(\tau)|\Delta\tau$$

with $c = |c_1| + |c_2|$. Define now

$$y(t) = \frac{|x(\sigma(t))|}{\sigma(t)}.$$

Then we obtain

$$y(t) \leqslant c + \int_{t_0}^{t}|q(\tau)||x^{\sigma}(\tau)|\Delta\tau = c + \int_{t_0}^{t}\sigma(\tau)|q(\tau)|y(\tau)\Delta\tau.$$

By the Gronwall lemma (Lemma 1), we obtain

$$y(t) \leqslant ce_{\sigma|q|}(t, t_0) \quad \text{for all } t \in \mathbb{T}_0,$$

i.e., (4.1). □

Corollary 2. *Any solution x of (1.1) satisfies*

$$x(\sigma(t)) = \mathrm{O}(\sigma(t)e_{\sigma|q|}(t, t_0)) \quad \text{as } t \to \infty. \tag{4.3}$$

For the continuous version of the next theorem we refer to [2, Theorem 5 in Chapter 6], while the discrete version can be found in [7, Theorem 3.1].

Theorem 2. *If*

$$\int_{t_0}^{\infty}\sigma(\tau)|q(\tau)|\Delta\tau < \infty, \tag{4.4}$$

then the general solution of (1.1) is asymptotic to $at + b$ as $t \to \infty$, where a or b may be zero, but not both simultaneously.

Proof. By (4.4), without loss of generality, we can assume that t_0 is such that

$$\int_{t_0}^{\infty}\sigma(\tau)|q(\tau)|\Delta\tau < \ln\left(\frac{3}{2}\right). \tag{4.5}$$

Let x be the solution of (1.1) with the initial conditions $x(t_0) = 1$ and $x^{\Delta}(t_0) = 1$. By Theorem 1, x satisfies (4.1) with $c = 1$. Thus

$$\left|\int_{t_0}^{t} q(\tau)x^{\sigma}(\tau)\Delta\tau\right| \leqslant \int_{t_0}^{t}\sigma(\tau)|q(\tau)|e_{\sigma|q|}(\tau, t_0)\Delta\tau = e_{\sigma|q|}(t, t_0) - 1 \leqslant \exp\left\{\int_{t_0}^{t}\sigma(\tau)|q(\tau)|\Delta\tau\right\} - 1$$
$$< \exp\left\{\ln\left(\frac{3}{2}\right)\right\} - 1 = \frac{1}{2},$$

where we used Lemma 2 and (4.5). In particular this means that the infinite integral $\int_{t_0}^{\infty} q(\tau)x^{\sigma}(\tau)\Delta\tau$ converges, and therefore, because of (4.2),

$$\alpha := \lim_{t\to\infty} x^{\Delta}(t) \quad \text{exists.}$$

Moreover, again by (4.2),

$$|x^{\Delta}(t)| \geqslant 1 - \left|\int_{t_0}^{t} q(\tau)x^{\sigma}(\tau)\Delta\tau\right| > \frac{1}{2}$$

so that $|\alpha| \geqslant \frac{1}{2} > 0$. Now we claim that

$$x(t) \sim \alpha t \quad \text{as } t\to\infty, \quad \text{i.e., } \lim_{t\to\infty}\frac{x(t)}{\alpha t} = 1. \tag{4.6}$$

To show this, we write $x^{\Delta}(t) = \alpha + \varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$, and then

$$x(t) = x(t_0) + \int_{t_0}^{t} x^{\Delta}(\tau)\Delta\tau = 1 + (t - t_0)\alpha + \int_{t_0}^{t}\varepsilon(\tau)\Delta\tau,$$

so

$$\frac{x(t)}{\alpha t} = 1 + \frac{1-\alpha t_0}{\alpha t} + \frac{1}{\alpha t}\int_{t_0}^{t}\varepsilon(\tau)\Delta\tau \to 1 \quad \text{as } t\to\infty,$$

where we used that $\frac{1}{t}\int_{t_0}^{t}\varepsilon(\tau)\Delta\tau \to 0$ as $t\to\infty$ due to L'Hôpital's rule [4, Theorem 1.119]. Hence (4.6) is established. Thus, for sufficiently large $t\in\mathbb{T}$,

$$\tilde{x}(t) := x(t)\int_{t}^{\infty}\frac{\Delta\tau}{x(\tau)x(\sigma(\tau))}$$

is well defined. It is an easy calculation involving the product rule and the quotient rule to show that $\tilde{x}$ is another solution of (1.1) such that the Wronskian of $\tilde{x}$ and x, defined by $\tilde{x}x^{\Delta} - x\tilde{x}^{\Delta}$, is constant equal to one. Therefore, for any nontrivial solution $\hat{x}$ of (1.1), there exist two constants $\beta,\gamma\in\mathbb{R}$ with $\beta^2+\gamma^2>0$ such that $\hat{x} = \beta x + \gamma\tilde{x}$. Since

$$\lim_{t\to\infty}\tilde{x}(t) = \lim_{t\to\infty}\frac{\int_t^{\infty}\frac{\Delta\tau}{x(\tau)x(\sigma(\tau))}}{\frac{1}{x(t)}} = \lim_{t\to\infty}\frac{1}{x^{\Delta}(t)} = \frac{1}{\alpha},$$

where we used again L'Hôpital's rule and the quotient rule, we find that

$$\hat{x}(t) = \beta x(t) + \gamma\tilde{x}(t) \sim \beta\alpha t + \frac{\gamma}{\alpha} \quad \text{as } t\to\infty,$$

which implies that the claimed statement holds with $a = \beta\alpha$ and $b = \gamma/\alpha$. □

In our next results we consider the two equations (1.1) and

$$y^{\Delta\Delta} + p(t)y^{\sigma} = 0. \tag{4.7}$$

We also put $r = p - q$.

Lemma 4. *Let $p,q\in C_{\mathrm{rd}}$, put $r = p - q$, and suppose x_1 and x_2 are two linearly independent solutions of (1.1). Let y be any solution of (4.7). Then there exist $c_1, c_2\in\mathbb{R}$ such that*

$$y(t) = c_1x_1(t) + c_2x_2(t) + c\int_{t_0}^{t}[x_2(\sigma(\tau))x_1(t) - x_1(\sigma(\tau))x_2(t)]r(\tau)y^{\sigma}(\tau)\Delta\tau \tag{4.8}$$

and

$$y(\sigma(t)) = c_1x_1(\sigma(t)) + c_2x_2(\sigma(t)) + c\int_{t_0}^{t}[x_2(\sigma(\tau))x_1(\sigma(t)) - x_1(\sigma(\tau))x_2(\sigma(t))]r(\tau)y^{\sigma}(\tau)\Delta\tau. \tag{4.9}$$

Proof. Let y be any solution of (4.7). Now let $d_1, d_2 \in \mathbb{R}$ and define

$$x = d_1 x_1 + d_2 x_2 \quad \text{and} \quad z = x - y.$$

Then

$$z^{\Delta\Delta}(t) + q(t)z^{\sigma}(t) = p(t)y^{\sigma}(t) - q(t)y^{\sigma}(t) = r(t)y^{\sigma}(t),$$

where $r = p - q$. Thus z solves the inhomogeneous equation

$$z^{\Delta\Delta} + q(t)z^{\sigma} = r(t)y^{\sigma}(t)$$

and hence by variation of parameters [4, Theorem 3.73] there exist $d_3, d_4 \in \mathbb{R}$ with

$$\begin{aligned} z(t) &= d_3 x_1(t) + d_4 x_2(t) + \int_{t_0}^{t} \frac{x_1(\sigma(\tau))x_2(t) - x_2(\sigma(\tau))x_1(t)}{W(x_1, x_2)(\sigma(\tau))} r(\tau) y^{\sigma}(\tau)\Delta\tau \\ &= d_3 x_1(t) + d_4 x_2(t) + d\int_{t_0}^{t} [x_1(\sigma(\tau))x_2(t) - x_2(\sigma(\tau))x_1(t)] r(\tau)y^{\sigma}(\tau)\Delta\tau, \end{aligned}$$

where the Wronskian $W(x_1, x_2)(t) \equiv 1/d$ is constant. Next, since $z = x - y$, we arrive at (4.8), where $c_1 = d_1 - d_3$ and $c_2 = d_2 - d_4$. Using (4.8) for $\sigma(t)$ instead of t and observing that

$$\int_{t}^{\sigma(t)} [x_1(\sigma(\tau))x_2(\sigma(t)) - x_2(\sigma(\tau))x_1(\sigma(t))] r(\tau) y(\sigma(\tau))\Delta\tau = 0,$$

we obtain (4.9). □

Theorem 3. *Let $p, q \in C_{\mathrm{rd}}$ and suppose x_1 and x_2 are two linearly independent solutions of (1.1) satisfying*

$$\int_{t_0}^{\infty} |p(t) - q(t)|[x_1^2(\sigma(t)) + x_2^2(\sigma(t))]\Delta t < \infty. \tag{4.10}$$

Then for any solution y of (4.7), there exist constants $a, b \in \mathbb{R}$ such that

$$y(t) = ax_1(t) + bx_2(t) + \mathrm{O}(|x_1(t)| + |x_2(t)|) \quad \text{as } t \to \infty. \tag{4.11}$$

Proof. With the setting and notation from Lemma 4, we arrive at (4.9). Thus

$$|y(\sigma(t))| \leqslant \tilde{c} d(\sigma(t)) + |c| \int_{t_0}^{t} d(\sigma(\tau))d(\sigma(t))|r(\tau)||y(\sigma(\tau))|\Delta\tau = d(\sigma(t))R(t),$$

where $\tilde{c} = \sqrt{c_1^2 + c_2^2}$, $d(t) = \sqrt{x_1^2(t) + x_2^2(t)}$, and

$$R(t) = \tilde{c} + |c| \int_{t_0}^{t} d(\sigma(\tau))|r(\tau)||y(\sigma(\tau))|\Delta\tau \leqslant \tilde{c} + |c| \int_{t_0}^{t} d^2(\sigma(\tau))|r(\tau)|R(\tau)\Delta\tau.$$

Now, by Corollary 1 and Lemma 2, we obtain

$$R(t) \leqslant \tilde{c} e_{|c|(d^{\sigma})^2|r|}(t, t_0) \leqslant \tilde{c} \exp\left\{ \int_{t_0}^{t} |c| d^2(\sigma(\tau))|r(\tau)|\Delta\tau \right\}$$

so that $\lim_{t\to\infty} R(t)$ exists and therefore both integrals

$$c_3 = \int_{t_0}^{\infty} x_1(\sigma(\tau))r(\tau)y(\sigma(\tau))\Delta\tau \quad \text{and} \quad c_4 = \int_{t_0}^{\infty} x_2(\sigma(\tau))r(\tau)y(\sigma(\tau))\Delta\tau$$

are finite. Hence, by using (4.8) and putting $a = c_1 + cc_4$ and $b = c_2 - cc_3$, we arrive at

$$y(t) = ax_1(t) + bx_2(t) - cx_1(t)i_2(t) + cx_2(t)i_1(t),$$

where

$$i_1(t) = \int_t^\infty x_1(\sigma(\tau))r(\tau)y(\sigma(\tau))\Delta\tau \quad \text{and} \quad i_2(t) = \int_t^\infty x_2(\sigma(\tau))r(\tau)y(\sigma(\tau))\Delta\tau$$

both tend to zero as $t \to \infty$. This proves (4.11). □

The following two corollaries are immediate from Theorem 3 and its proof.

Corollary 3. *Let $p, q \in C_{\text{rd}}$ and suppose*

$$\int_{t_0}^\infty |p(t) - q(t)|\Delta t < \infty. \tag{4.12}$$

Suppose x_1 and x_2 are two linearly independent solutions of (1.1) satisfying

$$M := \sup_{t \geqslant t_0} \sqrt{x_1^2(t) + x_2^2(t)} < \infty.$$

Then, with the notation from the proof of Theorem 3, any solution y of (4.7) satisfies the growth condition

$$|y(t)| \leqslant M\tilde{c}e_{|c|M^2|p-q|}(t, t_0).$$

Corollary 4. *Suppose (4.12) holds. If all solutions of (1.1) are bounded, then so are all solutions of (4.7).*

The continuous version of the above boundedness theorem is given in [2, Theorem 2 in Chapter 6]. Now we consider $\mathscr{L}^\alpha$-boundedness of solutions of (1.1) and (4.7). We say that

$$f \in \mathscr{L}^\alpha(t_0, \infty) \quad \text{if } \|f\|_\alpha := \left\{ \int_{t_0}^\infty |f(\tau)|^\alpha \Delta\tau \right\}^{1/\alpha} < \infty$$

provided $\alpha \in [1, \infty)$, and for $\alpha = \infty$ we write

$$f \in \mathscr{L}^\infty(t_0, \infty) \quad \text{if } \|f\|_\infty := \sup_{t \in \mathbb{T} \cap [t_0, \infty)} |f(t)| < \infty.$$

The next result is about $\alpha \in (1, \infty)$, and its continuous version can be found in [2, Exercise 6.8.3]. The subsequent corollary with $\alpha = 2$ is contained for differential equations in [2, Theorem 6 in Chapter 6]. Finally, when $\alpha = 1$, we give a result corresponding to the continuous result from [2, Exercise 6.8.5].

Theorem 4. *Let $p, q \in C_{\text{rd}}$ and suppose $r := p - q$ is bounded. Let $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$. If all solutions x of (1.1) satisfy $x^\sigma \in \mathscr{L}^\alpha(t_0, \infty) \cap \mathscr{L}^\beta(t_0, \infty)$, then all solutions y of (4.7) satisfy $y^\sigma \in \mathscr{L}^\alpha(t_0, \infty) \cap \mathscr{L}^\beta(t_0, \infty)$.*

Proof. With the setting and notation from Lemma 4, we arrive at (4.9). Therefore, by taking into account that there exists $c_3 > 0$ such that $|r(t)| \leqslant c_3$ for all $t \in \mathbb{T}$, we obtain

$$|y(\sigma(t))| \leqslant |c_1x_1(\sigma(t))| + |c_2x_2(\sigma(t))| + |cx_1(\sigma(t))|c_3 \int_{t_0}^t |x_2(\sigma(\tau))y(\sigma(\tau))|\Delta\tau$$
$$+ |cx_2(\sigma(t))|c_3 \int_{t_0}^t |x_1(\sigma(\tau))y(\sigma(\tau))|\Delta\tau.$$

Now we use the well-known inequality

$$(a + b)^\alpha \leqslant 2^{\alpha-1}(a^\alpha + b^\alpha) \quad \text{for } a, b \geqslant 0 \text{ and } \alpha \geqslant 1$$

twice, then use Hölder's inequality [4, Theorem 6.13], and then put $c_4 := \|x_1^\sigma\|_\beta$ and $c_5 := \|x_2^\sigma\|_\beta$ to obtain

$$|y(\sigma(t))|^\alpha \leqslant 2^{2(\alpha-1)}(|c_1x_1(\sigma(t))|^\alpha + |c_2x_2(\sigma(t))|^\alpha) + 2^{2(\alpha-1)}|cx_1(\sigma(t))c_3|^\alpha \left\{ \int_{t_0}^t |x_2(\sigma(\tau))y(\sigma(\tau))|\Delta\tau \right\}^\alpha$$
$$+ 2^{2(\alpha-1)}|cx_2(\sigma(t))c_3|^\alpha \left\{ \int_{t_0}^t |x_1(\sigma(\tau))y(\sigma(\tau))|\Delta\tau \right\}^\alpha$$
$$\leqslant 2^{2(\alpha-1)}(|c_1x_1(\sigma(t))|^\alpha + |c_2x_2(\sigma(t))|^\alpha) + 2^{2(\alpha-1)}\{|cc_3x_1(\sigma(t))c_5|^\alpha + |cc_3x_2(\sigma(t))c_4|^\alpha\}S(t),$$

where

$$S(t) = \int_{t_0}^{t} |y(\sigma(\tau))|^{\alpha} \Delta\tau.$$

By integrating the above inequality and putting $c_6 = \|x_1^{\sigma}\|_{\alpha}$ and $c_7 = \|x_2^{\sigma}\|_{\alpha}$, we obtain

$$S(t) \leqslant c_8 + c_9 \int_{t_0}^{t} [|x_1(\sigma(\tau))|^{\alpha} + |x_2(\sigma(\tau))|^{\alpha}] S(\tau)\Delta\tau,$$

where $c_8 = 2^{2(\alpha-1)}(|c_1c_6|^{\alpha} + |c_2c_7|^{\alpha})$ and $c_9 = 2^{2(\alpha-1)}\max\{|cc_5c_3|^{\alpha}, |cc_4c_3|^{\alpha}\}$. By Corollary 1 and Lemma 2, we find

$$S(t) \leqslant c_8 e_{c_9[|x_1^{\sigma}|^{\alpha}+|x_2^{\sigma}|^{\alpha}]}(t, t_0) \leqslant c_8 \exp\left\{ c_9 \int_{t_0}^{t} [|x_1(\sigma(\tau))|^{\alpha} + |x_2(\sigma(\tau))|^{\alpha}]\Delta\tau \right\}$$

and so

$$\int_{t_0}^{\infty} |y(\sigma(\tau))|^{\alpha} \Delta\tau \leqslant c_8 e^{c_9(c_6^{\alpha}+c_7^{\alpha})}$$

so that we have indeed $y^{\sigma} \in \mathscr{L}^{\alpha}(t_0, \infty)$. By a duality argument, it is clear that $y^{\sigma} \in \mathscr{L}^{\beta}(t_0, \infty)$, too, finishing the proof. □

Corollary 5. *Let $p, q \in C_{\mathrm{rd}}$ and suppose $p - q$ is bounded. If all solutions x of (1.1) satisfy $x^{\sigma} \in \mathscr{L}^2(t_0, \infty)$, then all solutions y of (4.7) satisfy $y^{\sigma} \in \mathscr{L}^2(t_0, \infty)$.*

Proof. Set $\alpha = \beta = 2$ in Theorem 4. □

Theorem 5. *Let $p, q \in C_{\mathrm{rd}}$ and suppose $p - q$ is bounded. If all solutions x of (1.1) satisfy $x^{\sigma} \in \mathscr{L}^1(t_0, \infty) \cap \mathscr{L}^{\infty}(t_0, \infty)$, then all solutions y of (4.7) satisfy $y^{\sigma} \in \mathscr{L}^1(t_0, \infty) \cap \mathscr{L}^{\infty}(t_0, \infty)$.*

Proof. With the setting and notation from Lemma 4, we arrive at (4.9). Therefore, by taking into account that there exists $c_3 > 0$ such that $|r(t)| \leqslant c_3$ for all $t \in \mathbb{T}$, we obtain

$$\begin{aligned}
|y(\sigma(t))| &\leqslant |c_1x_1(\sigma(t))| + |c_2x_2(\sigma(t))| + |cx_1(\sigma(t))|c_3 \int_{t_0}^{t} |x_2(\sigma(\tau))y(\sigma(\tau))|\Delta\tau \\
&\quad + |cx_2(\sigma(t))|c_3 \int_{t_0}^{t} |x_1(\sigma(\tau))y(\sigma(\tau))|\Delta\tau \\
&\leqslant |c_1x_1(\sigma(t))| + |c_2x_2(\sigma(t))| + |cx_1(\sigma(t))|c_3|x_2|_{\infty} \int_{t_0}^{t} |y(\sigma(\tau))|\Delta\tau \\
&\quad + |cx_2(\sigma(t))|c_3|x_1|_{\infty} \int_{t_0}^{t} |y(\sigma(\tau))|\Delta\tau \\
&\leqslant Ms(t) + Ms(t) \int_{t_0}^{t} |y(\sigma(\tau))|\Delta\tau,
\end{aligned}$$

where we put

$$M = \max\left\{|c_1|, |c_2|, |c|c_3\|x_2\|_{\infty}, |c|c_3\|x_1\|_{\infty}\right\} \quad \text{and} \quad s(t) = |x_1^{\sigma}(t)| + |x_2^{\sigma}(t)|.$$

Therefore, by Lemma 1, we have

$$|y(\sigma(t))| \leqslant Ms(t) + Ms(t) \int_{t_0}^{t} Ms(\tau)e_{Ms}(t, \sigma(\tau))\Delta\tau = Ms(t) + Ms(t)\{e_{Ms}(t, t_0) - e_{Ms}(t, t)\} = Ms(t)e_{Ms}(t, t_0),$$

where we used the formula [4, Theorem 2.39]

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Now we can conclude the following two statements. First, using Lemma 2, we obtain

$$|y(\sigma(t))| \leqslant M(\|x_1\|_\infty + \|x_2\|_\infty)e^{M(\|x_1^\sigma\|_1+\|x_2^\sigma\|_1)},$$

so $y^\sigma \in \mathscr{L}^\infty(t_0,\infty)$. Second, again using Lemma 2, we arrive at

$$\int_{t_0}^t |y(\sigma(\tau))|\Delta\tau \leqslant \int_{t_0}^t Ms(\tau)e_{Ms}(\tau,t_0)\Delta\tau = e_{Ms}(t,t_0) - 1 < e_{Ms}(t,t_0) \leqslant e^{M(\|x_1^\sigma\|_1+\|x_2^\sigma\|_1)},$$

so $y^\sigma \in \mathscr{L}^1(t_0,\infty)$. Altogether, $y^\sigma \in \mathscr{L}^1(t_0,\infty) \cap \mathscr{L}^\infty(t_0,\infty)$. □

5. Nonlinear dynamic equations

Now we consider nonlinear equations of the form (1.2) and show the following result.

Theorem 6. *Consider the equation*

$$(p(t)x^\Delta)^\Delta + q(t)(f \circ x^\sigma) = g(t) \tag{5.1}$$

and assume the following:

1. $p(t) \geqslant \delta > 0$ *for all* $t \geqslant t_0$;
2. $q \in C_{\mathrm{rd}}$ *satisfies* (4.4);
3. $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $|f(x)| \leqslant L|x|^\alpha$ *for* $x \in \mathbb{R}$, *where* $L > 0$ *and* $\alpha \in [0,1]$;
4. $g \in C_{\mathrm{rd}}$.

Then

$$x(\sigma(t)) = \mathrm{O}\left(\sigma(t) + \sigma(t)\int_{t_0}^t [\sigma(t) - \sigma(\tau)]|g(\tau)|\Delta\tau\right) \quad \textit{as } t \to \infty. \tag{5.2}$$

Proof. We integrate (5.1) between t_0 and t to arrive at

$$x^\Delta(t) = \frac{p(t_0)x^\Delta(t_0)}{p(t)} + \frac{1}{p(t)}\int_{t_0}^t [g(\tau) - q(\tau)f(x(\sigma(\tau)))]\Delta\tau.$$

Integrating this equation again between t_0 and $\sigma(t)$, we obtain

$$x(\sigma(t)) = x(t_0) + p(t_0)x^\Delta(t_0)\int_{t_0}^{\sigma(t)} \frac{\Delta\tau}{p(\tau)} + \int_{t_0}^{\sigma(t)} \frac{1}{p(s)}\int_{t_0}^s [g(\tau) - q(\tau)f(x(\sigma(\tau)))]\Delta\tau\Delta s.$$

We now use the assumptions, Lemma 3, and the elementary inequality

$$|x|^\alpha \leqslant 1 + |x| \quad \text{for } x \in \mathbb{R} \text{ and } \alpha \in [0,1]$$

to find

$$\begin{aligned}
|x(\sigma(t))| + 1 \leqslant |x(t_0)| + 1 + \frac{|p(t_0)x^\Delta(t_0)|}{\delta}[\sigma(t) - t_0] + \frac{1}{\delta}\int_{t_0}^{\sigma(t)}\int_{t_0}^s |g(\tau)|\Delta\tau\Delta s \\
+ \frac{L}{\delta}\int_{t_0}^{\sigma(t)}\int_{t_0}^s |q(\tau)||x(\sigma(\tau))|^\alpha\Delta\tau\Delta s = |x(t_0)| + 1 + \frac{|p(t_0)x^\Delta(t_0)|}{\delta}[\sigma(t) - t_0] \\
+ \frac{1}{\delta}\int_{t_0}^t [\sigma(t) - \sigma(\tau)]|g(\tau)|\Delta\tau + \frac{L}{\delta}\int_{t_0}^t [\sigma(t) - \sigma(\tau)]|q(\tau)||x(\sigma(\tau))|^\alpha\Delta\tau \\
\leqslant c[1 + \sigma(t) + A(t)] + \frac{L\sigma(t)}{\delta}\int_{t_0}^t |q(\tau)|[1 + |x(\sigma(\tau))|]\Delta\tau,
\end{aligned}$$

where we put

$$A(t)=\int_{t_0}^{t}[\sigma(t)-\sigma(\tau)]|g(\tau)|\Delta\tau \quad \text{and} \quad c=\max\left\{|x(t_0)|+1,\frac{|p(t_0)x^{\Delta}(t_0)|}{\delta},\frac{1}{\delta}\right\}.$$

Now, by Lemmas 1 and 2, we conclude

$$\begin{aligned}|x(\sigma(t))|+1 &\leqslant c[1+A(t)+\sigma(t)]+\frac{L\sigma(t)}{\delta}\int_{t_0}^{t}c[1+A(\tau)+\sigma(\tau)]|q(\tau)|e_{L|q|\sigma/\delta}(t,\sigma(\tau))\Delta\tau\\ &\leqslant c[1+A(t)+\sigma(t)]+\frac{L\sigma(t)}{\delta}\int_{t_0}^{t}c[1+A(\tau)+\sigma(\tau)]|q(\tau)|\exp\left\{\frac{L}{\delta}\int_{t_0}^{\tau}|q(s)|\sigma(s)\Delta s\right\}\Delta\tau\\ &\leqslant c[1+A(t)+\sigma(t)]+\frac{e^{M}cL\sigma(t)}{\delta}\int_{t_0}^{t}[1+A(\tau)+\sigma(\tau)]|q(\tau)|\Delta\tau,\end{aligned}$$

where the limit

$$M=\frac{L}{\delta}\int_{t_0}^{\infty}\sigma(\tau)|q(\tau)|\Delta\tau$$

is finite due to (4.4). Noticing that $0\leqslant A(\tau)\leqslant A(t)$ whenever $\tau\leqslant t$, we find

$$\begin{aligned}\frac{|x(\sigma(t))|}{\sigma(t)+\sigma(t)A(t)} \leqslant \frac{|x(\sigma(t))|+1}{\sigma(t)+\sigma(t)A(t)} &\leqslant \frac{c}{\sigma(t)}+\frac{c}{1+A(t)}+\frac{e^{M}cL}{\delta}\int_{t_0}^{t}\left[\frac{1+A(\tau)}{1+A(t)}+\frac{\sigma(\tau)}{1+A(t)}\right]|q(\tau)|\Delta\tau\\ &\leqslant 2c+\frac{2e^{M}cL}{\delta}\int_{t_0}^{t}\sigma(\tau)|q(\tau)|\Delta\tau\leqslant 2c+2cMe^{M},\end{aligned}$$

which shows that (5.2) holds. □

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