Asymptotic behaviour of solutions of rational difference systems

Martin Bohner$^a$ and Svetlin G. Georgiev$^b$

$^a$Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO, USA; $^b$Sorbonne University, Paris, France

ABSTRACT

In this article, we investigate the asymptotic behaviour of solutions of systems of rational difference equations in arbitrary dimensions. We give conditions for the parameters ensuring that the positive solutions of the considered system are bounded, unbounded, increasing, decreasing, and convergent, respectively.

ARTICLE HISTORY

Received 4 August 2016
Accepted 1 November 2016

KEYWORDS

Difference equation; boundedness; rational systems

AMS SUBJECT CLASSIFICATIONS

39A10; 39A11; 39A20

1. Introduction

In this article, we let $N \in \mathbb{N} \setminus \{1\}$ and investigate the solutions of the system of rational difference equations

\[
\begin{aligned}
x_1(n+1) &= \frac{\gamma_1}{x_2(n)}, \quad n \in \mathbb{N}_0, \quad x_1(0) > 0, \\
x_2(n+1) &= \frac{\gamma_2}{x_3(n)}, \quad n \in \mathbb{N}_0, \quad x_2(0) > 0, \\
&\vdots \\
x_{N-1}(n+1) &= \frac{\gamma_{N-1}}{x_N(n)}, \quad n \in \mathbb{N}_0, \quad x_{N-1}(0) > 0, \\
x_N(n+1) &= \frac{\alpha_0 + \sum_{i=1}^{N-1} \alpha_i x_i(n)}{\beta_0 + \sum_{i=1}^{N} \beta_i x_i(n)} = \frac{\alpha^T x(n)}{\beta^T x(n)}, \quad n \in \mathbb{N}_0, \quad x_N(0) > 0,
\end{aligned}
\]

with

\[
\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}, \quad x(n) = \begin{pmatrix} x_1(n) \\ \vdots \\ x_N(n) \end{pmatrix},
\]

and

\[
\alpha, \beta \geq 0, \quad \alpha, \beta \neq 0, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{N-1} \end{pmatrix} > 0.
\]

CONTACT

Svetlin G. Georgiev
svetlingeorgiev1@gmail.com

© 2016 Informa UK Limited, trading as Taylor & Francis Group
(Here, we write $z \geq 0$, $z \neq 0$, and $z > 0$ for a vector $z$ if all the entries of the vector $z$ are nonnegative, nonzero, and positive, respectively.) We give some conditions for the parameters of (1.1) ensuring that (1.1) has bounded, unbounded, increasing, decreasing, and convergent positive solutions, respectively. In particular, when $N = 3$, we give an answer of an open problem (see [6, Open Problem 1]). For related results, we refer to [3,5] and the references given therein. The set up of this paper is as follows. In Section 2, we formulate and prove our main results. Section 3 contains specifications of our main results to the important cases of $N = 2$ and $N = 3$ as well as a series of six examples illustrating our results.

2. Main results

Lemma 2.1: Assume (1.1) and (1.2). Then

$$\Delta x_i(0) = \frac{\gamma_i - x_i(0)x_{i+1}(0)}{x_{i+1}(0)} , \quad i \in \{1, 2, \ldots, N-1\},$$

$$\Delta x_N(0) = \frac{(\alpha - x_N(0)\beta)^T x(0)}{\beta^T x(0)}$$

and

$$x_{i+1}(n)x_{i+1}(n+1)\Delta x_i(n+1) = -\gamma_i \Delta x_{i+1}(n) , \quad i \in \{1, 2, \ldots, N-1\},$$

$$\beta^T x(n)\beta^T x(n+1)\Delta x_N(n+1)$$

$$= x^T(n)(\beta\alpha^T - \alpha\beta^T)K \begin{pmatrix} 0 \\ (-1)^{N-1}\Delta x_1(n) \\ (-1)^{N-2}\Delta x_2(n) \\ \vdots \\ (-1)^{N-N}\Delta x_N(n) \end{pmatrix}$$

for all $n \in \mathbb{N}_0$ with

$$K = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & (-1)^{N-1} & 0 & \ldots & 0 \\ 0 & 0 & (-1)^{N-2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (-1)^{N-N} \end{pmatrix}.$$

Proof: We have

$$\Delta x_i(0) = x_i(1) - x_i(0)$$

$$= \frac{\gamma_i}{x_{i+1}(0)} - x_i(0)$$

$$= \frac{\gamma_i - x_i(0)x_{i+1}(0)}{x_{i+1}(0)}, \quad i \in \{1, 2, \ldots, N-1\},$$

$$\Delta x_N(0) = x_N(1) - x_N(0)$$
\[
\begin{align*}
&= \alpha_0 + \sum_{i=1}^{N} \alpha_i x_i(0) - x_N(0) \\
&= \frac{\alpha_0 - x_N(0)\beta_0 + \sum_{i=1}^{N} (\alpha_i - x_N(0)\beta_i)x_i(0)}{\beta_0 + \sum_{i=1}^{N} \beta_i x_i(0)} \\
&= \frac{(\alpha - x_N(0)\beta)^T x(0)}{\beta^T x(0)}. 
\end{align*}
\]

We use the discrete quotient rule (see, e.g. [4, Theorem 2.1 (e)]) to obtain
\[
\begin{align*}
x_{i+1}(n)x_{i+1}(n+1)\Delta x_i(n+1) &= x_{i+1}(n)\Delta x_i - \gamma_i \Delta x_{i+1}(n) \\
&= -\gamma_i \Delta x_{i+1}(n)
\end{align*}
\]
for all \( i \in \{1, 2, \ldots, N-1\} \) and
\[
\begin{align*}
\beta^T x(n)\beta^T x(n+1)\Delta x_N(n+1) &= \alpha^T \Delta x(n)\beta^T x(n) - \alpha^T x(n)\beta^T \Delta x(n) \\
&= x^T(n)\beta\alpha^T \Delta x(n) - x^T(n)\alpha\beta^T \Delta x(n) \\
&= x^T(n)(\beta\alpha^T - \alpha\beta^T)\Delta x(n) \\
&= x^T(n)(\beta\alpha^T - \alpha\beta^T)K \begin{pmatrix} 0 \\ (-1)^{N-1}\Delta x_1(n) \\ (-1)^{N-2}\Delta x_2(n) \\ \vdots \\ (-1)^{N-N}\Delta x_N(n) \end{pmatrix},
\end{align*}
\]
which completes the proof.

First, we consider monotonicity of the solution of (1.1).

**Theorem 2.2:** Let \( N \) be odd and assume (1.1) and (1.2). If
\[
\begin{align*}
\gamma_i - x_i(0)x_{i+1}(0) &\geq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\gamma_i - x_i(0)x_{i+1}(0) &\leq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\alpha - x_N(0)\beta &\geq 0,
\end{align*}
\]
and
\[
(\beta\alpha^T - \alpha\beta^T)K \text{ has positive entries,}
\]
then
\[
\{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing for } i \in \left\{1, \ldots, \frac{N-1}{2}\right\}
\]
and
\[
\{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing for } i \in \left\{1, \ldots, \frac{N+1}{2}\right\}.
\]

**Proof:** By Lemma 2.1, we have
\[
\begin{align*}
\Delta x_i(0) &\geq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\Delta x_i(0) &\leq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\Delta x_N(0) &\geq 0.
\end{align*}
\]
Now assume that
\[
\begin{align*}
\Delta x_i(n) &\geq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\Delta x_i(n) &\leq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\Delta x_N(n) &\geq 0 
\end{align*}
\]
for some \( n \in \mathbb{N}_0 \). By Lemma 2.1, we conclude
\[
\begin{align*}
\Delta x_i(n+1) &\geq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\Delta x_i(n+1) &\leq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\Delta x_N(n+1) &\geq 0
\end{align*}
\]
This completes the proof.

As in Theorem 2.2, one can prove the following result.

**Theorem 2.3:** Let \( N \) be even and assume (1.1), (1.2), (2.1), and (2.3). If
\[
\alpha - x_N(0) \beta \leq 0,
\] (2.4)
then, for \( i \in \{1, 2, \ldots, N\} \),
\[
\{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing and } \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing.}
\]

**Theorem 2.4:** Let \( N \) be odd and assume (1.1), (1.2), (2.3), and (2.4). If
\[
\begin{align*}
\gamma_i - x_i(0)x_{i+1}(0) &\leq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\gamma_i - x_i(0)x_{i+1}(0) &\geq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, 
\end{align*}
\] (2.5)
then
\[
\{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing for } i \in \{1, \ldots, \frac{N-1}{2}\}
\]
and
\[
\{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing for } i \in \{1, \ldots, \frac{N+1}{2}\}.
\]

**Proof:** By Lemma 2.1, we have
\[
\begin{align*}
\Delta x_i(0) &\leq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\Delta x_i(0) &\geq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\Delta x_N(0) &\leq 0.
\end{align*}
\]
Now assume that
\[
\begin{align*}
\Delta x_i(n) &\leq 0 \quad \text{for odd } i \in \{1, \ldots, N-1\}, \\
\Delta x_i(n) &\geq 0 \quad \text{for even } i \in \{1, \ldots, N-1\}, \\
\Delta x_N(n) &\leq 0
\end{align*}
\]
for some \( n \in \mathbb{N}_0 \). By Lemma 2.1, we conclude

\[
\begin{align*}
\Delta x_i(n+1) \leq 0 & \quad \text{for odd } \quad i \in \{1, \ldots, N - 1\}, \\
\Delta x_i(n+1) \geq 0 & \quad \text{for even } \quad i \in \{1, \ldots, N - 1\}, \\
\Delta x_N(n+1) \leq 0,
\end{align*}
\]

which completes the proof. \(\square\)

As in Theorem 2.4, one can prove the following result.

**Theorem 2.5:** Let \( N \) be even and assume \( (1.1), (1.2), (2.2), (2.3), \) and \( (2.5) \). Then, for \( i \in \{1, 2, \ldots, \frac{N}{2}\} \),

\[
\{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing and } \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing.}
\]

**Remark 2.6:**

(a) Let \( N = 2, \alpha_1 \gamma_1 = 0, \alpha_0 = 0, \beta_1 \gamma_1 = 1, \beta_2 = 1, \beta_0 = 0, \alpha_1 \geq 0, \alpha_2 \geq 0, \beta_1 \geq 0, \gamma_1 \geq 0 \). If one of the subsequences \( \{x_2(2n)\}_{n \in \mathbb{N}_0} \) and \( \{x_2(2n + 1)\}_{n \in \mathbb{N}_0} \) lies in the interval \([0, 1]\) and the other in \([1, \infty)\), then, by [1, page 11], they must both converge monotonically to 1 and \( \{x_1(2n)\}_{n \in \mathbb{N}_0} \) and \( \{x_1(2n + 1)\}_{n \in \mathbb{N}_0} \) must both converge monotonically to \( \gamma_1 \).

(b) Let \( N = 2, \alpha_2 = 1, \alpha_0 = 0, \beta_1 \gamma_1 = 1, \beta_2 = 0, \beta_0 = 1, \alpha_1 \geq 0, \alpha_2 \geq 0, \beta_1 \geq 0, \gamma_1 \geq 0 \). In [2, page 28], it is proved that \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) is increasing (decreasing) and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) is decreasing (increasing).

(c) Let \( N = 2, \alpha_1 = 0, \alpha_0 = 1, \beta_1 \gamma_1 > 1, \beta_0 = 1, \beta_1 > 0, \gamma_1 > 0 \). In [2, Theorem 6.1], it is proved that \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) is increasing and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) is decreasing.

(d) Let \( N = 2, \alpha_2 = 1, \beta_1 \gamma_1 < 1, \alpha_1 \gamma_1 = 0, \alpha_0 = 1, \beta_2 = 0, \beta_0 = 1, \alpha_1 \geq 0, \beta_1 \geq 0, \gamma_1 \geq 0 \). In [2, Theorem 6.1], it is proved that \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) is decreasing and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) is increasing.

(e) Let \( N = 2, \alpha_2 > 1, \beta_1 \gamma_1 \leq 1, \alpha_1 \gamma_1 = 0, \alpha_0 = 1, \beta_2 = 0, \beta_0 = 1, \alpha_1 \geq 0, \beta_1 \geq 0, \gamma_1 \geq 0 \). In [2, Theorem 6.1], it is proved that \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) is decreasing and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) is increasing.

Next, we consider boundedness of the solution of (1.1).

**Theorem 2.7:** Assume \((1.1)\) and \((1.2)\). Let

\[
0 < p_i \leq \frac{\gamma_i}{q_{i+1}} \leq \frac{\gamma_i}{p_{i+1}} \leq q_i, \quad i \in \{1, \ldots, N - 1\},
\]

\[
0 < p_N \leq \frac{\alpha N q}{p} \leq \frac{\alpha N p}{q} \leq q_N,
\]

with \( p = \begin{pmatrix} 1 \\ p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \) and \( q = \begin{pmatrix} 1 \\ q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} \). If

\[
p_i \leq x_i(0) \leq q_i, \quad i \in \{1, \ldots, N\},
\]
then
\[ p_i \leq x_i(n) \leq q_i, \quad i \in \{1, \ldots, N\}, \quad \text{for all} \quad n \in \mathbb{N}_0. \quad (2.8) \]

**Proof:** Assume that
\[ p_i \leq x_i(n) \leq q_i, \quad i \in \{1, \ldots, N\}, \]
for some \( n \in \mathbb{N}_0 \). Then, using (2.6), we conclude
\[ p_i \leq x_i(n + 1) \leq q_i, \quad i \in \{1, \ldots, N\}, \]
which completes the proof. \( \square \)

**Remark 2.8:**

(a) Let \( N = 2, \alpha_0 = \alpha_2 = 0, \beta_1 \gamma_1 = 0, \beta_0 = \beta_2 = 1, \alpha_1 \geq 0, \beta_1 \geq 0, \gamma_1 \geq 0 \). In [1], it is proved that both \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) are bounded below and above by positive constants.

(b) Let \( N = 2, \alpha_0 = \beta_0 = 1, \beta_1 \gamma_1 = 0, \alpha_1 \geq 0, \alpha_2 \geq 0, \beta_1 \geq 0, \beta_2 \geq 0, \gamma_1 \geq 0 \). In [2], it is proved that both \( \{x_1(n)\}_{n \in \mathbb{N}_0} \) and \( \{x_2(n)\}_{n \in \mathbb{N}_0} \) are bounded below and above by positive constants.

Now, we consider convergence of the solution of (1.1).

**Theorem 2.9:** Assume (1.1), (1.2), (2.3), (2.6), and (2.7). If either the conditions (2.1) and (2.2) when \( N \) is odd, or (2.1) and (2.4) when \( N \) is even, or (2.4) and (2.5) when \( N \) is odd, or (2.2) and (2.5) when \( N \) is even, hold, then \( \{x_i(n)\}_{n \in \mathbb{N}_0}, \quad i \in \{1, \ldots, N\}, \) are convergent and
\[ p_i \leq \lim_{n \to \infty} x_i(n) \leq q_i, \quad i \in \{1, \ldots, N\}. \quad (2.9) \]

**Proof:** The inequalities (2.9) hold because by Theorems 2.2–2.5, the sequences \( \{x_i(n)\}_{n \in \mathbb{N}_0}, \quad i \in \{1, \ldots, N\}, \) are monotonic and (2.8) is fulfilled. \( \square \)

Finally, we consider unboundedness of the solution of (1.1).

**Theorem 2.10:** Let \( N \) be odd and assume (1.1), (1.2), (2.1), (2.2), (2.3),
\[ \beta_{2i+1} = 0, \quad i \in \left\{1, \ldots, \frac{N-1}{2}\right\}, \quad (2.10) \]
and
\[ \frac{\alpha_N}{\beta^T x(0)} > 1. \quad (2.11) \]

Then
\[ \lim_{n \to \infty} x_{2i}(n) = 0 \quad \text{for} \quad i \in \left\{1, \ldots, \frac{N-1}{2}\right\} \]
and
\[ \lim_{n \to \infty} x_{2i-1}(n) = \infty \quad \text{for} \quad i \in \left\{1, \ldots, \frac{N+1}{2}\right\}. \]

**Proof:** By Theorem 2.2, we have that
\[ \{x_{2i}(n)\}_{n \in \mathbb{N}_0} \] is decreasing for \( i \in \left\{1, \ldots, \frac{N-1}{2}\right\} \)
and 
\[ \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing for } i \in \left\{1, \ldots, \frac{N+1}{2}\right\}. \]

By the last equation of (1.1), we get 
\[ x_N(n+1) \geq \frac{\alpha_N}{\beta^T x(0)} x_N(n), \quad n \in \mathbb{N}_0, \]
whereupon 
\[ x_N(n) \geq \left( \frac{\alpha_N}{\beta^T x(0)} \right)^n x_N(0), \quad n \in \mathbb{N}_0. \]

Hence, by (2.11), 
\[ \lim_{n \to \infty} x_N(n) = \infty \]
and therefore
\[
\begin{align*}
\lim_{n \to \infty} x_{N-1}(n) &= \lim_{n \to \infty} \frac{\gamma_{N-1}}{x_N(n)} = 0, \\
\lim_{n \to \infty} x_{N-2}(n) &= \lim_{n \to \infty} \frac{\gamma_{N-2}}{x_{N-1}(n)} = \infty, \\
&\vdots \\
\lim_{n \to \infty} x_1(n) &= \lim_{n \to \infty} \frac{\gamma_1}{x_2(n)} = \infty,
\end{align*}
\]
which completes the proof. \(\square\)

**Remark 2.11:** Let \(N\) be odd. Then for any \(n \in \mathbb{N}\), we have
\[
\begin{align*}
x_1(n) &= \gamma_1 \gamma_3 \ldots \frac{\gamma_{N-2}}{\gamma_N} x_N(n-N+1), \\
x_2(n) &= \gamma_2 \gamma_4 \ldots \frac{1}{\gamma_{N-1}} x_N(n-N+2), \\
&\vdots \\
x_{N-2}(n) &= \frac{\gamma_{N-2}}{\gamma_N} x_N(n-2), \\
x_{N-1}(n) &= \frac{1}{\gamma_N} x_N(n-1).
\end{align*}
\]
Let
\[
\begin{align*}
A_1 &= \alpha_1 \frac{\gamma_1}{\gamma_3} \gamma_3 \ldots \frac{\gamma_{N-2}}{\gamma_N} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \ldots + \alpha_{N-2} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \alpha_N, \\
B_1 &= \beta_1 \frac{\gamma_1}{\gamma_3} \gamma_3 \ldots \frac{\gamma_{N-2}}{\gamma_N} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \ldots + \beta_{N-2} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \beta_N, \\
A_2 &= \alpha_2 \gamma_2 \gamma_4 \ldots \frac{\gamma_{N-2}}{\gamma_N} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \ldots + \alpha_{N-1} \gamma_{N-1}, \\
B_2 &= \beta_2 \gamma_2 \gamma_4 \ldots \frac{\gamma_{N-2}}{\gamma_N} \frac{\gamma_{N-2}}{\gamma_{N-1}} + \ldots + \beta_{N-1} \gamma_{N-1},
\end{align*}
\]
\[ A_1 = \frac{\gamma_1 \gamma_3 \ldots \gamma_{N-2}}{\gamma_{N-1}} x_N(n - N + 1) + \ldots + \frac{\gamma_{N-2}}{\gamma_{N-1}} x_N(n - 2) + \alpha_N x_N(n), \]
\[ B_1 = \frac{\beta_1 \gamma_1 \gamma_3 \gamma_4 \ldots \gamma_{N-2}}{\gamma_{N-1}} x_N(n - N + 1) + \ldots + \frac{\beta_{N-2}}{\gamma_{N-1}} x_N(n - 2) + \beta_N x_N(n), \]
\[ A_2 = \frac{\alpha_2 \gamma_2}{\gamma_3} \ldots \frac{1}{\gamma_{N-2} x_N(n - N + 2)} + \ldots + \alpha_{N-1} \gamma_{N-1} \frac{1}{x_N(n - 1)}, \]
\[ B_2 = \frac{\beta_2 \gamma_2}{\gamma_3} \ldots \frac{1}{\gamma_{N-2} x_N(n - N + 2)} + \ldots + \beta_{N-1} \gamma_{N-1} \frac{1}{x_N(n - 1)}. \]

Then, we can rewrite the last equation of (1.1) as
\[ x_N(n + 1) = \frac{\alpha_0 + A_1 + A_2 + B_1 + B_2}{\beta_0} . \]  

**Theorem 2.12:** Let \( N \) be odd and assume (1.1), (1.2), (2.3), (2.4), (2.5), and \( \alpha_{2i} = 0 \) for \( i \in \{1, \ldots, \frac{N-1}{2}\} \). If either
\[ (\beta_0 - A_1)^2 - 4B_1(B_2 - \alpha_0) < 0 \]  

or
\[ (\beta_0 - A_1)^2 - 4B_1(B_2 - \alpha_0) = 0 \quad \text{and} \quad A_1 - \beta_0 < 0 \]

or
\[ (\beta_0 - A_1)^2 - 4B_1(B_2 - \alpha_0) > 0, \quad B_2 - \alpha_0 > 0, \quad \text{and} \quad A_1 - \beta_0 < 0, \]

then
\[ \lim_{n \to \infty} x_{2i}(n) = \infty \quad \text{for} \quad i \in \left\{1, \ldots, \frac{N-1}{2}\right\} \]

and
\[ \lim_{n \to \infty} x_{2i-1}(n) = 0 \quad \text{for} \quad i \in \left\{1, \ldots, \frac{N+1}{2}\right\}. \]

**Proof:** By Theorem 2.4, we have that
\[ \{x_{2i}(n)\}_{n \in \mathbb{N}_0} \quad \text{is increasing for} \quad i \in \left\{1, \ldots, \frac{N-1}{2}\right\} \]

and
\[ \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \quad \text{is decreasing for} \quad i \in \left\{1, \ldots, \frac{N+1}{2}\right\}. \]

Therefore,
\[ m = \lim_{n \to \infty} x_N(n) \]

exists. Using (2.12) and \( \alpha_{2i} = 0 \) for \( i \in \{1, \ldots, \frac{N-1}{2}\} \), we get
\[ m = \frac{\alpha_0 + A_1 m + A_2 \frac{1}{m}}{\beta_0 + B_1 m + B_2 \frac{1}{m}} = \frac{\alpha_0 m + A_1 m^2}{\beta_0 m + B_1 m^2 + B_2} . \]
i.e.

\[ m(B_1 m^2 + \beta_0 m + B_2) = \alpha_0 m + A_1 m^2. \]

The last equation has the unique nonnegative solution \( m = 0 \). Hence,

\[
\lim_{n \to \infty} x_N(n) = 0
\]

and therefore

\[
\lim_{n \to \infty} x_{N-1}(n) = \lim_{n \to \infty} \frac{\gamma_{N-1}}{x_N(n-1)} = \infty,
\]

\[
\lim_{n \to \infty} x_{N-2}(n) = \lim_{n \to \infty} \frac{\gamma_{N-2}}{x_{N-1}(n-1)} = 0,
\]

\[
\vdots
\]

\[
\lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} \frac{\gamma_1}{x_2(n-1)} = 0,
\]

which completes the proof.

\[ \square \]

**Remark 2.13:** Let \( N \) be even. Then for any \( n \in \mathbb{N} \), we have

\[
x_1(n) = \frac{\gamma_1}{x_2} \frac{\gamma_3}{y_4} \cdots \frac{\gamma_{N-1}}{x_N(n-N+1)} \frac{1}{y_N} ,
\]

\[
x_2(n) = \frac{\gamma_2}{y_4} \frac{\gamma_4}{y_5} \cdots \frac{\gamma_{N-2}}{x_N(n-N+2)} ,
\]

\[
\vdots
\]

\[
x_{N-2}(n) = \frac{\gamma_{N-2}}{x_N(n-2)} ,
\]

\[
x_{N-1}(n) = \frac{\gamma_{N-1}}{x_N(n-1)} \frac{1}{x_N(n-1)} .
\]

Let

\[
C_1 = \alpha_1 \frac{\gamma_1}{x_2} \frac{\gamma_3}{y_4} \cdots \frac{\gamma_{N-1}}{x_N(n-N+1)} + \cdots + \alpha_{N-1} \gamma_{N-1},
\]

\[
C_2 = \beta_1 \frac{\gamma_1}{x_2} \frac{\gamma_3}{y_4} \cdots \frac{\gamma_{N-1}}{x_N(n-N+1)} + \cdots + \beta_{N-1} \gamma_{N-1},
\]

\[
D_1 = \alpha_2 \frac{\gamma_2}{y_4} \frac{\gamma_4}{y_5} \cdots \frac{\gamma_{N-2}}{x_N(n-N+2)} + \cdots + \alpha_N ,
\]

\[
D_2 = \beta_2 \frac{\gamma_2}{y_4} \frac{\gamma_4}{y_5} \cdots \frac{\gamma_{N-2}}{x_N(n-N+2)} + \cdots + \beta_N ,
\]

\[
\overline{C}_1 = \alpha_1 \frac{\gamma_1}{x_2} \frac{\gamma_3}{y_4} \cdots \frac{\gamma_{N-1}}{x_N(n-N+1)} \frac{1}{x_N(n-1)} + \cdots + \alpha_{N-1} \gamma_{N-1} \frac{1}{x_N(n-1)},
\]

\[
\overline{C}_2 = \beta_1 \frac{\gamma_1}{x_2} \frac{\gamma_3}{y_4} \cdots \frac{\gamma_{N-1}}{x_N(n-N+1)} \frac{1}{x_N(n-1)} + \cdots + \beta_{N-1} \gamma_{N-1} \frac{1}{x_N(n-1)},
\]

\[
\overline{D}_1 = \alpha_2 \frac{\gamma_2}{y_4} \frac{\gamma_4}{y_5} \cdots \frac{\gamma_{N-2}}{x_N(n-N+2)} \frac{x_N(n-N+2)}{x_N(n-1)} + \cdots + \alpha_N x_N(n) ,
\]

\[
\overline{D}_2 = \beta_2 \frac{\gamma_2}{y_4} \frac{\gamma_4}{y_5} \cdots \frac{\gamma_{N-2}}{x_N(n-N+2)} \frac{x_N(n-N+2)}{x_N(n-1)} + \cdots + \beta_N x_N(n) .
\]
Then, we can rewrite the last equation of (1.1) as
\[ x_N(n + 1) = \frac{\alpha_0 + C_1 + D_1}{\beta_0 + C_2 + D_2}. \] (2.16)

**Theorem 2.14:** Let \( N \) be even and assume (1.1), (1.2), (2.1), (2.3), (2.4), and \( \alpha_{2i-1} = 0, \ i \in \{1, \ldots, \frac{N}{2}\} \). If either
\[ (\beta_0 - D_1)^2 - 4D_2(C_2 - \alpha_0) < 0 \] (2.17)
or
\[ (\beta_0 - D_1)^2 - 4D_2(C_2 - \alpha_0) = 0 \text{ and } D_1 - \beta_0 < 0 \] (2.18)
or
\[ (\beta_0 - D_1)^2 - 4D_2(C_2 - \alpha_0) > 0, \ C_2 - \alpha_0 > 0 \ \text{and} \ D_1 - \beta_0 < 0, \] (2.19)

then
\[ \lim_{n \to \infty} x_{2i}(n) = 0 \text{ and } \lim_{n \to \infty} x_{2i-1}(n) = \infty \text{ for } i \in \{1, \ldots, \frac{N}{2}\}. \]

**Proof:** By Theorem 2.3, we have that, for \( i \in \{1, \ldots, \frac{N}{2}\} \),
\[ \{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing and } \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing}. \]

Therefore,
\[ m = \lim_{n \to \infty} x_N(n) \]
exists. Taking into account also (2.16), we get
\[ m = \frac{\alpha_0 m + D_1 m^2}{\beta_0 m + C_2 + D_2 m^2}, \]
i.e.
\[ m(D_2 m^2 + \beta_0 m + C_2) = \alpha_0 m + D_1 m^2. \]

Using (2.17), (2.18), and (2.19), the last equation has the unique nonnegative solution \( m = 0 \). Hence,
\[ \lim_{n \to \infty} x_N(n) = 0 \]
and therefore
\[ \lim_{n \to \infty} x_{N-1}(n) = \lim_{n \to \infty} \frac{\gamma_{N-1}}{x_N(n)} = \infty, \]
\[ \lim_{n \to \infty} x_{N-2}(n) = \lim_{n \to \infty} \frac{\gamma_{N-2}}{x_{N-1}(n)} = 0, \]
\[ \vdots \]
\[ \lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} \frac{\gamma_1}{x_2(n)} = \infty, \]
which completes the proof. \( \square \)
Remark 2.15:

(a) Let $N = 2$, $\alpha_0 = \alpha_1 = \beta_1 = 0$, $\beta_0 = \beta_1 \gamma_1 = 1$, $0 \leq \alpha_2 \leq 1$, $\beta_1 \geq 0$, $\gamma_1 \geq 0$. In [1], it is proved that $x_1(n) \to \infty$ and $x_2(n) \to 0$ as $n \to \infty$.

(b) Let $N = 2$, $\alpha_1 \gamma_1 = \alpha_2 = \beta_0 = 0$, $\beta_1 = 1$, $0 \leq \gamma_1 < 1$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $\alpha_0 \leq 1$. In [1], it is proved that $x_1(n) \to \infty$ and $x_2(n) \to 0$ as $n \to \infty$.

(c) Let $N = 2$, $\alpha_1 = \alpha_2 = \beta_0 = 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_1 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $x_1(n) \to \infty$ and $x_2(n) \to 0$ as $n \to \infty$.

(d) Let $N = 2$, $0 \leq \alpha_0 \leq 1$, $\alpha_1 = \alpha_2 = 0$, $\beta_1 \gamma_1 = \beta_0 = 1$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $x_1(n) \to \infty$ and $x_2(n) \to 0$ as $n \to \infty$.

Theorem 2.16: Let $N$ be even and assume (1.1), (1.2), (2.2), (2.3), (2.5), (2.11), and

$$\beta_{2i} = 0, \quad i \in \left\{1, \ldots, \frac{N}{2}\right\}. \quad (2.20)$$

Then

$$\lim_{n \to \infty} x_{2i}(n) = \infty \text{ and } \lim_{n \to \infty} x_{2i-1}(n) = 0 \text{ for } i \in \left\{1, \ldots, \frac{N}{2}\right\}. \quad (2.21)$$

Proof: By Theorem 2.5, we have that, for $i \in \left\{1, \ldots, \frac{N}{2}\right\}$,

$$\{x_{2i}(n)\}_{n \in \mathbb{N}_0} \text{ is increasing and } \{x_{2i-1}(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing.}$$

By the last equation of (1.1), we get

$$x_N(n + 1) \geq \frac{\alpha_n}{\beta^T x(0)} x_N(n), \quad n \in \mathbb{N}_0,$$

whereupon

$$x_N(n) \geq \left(\frac{\alpha_n}{\beta^T x(0)}\right)^n x_N(0), \quad n \in \mathbb{N}_0.$$  

Hence, by (2.11),

$$\lim_{n \to \infty} x_N(n) = \infty$$

and therefore

$$\lim_{n \to \infty} x_{N-1}(n) = \lim_{n \to \infty} \frac{\gamma_{N-1}}{x_N(n)} = 0,$$

$$\lim_{n \to \infty} x_{N-2}(n) = \lim_{n \to \infty} \frac{\gamma_{N-2}}{x_{N-1}(n)} = \infty,$$

$$\vdots$$

$$\lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} \frac{\gamma_1}{x_2(n)} = 0,$$

which completes the proof. \hfill \square

Remark 2.17:

(a) Let $N = 2$, $\alpha_0 = \beta_0 = \beta_2 = 0$, $\beta_1 \gamma_1 = 1$, $\alpha_1 \gamma_1 > \frac{1}{2}$, $\beta_1 \geq 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\gamma_1 \geq 0$. In [1], it is proved that $x_1(n) \to 0$ and $x_2(n) \to \infty$ as $n \to \infty$. 

...
(b) Let $N = 2$, $\alpha_0 \geq 1$, $\alpha_2 = 1$, $\beta_1 \gamma_1 = 1$, $\alpha_1 = \beta_0 = \beta_2 = 0$, $\beta_1 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $x_1(n) \to 0$ and $x_2(n) \to \infty$ as $n \to \infty$.

(c) Let $N = 2$, $\alpha_2 \geq 1$, $\alpha_0 = 1$, $\beta_1 \gamma_1 = 0$, $\beta_0 = 1$, $\beta_2 = 0$, $\beta_1 \geq 0$, $\alpha_1 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $x_1(n) \to 0$ and $x_2(n) \to \infty$ as $n \to \infty$.

(d) Let $N = 2$, $\alpha_2 > 1$, $x_1(0), x_2(0) > \frac{\beta_1 \gamma_1}{\alpha_2 - 1}$, $\alpha_0 = 1$, $\beta_0 = 1$, $\beta_2 = 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $x_1(n) \to 0$ and $x_2(n) \to \infty$ as $n \to \infty$. Item[(e)]

Let $N = 2$, $\alpha_1 = \alpha_2 = \beta_0 = 0$, $\beta_2 = 1$, $\gamma_1 \geq 1$, $\alpha_0 \geq 0$, $\beta_1 \geq 0$. In [1], it is proved that $\{x_1(n)\}_{n \in \mathbb{N}_0}$ is bounded and $\{x_2(n)\}_{n \in \mathbb{N}_0}$ is unbounded.

(f) Let $N = 2$, $\alpha_2 \geq 1$, $\alpha_0 = 0$, $\beta_1 \gamma_1 = 1$, $\beta_2 = 0$, $\beta_0 = 1$, $\alpha_1 \geq 0$, $\gamma_1 \geq 0$, $\beta_1 \geq 0$. In [1], it is proved that $\{x_1(n)\}_{n \in \mathbb{N}_0}$ is bounded and $\{x_2(n)\}_{n \in \mathbb{N}_0}$ is unbounded.

(g) Let $N = 2$, $\beta_1 \gamma_1 < 1$, $\alpha_1 \gamma_1 = 0$, $\alpha_0 = 1$, $\beta_2 = 1$, $\beta_0 = 0$, $\beta_1 \geq 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\gamma_1 \geq 0$. In [2], it is proved that $\{x_1(n)\}_{n \in \mathbb{N}_0}$ is bounded and $\{x_2(n)\}_{n \in \mathbb{N}_0}$ is unbounded.

3. Examples

3.1. Two-dimensional rational systems

For $N = 2$, the system (1.1) takes the form

$$\begin{cases}
    x_1(n+1) = \frac{\gamma_1}{x_2(n)}, & n \in \mathbb{N}_0, \quad x_1(0) > 0, \\
    x_2(n+1) = \frac{\alpha_0 + \alpha_1 x_1(n) + \alpha_2 x_2(n)}{\beta_0 + \beta_1 x_1(n) + \beta_2 x_2(n)}, & n \in \mathbb{N}_0, \quad x_2(0) > 0.
\end{cases}
$$ (3.1)

Our main results can be reflected for (3.1) as follows.

(1) Assume

$$\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \geq 0, \quad (\alpha_0 + \alpha_1 + \alpha_2)(\beta_0 + \beta_1 + \beta_2) > 0,$$

$$\gamma_1 \geq x_1(0)x_2(0),$$

$$\alpha_0 \beta_1 \geq \alpha_1 \beta_0, \quad \alpha_2 \beta_0 \geq \alpha_0 \beta_2, \quad \alpha_2 \beta_1 \geq \alpha_1 \beta_2,$$

and

$$\alpha_0 \leq x_2(0)\beta_0, \quad \alpha_1 \leq x_2(0)\beta_1, \quad \alpha_2 \leq x_2(0)\beta_2.$$ (3.5)

Then $\{x_1(n)\}_{n \in \mathbb{N}_0}$ is increasing and $\{x_2(n)\}_{n \in \mathbb{N}_0}$ is decreasing.

(2) Assume (3.2), (3.4), $\gamma_1 > 0$,

$$\alpha_0 \geq x_2(0)\beta_0, \quad \alpha_1 \geq x_2(0)\beta_1, \quad \alpha_2 \geq x_2(0)\beta_2,$$

and

$$\gamma_1 \leq x_1(0)x_2(0).$$ (3.7)

Then $\{x_1(n)\}_{n \in \mathbb{N}_0}$ is decreasing and $\{x_2(n)\}_{n \in \mathbb{N}_0}$ is increasing.
(3) Assume (3.2) and
\[
\begin{align*}
0 < p_1 &\leq \frac{\gamma_1}{q_2}, \quad q_1, \\
0 < p_2 &\leq \frac{\alpha_0 + \alpha_1 p_1 + \alpha_2 p_2}{\beta_0 + \beta_1 q_1 + \beta_2 q_2} \leq \frac{\alpha_0 + \alpha_1 q_1 + \alpha_2 q_2}{\beta_0 + \beta_1 p_1 + \beta_2 p_2} \leq q_2, \\
p_1 &\leq x_1(0) \leq q_1, \quad p_2 \leq x_2(0) \leq q_2.
\end{align*}
\] (3.8)

Then

\[ p_1 \leq x_1(n) \leq q_1 \quad \text{and} \quad p_2 \leq x_2(n) \leq q_2 \quad \text{for all} \quad n \in \mathbb{N}_0. \]

(4) Assume (3.2), (3.4), and (3.8). If either (3.3) and (3.5) or (3.6) and (3.7) hold, then \{x_1(n)\}_{n \in \mathbb{N}_0} and \{x_2(n)\}_{n \in \mathbb{N}_0} are convergent and

\[ p_1 \leq \lim_{n \to \infty} x_1(n) \leq q_1 \quad \text{and} \quad p_2 \leq \lim_{n \to \infty} x_2(n) \leq q_2. \]

(5) Assume (3.2), (3.3), (3.4), (3.5),
\[ \alpha_0 = \alpha_1 = 0, \quad \beta_2 = 0, \] (3.9)
and
\[ \frac{\alpha_2}{\beta_0 + \beta_1 x_1(0)} < 1. \] (3.10)

Then

\[ \lim_{n \to \infty} x_1(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} x_2(n) = 0. \]

(6) Assume (3.2), (3.4), (3.6), (3.7), (3.9), and
\[ \frac{\alpha_2}{\beta_0 + \beta_1 x_1(0)} > 1. \] (3.11)

Then

\[ \lim_{n \to \infty} x_1(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} x_2(n) = \infty. \]

**Example 3.1:** The constants
\[ \alpha_0 = \beta_0 = 4, \quad \alpha_1 = \beta_1 = \frac{1}{2}, \quad \alpha_2 = 3, \quad \beta_2 = 1, \quad \gamma_1 = 2, \]
\[ x_1(0) = 2, \quad x_2(0) = 1, \quad p_1 = p_2 = 1, \quad q_1 = q_2 = 2 \]
satisfy (3.2), (3.3), (3.4), (3.6), (3.7), and (3.8). The solution of the system
\[
\begin{align*}
x_1(n + 1) &= \frac{2}{x_2(n)}, \quad n \in \mathbb{N}_0, \quad x_1(0) = 2, \\
x_2(n + 1) &= \frac{4 + \frac{1}{4} x_1(n) + \frac{3}{4} x_2(n)}{4 + \frac{1}{4} x_1(n) + x_2(n)}, \quad n \in \mathbb{N}_0, \quad x_2(0) = 1
\end{align*}
\] (3.12)
is depicted in Figure 1, and it satisfies
\[ \lim_{n \to \infty} x_1(n) = \frac{2}{x^*} \approx 1.3503, \quad \lim_{n \to \infty} x_2(n) = x^* \approx 1.4812, \]
where \(x^*\) is the (single) positive root of the polynomial \(x^3 + x^2 - 3x - 1\).
Example 3.2: The constants
\[
\begin{align*}
\alpha_0 &= \beta_0 = \beta_1 = \gamma_1 = 1, & \alpha_1 &= \alpha_2 = \beta_2 = 0, \\
x_1(0) &= x_2(0) = 1
\end{align*}
\]
satisfy (3.2), (3.3), (3.4), (3.5), and (3.10).

Example 3.3: The constants
\[
\begin{align*}
\alpha_0 &= \beta_0 = \gamma_1 = 1, & \alpha_1 &= \beta_1 = \beta_2 = 0, & \alpha_2 &= 4, \\
x_1(0) &= x_2(0) = 1
\end{align*}
\]
satisfy (3.2), (3.4), (3.6), (3.7), (3.9), and (3.11).

### 3.2. Three-dimensional rational systems

For \( N = 3 \), the system (1.1) takes the form

\[
\begin{align*}
x_1(n + 1) &= \frac{x_1(n)}{x_2(n)}, & n \in \mathbb{N}_0, & x_1(0) > 0, \\
x_2(n + 1) &= \frac{x_2(n)}{x_3(n)}, & n \in \mathbb{N}_0, & x_2(0) > 0, \\
x_3(n + 1) &= \frac{\alpha_0 + \alpha_1 x_1(n) + \alpha_2 x_2(n) + \alpha_3 x_3(n)}{\beta_0 + \beta_1 x_1(n) + \beta_2 x_2(n) + \beta_3 x_3(n)}, & n \in \mathbb{N}_0, & x_3(0) > 0.
\end{align*}
\]

Our main results can be reflected for (3.13) as follows.
(1) Assume
\[
\begin{align*}
\gamma_1 & \geq x_1(0)x_2(0), \quad \gamma_2 \leq x_2(0)x_3(0), \\
\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 & \geq 0, \quad \gamma_1, \gamma_2 > 0, \\
(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)(\beta_0 + \beta_1 + \beta_2 + \beta_3) & > 0,
\end{align*}
\]
and
\[
\begin{align*}
\alpha_0 & \geq x_3(0)\beta_0, \quad \alpha_1 \geq x_3(0)\beta_1, \quad \alpha_2 \geq x_3(0)\beta_2, \quad \alpha_3 \geq x_3(0)\beta_3.
\end{align*}
\]
Then
\[
\begin{align*}
\{x_1(n)\}_{n \in \mathbb{N}_0} \text{ is increasing}, \\
\{x_2(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing},
\end{align*}
\]
and
\[
\{x_3(n)\}_{n \in \mathbb{N}_0} \text{ is increasing}.
\]

(2) Assume (3.15),
\[
\begin{align*}
\alpha_0 & \leq x_3(0)\beta_0, \quad \alpha_1 \leq x_3(0)\beta_1, \quad \alpha_2 \leq x_3(0)\beta_2, \quad \alpha_3 \leq x_3(0)\beta_3
\end{align*}
\]
and
\[
\gamma_1 \leq x_1(0)x_2(0), \quad \gamma_2 \geq x_2(0)x_3(0).
\]
Then
\[
\begin{align*}
\{x_1(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing}, \\
\{x_2(n)\}_{n \in \mathbb{N}_0} \text{ is increasing},
\end{align*}
\]
and
\[
\{x_3(n)\}_{n \in \mathbb{N}_0} \text{ is decreasing}.
\]

(3) Assume (3.15),
\[
\begin{align*}
0 < p_1 & \leq \frac{\gamma_1}{q_2} \leq \frac{\gamma_1}{p_2} \leq q_1, \\
0 < p_2 & \leq \frac{\gamma_2}{q_3} \leq \frac{\gamma_2}{p_3} \leq q_2, \\
0 < p_3 & \leq \frac{\gamma_3}{q_4} \leq \frac{\gamma_3}{p_4} \leq q_3
\end{align*}
\]
and
\[
\begin{align*}
p_1 & \leq x_1(0) \leq q_1, \quad p_2 \leq x_2(0) \leq q_2, \quad p_3 \leq x_3(0) \leq q_3.
\end{align*}
\]
for all \( n \in \mathbb{N}_0 \).

(4) Assume (3.15), (3.19), (3.20). If either (3.14) and (3.16) or (3.17) and (3.18) hold, then \( \{x_1(n)\}_{n \in \mathbb{N}_0} \), \( \{x_2(n)\}_{n \in \mathbb{N}_0} \), and \( \{x_3(n)\}_{n \in \mathbb{N}_0} \) are convergent and

\[
p_1 \leq \lim_{n \to \infty} x_1(n) \leq q_1, \quad p_2 \leq \lim_{n \to \infty} x_2(n) \leq q_2, \quad p_3 \leq \lim_{n \to \infty} x_3(n) \leq q_3.
\]

(5) Assume (3.14), (3.15), (3.16), \( \beta_3 = 0 \), and

\[
\frac{\alpha_3}{\beta_0 + \beta_1 x_1(0) + \beta_2 x_2(0)} > 1. \tag{3.21}
\]

Then

\[
\lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} x_3(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} x_2(n) = 0.
\]

(6) Assume (3.14), (3.15), (3.16),

\[
\alpha_0 = \alpha_1 = \alpha_2 = \beta_2 = 0,
\]

and

\[
\frac{\alpha_3}{\beta_0 + \beta_1 x_1(0) + \beta_3 x_3(0)} < 1. \tag{3.22}
\]

Then

\[
\lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} x_3(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} x_2(n) = \infty.
\]

**Example 3.4:** The constants

\[
\alpha_0 = \beta_0 = 4, \quad \alpha_1 = \alpha_3 = \beta_1 = \beta_3 = \frac{1}{2}, \quad \alpha_2 = 3, \quad \beta_2 = 1, \quad \gamma_1 = \gamma_2 = 2, \quad x_1(0) = 1, \quad x_2(0) = 2, \quad x_3(0) = 1, \quad p_1 = p_2 = p_3 = 1, \quad q_1 = q_2 = q_3 = 2
\]

satisfy (3.14), (3.16), (3.18), (3.19), and (3.20). The solution of the system

\[
\begin{align*}
  x_1(n+1) &= \frac{2}{x_2(n)}, & n \in \mathbb{N}_0, & x_1(0) = 1, \\
  x_2(n+1) &= \frac{2}{x_3(n)}, & n \in \mathbb{N}_0, & x_2(0) = 2, \\
  x_3(n+1) &= \frac{4 + \frac{1}{2} x_1(n) + 3 x_2(n) + \frac{1}{2} x_3(n)}{4 + \frac{1}{2} x_1(n) + x_2(n) + \frac{1}{2} x_3(n)}, & n \in \mathbb{N}_0, & x_3(0) = 1
\end{align*}
\]

is depicted in Figure 2, and it satisfies

\[
\lim_{n \to \infty} x_1(n) = \lim_{n \to \infty} x_2(n) = \lim_{n \to \infty} x_3(n) = \sqrt{2}.
\]

**Example 3.5:** The constants

\[
\alpha_0 = \alpha_1 = \alpha_2 = \beta_0 = \beta_1 = \beta_2 = 1, \quad \alpha_3 = 5, \quad \beta_3 = 0, \quad \gamma_1 = 2, \quad \gamma_2 = 1, \quad x_1(0) = 2, \quad x_2(0) = x_3(0) = 1
\]

satisfy (3.14), (3.16) and (3.21).
Example 3.6: The constants

\[ \alpha_0 = \alpha_1 = \alpha_2 = \beta_1 = \beta_3 = 0, \quad \alpha_3 = \beta_0 = \beta_2 = 1, \quad \gamma_1 = 2, \quad \gamma_2 = 1, \]

\[ x_1(0) = 2, \quad x_2(0) = x_3(0) = 1 \]

satisfy (3.14), (3.15) and (3.22).

Acknowledgements

Svetlin Georgiev would like to thank Professor Mokhtar Kirane (University of La Rochelle, France) for his suggestion to examine [6, Open Problem 1]).

Disclosure statement

No potential conflict of interest was reported by the authors.

References


