

Asymptotic expansions and analytic dynamic equations

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Received 8 October 2004, revised and accepted 16 March 2005

Published online 29 June 2005

Key words Time scale, perturbation result, dichotomy condition, growth condition

MSC (2000) 39A10, 39A12, 34E05, 34E10

Time scales have been introduced in order to unify the theories of differential and difference equations and in order to extend these cases to many other so-called dynamic equations. In this paper we consider a linear dynamic equation on a time scale together with a perturbed equation. We show that, if certain exponential dichotomy conditions are satisfied, then for any solution of the perturbed equation there exists a solution of the unperturbed equation that asymptotically differs from the solution of the perturbed equation no more than the order of the perturbation term. In order to show this perturbation theorem, we use many properties of the exponential function on time scales and derive several bounds for certain monomials that appear in the dynamic version of Taylor's formula.

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1 Introduction

Previously, we [1] have investigated the asymptotic behavior of solutions of certain kinds of *dynamic equations on time scales* of the form

$$x^\Delta = A(t)x. \quad (1)$$

Under suitable conditions on $A(t)$, an $n \times n$ -matrix of functions which are at least rd-continuous on \mathbb{T} , we have shown (see [1, Theorem 4.1]) that there exist fundamental solutions of the form

$$X(t) = P(t)[I + E(t)]D(t), \quad (2)$$

where $P(t)$ is an invertible matrix of explicit, elementary functions, $D(t)$ is a diagonal matrix of functions which are solutions of scalar (one-dimensional) dynamic equations (often also expressible using elementary functions such as powers, exponentials, and logarithms), and the matrix $E(t)$ is an $n \times n$ -matrix of functions which tend to zero as $t \rightarrow \infty$, written as

$$E(t) = o(1) \quad \text{as } t \rightarrow \infty$$

using the Hardy–Littlewood, or Landau, “o” symbol. Now we are interested in obtaining more precise information about $E(t)$. For this, it is natural to assume (when comparing to known results for special cases of differential, difference, q -difference and other functional equations) that $A(t)$ has an analytic structure, i.e.,

$$A(t) = t^r \sum_{k=0}^{\infty} \frac{A_k}{t^k} \quad \text{with } r \in \mathbb{Z}$$

and with the series converging for $t > t_0$, and to consider solutions (2) with $E(t)$ replaced by an expression of the form

$$\hat{E}(t) = \sum_{k=1}^{\infty} \frac{E_k}{t^k}. \quad (3)$$

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Under certain extra assumptions on $A(t)$, it can be shown in such cases that there exist “solutions” of this type, but they are generally only *formal* expressions in the following sense: If one substitutes the series (3) into (2), then operating on the terms of the series without regard to convergence, both sides of (1) reduce to formal power series and the coefficients E_k are such that both sides are 0, as formal power series. However, in “most” cases, the terms E_k grow so quickly that (3) as a power series diverges for all t .

In the case of analytic differential equations, Poincaré defined what is now called an asymptotic expansion in the following way: If there exists an actual function $E(t)$ such that for all $N \in \mathbb{N}$ there exists a constant $M = M(N)$ satisfying for all sufficiently large $t \geq t_1$

$$\left| E(t) - \sum_{k=0}^N \frac{E_k}{t^k} \right| \leq \frac{M}{t^{N+1}}, \quad (4)$$

then one says that $\hat{E}(t)$ is an asymptotic expansion (in the sense of Poincaré) for $E(t)$ as $t \rightarrow \infty$, and this is expressed symbolically as

$$E(t) \sim \hat{E}(t) \quad \text{as } t \rightarrow \infty. \quad (5)$$

(We also write in this case (as in (2)) $X(t) \sim \hat{X}(t) = P(t)[I + \hat{E}(t)]D(t)$ if $P^{-1}(t)X(t)D^{-1}(t) \sim I + \hat{E}(t)$ as $t \rightarrow \infty$ and remark that such expressions will then be proven for each column of $X(t)$ separately.) Actually, in the context of analytic differential or difference equations (1) it is more useful and important to strengthen this definition by considering a sector S in the complex plane in which $E(t)$ is holomorphic and (5) holds in every *closed* subsector. But here we will consider such asymptotic solutions only along rays and in particular along the positive real axis.

Our main purpose in this paper is to prove such a result which generalizes some of the known so-called *asymptotic existence theorems* and to identify which extra conditions on the time scale \mathbb{T} and in particular on the *graininess* $\mu(t) = \sigma(t) - t$ we require. One could consider more general types of formal solutions by replacing, for example, t^k by $t^{k/p}$ for some $p \in \mathbb{N}$ or some more general type of basis functions which are particularly related to a special time scale \mathbb{T} , but we will not pursue such questions here.

2 The time scales calculus

For an introduction to the theory of time scales we refer to the original work by Hilger [2], to the paper by Aulbach and Hilger [3], and to the recently appeared books by Bohner and Peterson [4,5]. In this section we will give a short overview on those parts of the theory that are important for our present treatment of asymptotic behavior of dynamic equations on time scales. By a time scale we simply mean a nonempty closed subset of \mathbb{R} . In this paper we only consider time scales \mathbb{T} that contain (without loss of generality) 0 and that are unbounded above, i.e.,

$$\mathbb{T} \subset \mathbb{R} \text{ is closed, } 0 \in \mathbb{T}, \text{ and } \lim_{m \rightarrow \infty} t_m = \infty \text{ for some set } \{t_m : m \in \mathbb{N}\} \subset \mathbb{T}, \quad (6)$$

and then the (forward and backward) *jump operators*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well-defined for each $t \in \mathbb{T}$. The point $t \in \mathbb{T}$ is called *left-dense*, *left-scattered*, *right-dense*, and *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. The *graininess* $\mu : \mathbb{T} \rightarrow \mathbb{R}_0^+$ is defined by $\mu(t) = \sigma(t) - t$. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is *differentiable* at $t \in \mathbb{T}$ provided

$$f^\Delta(t) := \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \rightarrow t, \quad s \in \mathbb{T} \setminus \{\sigma(t)\}$$

exists. The function f is called differentiable on \mathbb{T} if $f^\Delta(t)$ exists for all $t \in \mathbb{T}$. We shall write f^σ for $f \circ \sigma$. Some fundamental results are as follows:

- $f^\sigma = f + \mu f^\Delta$;
- $(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g$ (“Product Rule”);
- $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(g g^\sigma)$ (“Quotient Rule”).

From the product rule it is easy to derive the formula [4, Theorem 1.24]

$$f^\Delta(t) = \sum_{\nu=0}^m (\sigma(t) - s)^\nu (t - s)^{m-\nu} \quad \text{if } f(t) = (t - s)^{m+1}. \tag{7}$$

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous*, if it has a left-sided limit in each left-dense $t \in \mathbb{T}$ and if it is continuous in each right-dense $t \in \mathbb{T}$. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ with $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$ is said to be an *antiderivative* of f on \mathbb{T} , and in this case we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

The main existence theorem for antiderivatives is as follows:

- Rd-continuous functions possess antiderivatives.

We call an $n \times n$ -matrix-valued function A *rd-continuous* if each entry is rd-continuous, and A is called *regressive* if

$$I + \mu(t)A(t) \quad \text{is invertible for all } t \in \mathbb{T}.$$

Then the following theorem is valid:

- (1) has a fundamental system of solutions if A is rd-continuous and regressive.

In the scalar case, given that $p : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous and regressive, the unique solution of $y^\Delta = p(t)y$, $y(t_0) = 1$ is called the *exponential function* and denoted by $e_p(\cdot, t_0)$. The following properties of exponential functions will be used in this paper:

- $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- $e_p(\sigma(t), s) = [1 + \mu(t)p(t)]e_p(t, s)$;
- $e_p(s, t) = e_p^{-1}(t, s)$;
- $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- $\int_a^b p(t)e_p(t, c)\Delta t = e_p(a, c) - e_p(b, c)$;
- $\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b)$.
- $e_p(a, b) \geq 0$ for all $a, b \in \mathbb{T}$ if $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$.

3 Preliminaries

From [1, Theorem 4.1] (see also [6]) we recall that if, after a suitable preliminary transformation $P(t)$ which we now suppress, $A(t)$ has the form

$$A(t) = \Lambda(t) + R(t), \tag{8}$$

where $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ is rd-continuous and regressive and satisfies the *dichotomy conditions* [1, (V₄^{*})] and R satisfies the *growth condition*

$$\int_{t_0}^\infty \left| \frac{R(t)}{1 + \mu(t)\lambda_k(t)} \right| \Delta t < \infty \quad \text{for all } 1 \leq k \leq n,$$

then (1) has solution vectors of the form

$$x_k(t) = w_k(t)y_k(t) \quad \text{for all } 1 \leq k \leq n, \tag{9}$$

where w_k is a solution of the scalar equation

$$w_k^\Delta = \lambda_k(t)w_k, \tag{10}$$

and y_k is a solution of (we put $U \ominus V = (U - V)(I + \mu V)^{-1}$ for regressive V)

$$y_k^\Delta = B_k(t)y \quad \text{with } B_k = (R + \Lambda) \ominus \lambda_k I \tag{11}$$

satisfying $y_k(t) = e_k + o(1)$ as $t \rightarrow \infty$. We will assume that $1 \leq k \leq n$ is arbitrary, but from now on fixed, and we assume also that (11) has a formal solution vector of the form

$$\hat{y}_k(t) = \sum_{\nu=0}^{\infty} \frac{f_{\nu}}{t^{\nu}}, \quad f_0 = e_k. \quad (12)$$

This means that substituting (12) into (11) and equating coefficients of like powers of t produces identities which the coefficients f_{ν} are required to satisfy. These identities often may be solved recursively to calculate the coefficients. Under certain conditions on $\Lambda(t)$, $R(t)$, $\lambda_k(t)$, and $\mu(t)$, such formal solutions can be shown to exist, but we will not impose such conditions here. (See [7] for a discussion of sufficient conditions for the special case of functional equations of the form $y(\phi(t)) = A(t)y(t)$.)

Now letting $y_N(t) = \sum_{\nu=0}^N \frac{f_{\nu}}{t^{\nu}}$, one sees that y_N satisfies the nonhomogeneous system

$$y^{\Delta} = B_k(t)y + b_N(t) \quad \text{with} \quad b_N(t) = B_k(t) \sum_{\nu=N+1}^{\infty} \frac{f_{\nu}}{t^{\nu}} - \left(\sum_{\nu=N+1}^{\infty} \frac{f_{\nu}}{t^{\nu}} \right)^{\Delta}. \quad (13)$$

From the formal identities which determine the f_{ν} , it can be shown that

$$b_N(t) = O\left(\frac{1}{t^{N+1}}\right) \quad \text{as} \quad t \rightarrow \infty,$$

and in the k th position we even have

$$(b_N(t))_k = O\left(\frac{1}{t^{N+2}}\right) \quad \text{as} \quad t \rightarrow \infty.$$

For each $j \neq k$ consider the quantities

$$\operatorname{Re}(\lambda_j \ominus \lambda_k),$$

and we now assume that there exists a constant $c > 0$ such that either

$$\operatorname{Re}(\lambda_j \ominus \lambda_k)(t) \geq c \quad \text{for all} \quad t \geq t_0 \quad (14)$$

or

$$\operatorname{Re}(\lambda_j \ominus \lambda_k)(t) \leq -c \quad \text{for all} \quad t \geq t_0. \quad (15)$$

Now let P_1 denote the diagonal matrix with j th diagonal element equal to 0 in case (14), 1 in case (15), and the k th diagonal element equal to 0. Then, with $P_2 = I - P_1$, note that P_1 and P_2 are supplementary projections and, moreover, the dichotomy conditions [1, (V_4^*)] hold for this choice of P_1 and P_2 . From (14), (15), and the definitions of P_1 and P_2 it follows that there exists $K > 0$ such that

$$|Y(t)P_1Y^{-1}(\sigma(s))| \leq Ke_c(s, t) \quad \text{for} \quad s \leq t$$

and

$$|Y(t)P_2Y^{-1}(\sigma(s))| \leq Ke_c(t, \sigma(s)) \quad \text{for} \quad s \geq t.$$

Hence (11) satisfies an exponential dichotomy condition. It remains to show that (11) has a solution y satisfying

$$|y(t) - y_N(t)| \leq \frac{M}{t^{N+1}},$$

and to do this we define

$$y(t) = y_N(t) - \int_{t_0}^t Y(t)P_1Y^{-1}(\sigma(s))b_N(s)\Delta s + \int_t^{\infty} Y(t)P_2Y^{-1}(\sigma(s))b_N(s)\Delta s,$$

which is easily shown (using the product rule) to be a solution of (11). To estimate the integrals and complete the argument, we digress and first investigate properties of scalar exponential functions, and then prove a slightly more general perturbation result to complete the analysis.

4 The perturbation result

Important for the estimation of scalar exponential functions are the following auxiliary functions $h_k : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}, k \in \mathbb{N}_0$, defined in [8] by

$$h_0(t, s) \equiv 1 \quad \text{and} \quad h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau \quad \text{for } k \in \mathbb{N}_0. \tag{16}$$

By Taylor’s formula [4, Theorem 1.113], we have for an n -times differentiable function $f : \mathbb{T} \rightarrow \mathbb{R}$

$$f(t) = \sum_{k=0}^{n-1} h_k(t, s) f^{\Delta^k}(s) + \int_s^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau \tag{17}$$

(here, $\rho^0(t) = t$ and $\rho^{k+1}(t) = \rho(\rho^k(t))$ for $k \in \mathbb{N}_0$). We begin by establishing some inequalities satisfied by the functions h_k .

Theorem 4.1 *Let $k \in \mathbb{N}_0$. Then we have*

$$0 \leq h_k(t, s) \leq \frac{(t-s)^k}{k!} \quad \text{for all } t \geq s. \tag{18}$$

Proof. Clearly (18) is true for $k = 0$. If (18) is true for $k = m \in \mathbb{N}_0$, then for $t \geq s$

$$\begin{aligned} h_{m+1}(t, s) &\stackrel{(16)}{=} \int_s^t h_m(\tau, s) \Delta\tau \\ &\leq \frac{1}{m!} \int_s^t (\tau-s)^m \Delta\tau \\ &= \frac{1}{(m+1)!} \int_s^t \sum_{\nu=0}^m (\tau-s)^m \Delta\tau \\ &= \frac{1}{(m+1)!} \int_s^t \sum_{\nu=0}^m (\tau-s)^\nu (\tau-s)^{m-\nu} \Delta\tau \\ &\leq \frac{1}{(m+1)!} \int_s^t \sum_{\nu=0}^m (\sigma(\tau)-s)^\nu (\tau-s)^{m-\nu} \Delta\tau \\ &\stackrel{(7)}{=} \frac{(t-s)^{m+1}}{(m+1)!} \end{aligned}$$

so that (18) is true for $k = m + 1$. By the principle of mathematical induction, (18) is true for all $k \in \mathbb{N}_0$. □

If we restrict the time scales under consideration, then we are able to provide a useful lower bound for the functions h_k . For this purpose, let us consider time scales that satisfy the assumption

$$\text{there exists } \alpha > 1 \text{ with } \sigma(t) \leq \alpha t \quad \text{for all } t \in \mathbb{T}. \tag{19}$$

Note that when $\mathbb{T} = \mathbb{R}$, then $h_k(t, s) = (t-s)^k/k!$, which gives a universal upper bound, but for a lower bound as given in the next result we require (19). Note also that any of the time scales $\mathbb{R}, \mathbb{Z}, h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with $h > 0, q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$ satisfies (19).

Theorem 4.2 *Assume (19). Let $k \in \mathbb{N}_0$. Then we have*

$$\frac{t^k}{k! \alpha^{k(k-1)/2}} - \left(\frac{t^k}{k!} - \frac{(t-s)^k}{k!} \right) \leq h_k(t, s) \leq \frac{(t-s)^k}{k!} \quad \text{for all } t \geq s \geq 0. \tag{20}$$

In particular, if $s = 0$, then

$$\frac{t^k}{k! \alpha^{k(k-1)/2}} \leq h_k(t, 0) \leq \frac{t^k}{k!} \quad \text{for all } t \geq 0. \tag{21}$$

Proof. We first prove (21). Clearly (21) is true for $k = 0$. Assuming (21) is true for $k = m \in \mathbb{N}_0$, we have for $t \geq 0$

$$\begin{aligned}
 h_{m+1}(t, 0) &\stackrel{(16)}{=} \int_0^t h_m(\tau, 0) \Delta\tau \\
 &\geq \frac{1}{m! \alpha^{m(m-1)/2}} \int_0^t \tau^m \Delta\tau \\
 &= \frac{1}{(m+1)! \alpha^{m(m-1)/2}} \int_0^t \sum_{\nu=0}^m \tau^m \Delta\tau \\
 &\stackrel{(19)}{\geq} \frac{1}{(m+1)! \alpha^{m(m-1)/2}} \int_0^t \sum_{\nu=0}^m \left(\frac{\sigma(\tau)}{\alpha}\right)^\nu \tau^{m-\nu} \Delta\tau \\
 &\geq \frac{1}{(m+1)! \alpha^{m(m-1)/2} \alpha^m} \int_0^t \sum_{\nu=0}^m (\sigma(\tau))^\nu \tau^{m-\nu} \Delta\tau \\
 &= \frac{1}{(m+1)! \alpha^{m(m+1)/2}} \int_0^t \sum_{\nu=0}^m (\sigma(\tau))^\nu \tau^{m-\nu} \Delta\tau \\
 &\stackrel{(7)}{=} \frac{t^{m+1}}{(m+1)! \alpha^{m(m+1)/2}}
 \end{aligned}$$

so that (21) is true for $k = m + 1$. By the principle of mathematical induction, (21) is true for all $k \in \mathbb{N}_0$.

To show (20), let $t \geq s \geq 0$. We use Taylor's formula (17) for $f(t) = h_k(t, 0)$. Note that $f^{\Delta^\nu}(t) = h_{k-\nu}(t, 0)$ for all $0 \leq \nu \leq k$ and $f^{\Delta^{k+1}}(t) \equiv 0$. Therefore an application of (17) yields

$$\begin{aligned}
 h_k(t, 0) &= f(t) = \sum_{\nu=0}^k h_\nu(t, s) f^{\Delta^\nu}(s) \\
 &= \sum_{\nu=0}^k h_\nu(t, s) h_{k-\nu}(s, 0) \\
 &= h_k(t, s) + \sum_{\nu=0}^{k-1} h_\nu(t, s) h_{k-\nu}(s, 0) \\
 &\stackrel{(18)}{\leq} h_k(t, s) + \sum_{\nu=0}^{k-1} \frac{(t-s)^\nu}{\nu!} \frac{s^{k-\nu}}{(k-\nu)!} \\
 &= h_k(t, s) + \frac{1}{k!} \sum_{\nu=0}^{k-1} \binom{k}{\nu} (t-s)^\nu s^{k-\nu} \\
 &= h_k(t, s) + \frac{t^k - (t-s)^k}{k!}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 h_k(t, s) &\geq h_k(t, 0) - \frac{t^k - (t-s)^k}{k!} \\
 &\stackrel{(21)}{\geq} \frac{t^k}{k! \alpha^{k(k-1)/2}} - \frac{t^k - (t-s)^k}{k!},
 \end{aligned}$$

which proves inequality (20). \square

Now we present the main perturbation theorem of this paper, which is somewhat more general than the one described in Section 3.

Theorem 4.3 *Assume \mathbb{T} satisfies condition (19). Let B be an rd-continuous and regressive $n \times n$ -matrix-valued function. Suppose b is a regressive n -vector-valued function satisfying*

$$b(t) = O(1/t^N) \quad \text{as } t \rightarrow \infty \quad \text{for some } N \in \mathbb{N}. \quad (22)$$

Let Y be a fundamental solution matrix for $Y^\Delta = B(t)Y$ and assume that there exist $K, c, d > 0$ and supplementary projections P_1 and P_2 satisfying the exponential dichotomy conditions

$$|Y(t)P_1Y^{-1}(\sigma(s))| \leq Ke_c(s, t) \quad \text{whenever } s \leq t \tag{23}$$

and

$$|Y(t)P_2Y^{-1}(\sigma(s))| \leq Ke_d(t, \sigma(s)) \quad \text{whenever } s \geq t. \tag{24}$$

If x is a solution of the perturbed equation

$$x^\Delta = B(t)x + b(t), \tag{25}$$

then there exists a solution y of the unperturbed equation

$$y^\Delta = B(t)y \tag{26}$$

satisfying

$$|x(t) - y(t)| = O(1/t^N) \quad \text{as } t \rightarrow \infty. \tag{27}$$

Proof. Suppose x solves (25). Let $t_0 \in \mathbb{T}$ with $t_0 > 0$. Then we define y as in Section 3 by

$$y(t) = x(t) - \int_{t_0}^t Y(t)P_1Y^{-1}(\sigma(s))b(s)\Delta s + \int_t^\infty Y(t)P_2Y^{-1}(\sigma(s))b(s)\Delta s.$$

Now we show (27). First, (22) implies that there exists $M > 0$ with

$$|b(t)| \leq \frac{M}{t^N} \quad \text{for large } t \in \mathbb{T}.$$

We conclude for $T \geq t$, by using (24) and [4, Theorem 2.39],

$$\begin{aligned} \left| \int_t^T Y(t)P_2Y^{-1}(\sigma(s))b(s)\Delta s \right| &\leq \int_t^T |Y(t)P_2Y^{-1}(\sigma(s))||b(s)|\Delta s \\ &\leq \int_t^T Ke_d(t, \sigma(s)) \frac{M}{s^N} \Delta s \\ &\leq \frac{MK}{t^N} \int_t^T e_d(t, \sigma(s))\Delta s \\ &= \frac{MK}{dt^N} \int_t^T de_d(t, \sigma(s))\Delta s \\ &= \frac{MK}{dt^N} [e_d(t, t) - e_d(t, T)] \\ &= \frac{MK}{dt^N} [1 - e_d(t, T)] \\ &\leq \frac{MK}{dt^N}. \end{aligned}$$

Thus indeed

$$\int_t^\infty Y(t)P_2Y^{-1}(\sigma(s))b(s)\Delta s = O\left(\frac{1}{t^N}\right) \quad \text{as } t \rightarrow \infty.$$

We will split the first integral into two parts as follows. Since

$$\lim_{\gamma \rightarrow 0^+} \left[\alpha^{N(1-N)/2} - 1 + (1 - \gamma)^N \right] = \alpha^{N(1-N)/2} > 0,$$

there exists $\gamma > 0$ with

$$\beta := \alpha^{N(1-N)/2} - 1 + (1 - \gamma)^N > 0.$$

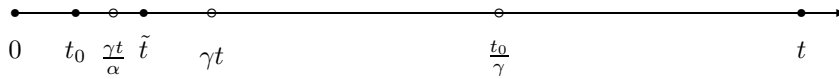


Fig. 1 Choosing \tilde{t}

Without loss of generality, let $t > t_0/\gamma$. Clearly, (19) implies that $[s, \alpha s) \cap \mathbb{T} \neq \emptyset$ for each $s \geq t_0$ (for otherwise we can take $t_1 \in [t_0, s)$ and $t_2 \geq \alpha s$ with $(t_1, s) \cap \mathbb{T} = \emptyset$, $[\alpha s, t_2) \cap \mathbb{T} = \emptyset$ and $t_1, t_2 \in \mathbb{T}$ so that $\sigma(t_1) = t_2$, but this implies the contradiction $\alpha t_1 \geq \sigma(t_1) = t_2 \geq \alpha s > \alpha t_1$). Hence we can pick (for an illustration see Fig. 1)

$$\tilde{t} \in \left[\frac{\gamma t}{\alpha}, \gamma t \right) \cap \mathbb{T} \quad (28)$$

so that we may use (23) to obtain

$$\begin{aligned} & \left| \int_{t_0}^t Y(t)P_1Y^{-1}(\sigma(s))b(s)\Delta s \right| \leq \int_{t_0}^t |Y(t)P_1Y^{-1}(\sigma(s))||b(s)|\Delta s \\ & \leq \int_{t_0}^t Ke_c(s, t) \frac{M}{s^N} \Delta s = \frac{KM}{c} \int_{t_0}^t ce_c(s, t) \frac{\Delta s}{s^N} \\ & = \frac{KM}{c} \int_{t_0}^{\tilde{t}} ce_c(s, t) \frac{\Delta s}{s^N} + \frac{KM}{c} \int_{\tilde{t}}^t ce_c(s, t) \frac{\Delta s}{s^N} \\ & \leq \frac{KM}{ct_0^N} \int_{t_0}^{\tilde{t}} ce_c(s, t) \Delta s + \frac{KM}{c\tilde{t}^N} \int_{\tilde{t}}^t ce_c(s, t) \Delta s \\ & = \frac{MK}{ct_0^N} [e_c(\tilde{t}, t) - e_c(t_0, t)] + \frac{MK}{c\tilde{t}^N} [e_c(t, t) - e_c(\tilde{t}, t)] \\ & \leq \frac{MK}{c\tilde{t}^N} + \frac{MK}{ct_0^N} [e_c(\tilde{t}, t) + e_c(t_0, t)] \\ & \stackrel{(28)}{\leq} \frac{MK\alpha^N}{c\gamma^N t^N} + \frac{MK}{ct_0^N} [e_c(\tilde{t}, t) + e_c(t_0, t)]. \end{aligned}$$

We will show that this last expression is $O(1/t^N)$ as $t \rightarrow \infty$. Note now that

$$e_c(t, 0) \stackrel{(17)}{=} \sum_{k=0}^{\infty} c^k h_k(t, 0) \geq c^N h_N(t, 0) \stackrel{(21)}{\geq} c^N \frac{t^N}{N! \alpha^{N(N-1)/2}}.$$

Hence $e_c(t_0, t) = e_c(t_0, 0)e_c(0, t) = e_c(t_0, 0)/e_c(t, 0) = O(1/t^N)$ as $t \rightarrow \infty$. Similarly, by Theorem 4.2,

$$\begin{aligned} e_c(t, \tilde{t}) & \stackrel{(17)}{=} \sum_{k=0}^{\infty} c^k h_k(t, \tilde{t}) \geq c^N h_N(t, \tilde{t}) \\ & \stackrel{(20)}{\geq} c^N \left[\frac{t^N}{N! \alpha^{N(N-1)/2}} - \frac{t^N}{N!} + \frac{(t - \tilde{t})^N}{N!} \right] \\ & \stackrel{(28)}{\geq} \frac{c^N}{N!} \left[\frac{t^N}{\alpha^{N(N-1)/2}} - t^N + (t - \gamma t)^N \right] \\ & = \frac{\beta c^N t^N}{N!} \end{aligned}$$

so that $e_c(\beta t, t) = O(1/t^N)$ as $t \rightarrow \infty$, too, and the proof is complete. \square

Remark 4.4 Since $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}_0}$ all satisfy (19), this result generalizes classically known results for asymptotic expansions of formal solutions of analytic differential, difference, and q -difference equations along a ray which is not a so-called ‘‘Stokes direction’’ for the equation. For time scales $\mathbb{N}!$, $\mathbb{N}^{\mathbb{N}}$, and $\mathbb{N}^{\mathbb{N}^2}$ which do not satisfy (19), it is not

known whether corresponding results hold or not. Certainly the exponential functions on such time scales grow much more slowly and this might present problems. Also in the classically-studied cases of \mathbb{R} , \mathbb{Z} , and $q^{\mathbb{N}_0}$, formal solutions are usually expressed using power series $\sum_{k=0}^{\infty} y_k/t^k$. It could be that other basis functions adapted to the time scale would be more appropriate in case (19) is not satisfied. The investigation of these questions are left as open problems.

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