Submitted to J. Difference Equ. Appl. (July 20, 2005).
Report sent on Sep 14, 2005. Revised version submitted on Sep 15, 2005

AN INTRODUCTION TO COMPLEX FUNCTIONS ON PRODUCTS OF TWO TIME SCALES

MARTIN BOHNER AND GUSEIN SH. GUSEINOV

University of Missouri–Rolla, Department of Mathematics and Statistics, Rolla, Missouri 65401, USA. *E-mail:* bohner@umr.edu

Atilim University, Department of Mathematics, 06836 Incek, Ankara, Turkey. *E-mail:* guseinov@atilim.edu.tr

ABSTRACT. In this paper we study the concept of analyticity for complex-valued functions of a complex time scale variable, derive a time scale counterpart of the classical Cauchy–Riemann equations, introduce complex line delta and nabla integrals along time scales curves, and obtain a time scale version of the classical Cauchy integral theorem.

Keywords. Time scales, delta analytic functions, time scale curves, complex line delta and nabla integrals, Cauchy's integral theorem.

AMS (MOS) Subject Classification. 30G25, 30G30, 39A10.

DEDICATION

This paper is dedicated to the memory of Professor Bernd Aulbach.

1. INTRODUCTION

Discrete analytic (or holomorphic) functions being analytic functions on the Gaussian integers $\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$ were introduced by Isaacs [13]. He introduced two kinds of difference equations, both of which are discrete counterparts of the Cauchy–Riemann equations in one complex variable. He thus defined two classes of discrete analytic functions and called them monodiffric functions of the first and second kind, respectively. In [14], he continued the study of monodiffric functions of the first kind. In [11], Ferrand investigated monodiffric functions of the second kind, which she called "preholomorphic". Later, this class was studied quite extensively by Duffin [9], Duffin and Peterson [10], Zeilberger [16], Zeilberger and Dym [17], and others.

In this paper we introduce a concept of analytic functions on an arbitrary time scale complex plane $\mathbb{T}_1 + i\mathbb{T}_2$, where $\mathbb{T}_1$ and $\mathbb{T}_2$ are arbitrary time scales. Note that a

MB acknowledges financial support by a University of Missouri Research Board grant.

time scale is an arbitrary nonempty closed subset of the reals $\mathbb{R}$, which in particular may be the reals $\mathbb{R}$ itself as well as the integers $\mathbb{Z}$. Therefore we unify and extend the concepts of continuous and discrete analytic functions. For a general introduction to the calculus of time scales, we refer the reader to the original works of Aulbach and Hilger [1, 2, 3, 12] and the textbooks by Bohner and Peterson [7, 8]. Bernd Aulbach, to whom this paper is dedicated, can be considered, together with Stefan Hilger, as the founder of time scales calculus. From the beginning until his sudden and unexpected death on January 14, 2005, Bernd Aulbach supported and followed with close interest the activities of time scales research.

The paper is organized as follows. In Section 2, we follow [4] and present a definition of completely delta differentiability for functions of two real time scale variables, needed in the sequel. In Section 3, we introduce a concept of delta differentiability (or delta analyticity) for complex-valued functions of a complex time scale variable and derive a time scale version of the classical Cauchy–Riemann equations for usual analytic functions of a (continuous) complex variable. We show that our definition of delta analyticity coincides in the case $\mathbb{R}+i\mathbb{R}=\mathbb{C}$ with the usual analyticity and in the case of the Gaussian integers $\mathbb{Z}+i\mathbb{Z}=\mathbb{Z}[i]$ with the definition of monodiffric functions of the first kind given by Isaacs. Section 4 treats curves in the time scale complex plane and offers an integral formula for computing their lengths. In Section 5, we define complex line delta and nabla integrals and give sufficient conditions for the existence of these integrals and also provide formulas for their evaluation. Next we present several properties of complex line delta integrals. Finally, in Section 6, we establish for complex delta analytic functions a version of the classical Cauchy integral theorem. To do so, the concepts of connectedness, domain, and fence of a set are introduced for sets in the time scale complex plane.

2. FUNCTIONS OF TWO REAL TIME SCALE VARIABLES

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. Let us set $\mathbb{T}_1\times\mathbb{T}_2=\{(x,y):\ x\in\mathbb{T}_1,\ y\in\mathbb{T}_2\}$. The set $\mathbb{T}_1\times\mathbb{T}_2$ is a complete metric space with the metric (distance) d defined by

$$d\left((x,y),(x',y')\right)=\sqrt{(x-x')^2+(y-y')^2}\quad\text{for}\quad (x,y),(x',y')\in\mathbb{T}_1\times\mathbb{T}_2.$$

For a given $\delta>0$, the δ-neighborhood $U_\delta(x_0,y_0)$ of a given point $(x_0,y_0)\in\mathbb{T}_1\times\mathbb{T}_2$ is the set of all points $(x,y)\in\mathbb{T}_1\times\mathbb{T}_2$ such that $d((x_0,y_0),(x,y))<\delta$. Let σ_1 and σ_2 be the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Let $u:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ be a function. The first order partial delta derivatives of u at a point $(x_0,y_0)\in\mathbb{T}_1^\kappa\times\mathbb{T}_2^\kappa$ are defined to be

$$\frac{\partial u(x_0,y_0)}{\Delta_1 x}=\lim_{x\to x_0,x\neq\sigma_1(x_0)}\frac{u(\sigma_1(x_0),y_0)-u(x,y_0)}{\sigma_1(x_0)-x}$$

and

$$\frac{\partial u(x_0, y_0)}{\Delta_2 y} = \lim_{y \to y_0, y \neq \sigma_2(y_0)} \frac{u(x_0, \sigma_2(y_0)) - u(x_0, y)}{\sigma_2(y_0) - y}.$$

Definition 2.1 (see [4])**.** We say that a function $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is *completely delta differentiable* at a point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ if there exist numbers A_1 and A_2 independent of $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ (but, in general, dependent on (x_0, y_0)) such that

$$u(x_0, y_0) - u(x, y) = A_1(x_0 - x) + A_2(y_0 - y) + \alpha_1(x_0 - x) + \alpha_2(y_0 - y), \tag{2.1}$$

$$\begin{aligned} u(\sigma_1(x_0), y_0) - u(x, y) &= A_1[\sigma_1(x_0) - x] + A_2(y_0 - y) \\ &\quad + \beta_{11}[\sigma_1(x_0) - x] + \beta_{12}(y_0 - y), \end{aligned} \tag{2.2}$$

$$\begin{aligned} u(x_0, \sigma_2(y_0)) - u(x, y) &= A_1(x_0 - x) + A_2[\sigma_2(y_0) - y] \\ &\quad + \beta_{21}(x_0 - x) + \beta_{22}[\sigma_2(y_0) - y] \end{aligned} \tag{2.3}$$

for all $(x, y) \in U_\delta(x_0, y_0)$, where $\delta > 0$ is sufficiently small, $\alpha_j = \alpha_j(x_0, y_0; x, y)$ and $\beta_{jk} = \beta_{jk}(x_0, y_0; x, y)$ are defined on $U_\delta(x_0, y_0)$ such that they are equal to zero at $(x, y) = (x_0, y_0)$ and

$$\lim_{(x,y)\to(x_0,y_0)} \alpha_j(x_0, y_0; x, y) = \lim_{(x,y)\to(x_0,y_0)} \beta_{jk}(x_0, y_0; x, y) = 0$$

for $j, k \in \{1, 2\}$.

Note that in case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the neighborhood $U_\delta(x_0, y_0)$ contains the single point (x_0, y_0) for $\delta < 1$. Therefore, in this case, the condition (2.1) disappears, while the conditions (2.2) and (2.3) hold with $\beta_{jk} = 0$ and with

$$A_1 = u(x_0 + 1, y_0) - u(x_0, y_0) = \frac{\partial u(x_0, y_0)}{\Delta_1 x}$$

and

$$A_2 = u(x_0, y_0 + 1) - u(x_0, y_0) = \frac{\partial u(x_0, y_0)}{\Delta_2 y}.$$

This shows that each function $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is completely delta differentiable at every point.

It follows from Definition 2.1 that if the function $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is completely delta differentiable at the point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$, then it is continuous at that point and has at (x_0, y_0) the first order partial delta derivatives equal to A_1 and A_2, namely

$$\frac{\partial u(x_0, y_0)}{\Delta_1 x} = A_1 \quad \text{and} \quad \frac{\partial u(x_0, y_0)}{\Delta_2 y} = A_2.$$

Remark 2.2. In general, the product of two completely delta differentiable functions need not be completely delta differentiable.

3. FUNCTIONS OF A COMPLEX TIME SCALE VARIABLE. CAUCHY–RIEMANN EQUATIONS

For given time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, let us set

$$\mathbb{T}_1 + i\mathbb{T}_2 = \{z = x + iy : \ x \in \mathbb{T}_1, \ y \in \mathbb{T}_2\}, \tag{3.1}$$

where $i = \sqrt{-1}$ is the imaginary unit. The set $\mathbb{T}_1 + i\mathbb{T}_2$ is called the *time scale complex plane* and is a complete metric space with the metric d defined by

$$d(z, z') = |z-z'| = \sqrt{(x - x')^2 + (y - y')^2}, \quad z = x+iy, z' = x'+iy' \in \mathbb{T}_1+i\mathbb{T}_2. \tag{3.2}$$

Any function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ can be represented in the form

$$f(z) = u(x, y) + iv(x, y) \quad \text{for} \quad z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2,$$

where $u : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is the real part of f and $v : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is the imaginary part of f.

Let σ_1 and σ_2 be the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. For $z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2$, let us set

$$z^{\sigma_1} = \sigma_1(x) + iy \quad \text{and} \quad z^{\sigma_2} = x + i\sigma_2(y).$$

Definition 3.1. We say that a complex-valued function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ is *delta differentiable* (or *delta analytic*) at a point $z_0 = x_0 + iy_0 \in \mathbb{T}_1^\kappa + i\mathbb{T}_2^\kappa$ if there exists a complex number A (depending in general on z_0) such that

$$f(z_0) - f(z) = A(z_0 - z) + \alpha(z_0 - z) \tag{3.3}$$

$$f(z_0^{\sigma_1}) - f(z) = A(z_0^{\sigma_1} - z) + \beta(z_0^{\sigma_1} - z) \tag{3.4}$$

$$f(z_0^{\sigma_2}) - f(z) = A(z_0^{\sigma_2} - z) + \gamma(z_0^{\sigma_2} - z) \tag{3.5}$$

for all $z \in U_\delta(z_0)$, where $U_\delta(z_0)$ is a δ-neighborhood of z_0 in $\mathbb{T}_1 + i\mathbb{T}_2$, $\alpha = \alpha(z_0, z)$, $\beta = \beta(z_0, z)$ and $\gamma = \gamma(z_0, z)$ are defined for $z \in U_\delta(z_0)$, they are equal to zero at $z = z_0$, and

$$\lim_{z\to z_0} \alpha(z_0, z) = \lim_{z\to z_0} \beta(z_0, z) = \lim_{z\to z_0} \gamma(z_0, z) = 0.$$

Then the number A is called the *delta derivative* (or *Δ-derivative*) of f at z_0 and is denoted by $f^\Delta(z_0)$.

Theorem 3.2. *Let the function $f : \mathbb{T}_1 + i\mathbb{T}_2 \to \mathbb{C}$ have the form*

$$f(z) = u(x, y) + iv(x, y) \quad \textit{for} \quad z = x + iy \in \mathbb{T}_1 + i\mathbb{T}_2.$$

Then a necessary and sufficient condition for f to be Δ-differentiable (as a function of the complex variable z) at the point $z_0 = x_0 + iy_0 \in \mathbb{T}_1^\kappa + i\mathbb{T}_2^\kappa$ is that the functions

u and v be completely Δ-differentiable (as functions of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) at the point (x_0, y_0) and satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\Delta_1 x} = \frac{\partial v}{\Delta_2 y} \quad \text{and} \quad \frac{\partial u}{\Delta_2 y} = -\frac{\partial v}{\Delta_1 x} \tag{3.6}$$

at (x_0, y_0). If these equations are satisfied, then $f^\Delta(z_0)$ can be represented in any of the forms

$$f^\Delta(z_0) = \frac{\partial u}{\Delta_1 x} + i\frac{\partial v}{\Delta_1 x} = \frac{\partial v}{\Delta_2 y} - i\frac{\partial u}{\Delta_2 y} = \frac{\partial u}{\Delta_1 x} - i\frac{\partial u}{\Delta_2 y} = \frac{\partial v}{\Delta_2 y} + i\frac{\partial v}{\Delta_1 x}, \tag{3.7}$$

where the partial derivatives are evaluated at (x_0, y_0).

Proof. First we show necessity. Assume that f is Δ-differentiable at $z_0 = x_0 + iy_0$ with $f^\Delta(z_0) = A$. Then (3.3)–(3.5) are satisfied. Letting

$$f = u + iv, \quad A = A_1 + iA_2, \quad \alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2, \quad \gamma = \gamma_1 + i\gamma_2,$$

we get from (3.3)–(3.5), equating the real and imaginary parts of both sides in each of these equations,

$$\begin{cases} u(x_0, y_0) - u(x, y) = A_1(x_0 - x) - A_2(y_0 - y) + \alpha_1(x_0 - x) - \alpha_2(y_0 - y) \\ u(\sigma_1(x_0), y_0) - u(x, y) = A_1[\sigma_1(x_0) - x] - A_2(y_0 - y) \\ \qquad\qquad +\beta_1[\sigma_1(x_0) - x] - \beta_2(y_0 - y) \\ u(x_0, \sigma_2(y_0)) - u(x, y) = A_1(x_0 - x) - A_2[\sigma_2(y_0) - y] \\ \qquad\qquad +\gamma_1(x_0 - x) - \gamma_2[\sigma_2(y_0) - y] \end{cases}$$

and

$$\begin{cases} v(x_0, y_0) - v(x, y) = A_2(x_0 - x) + A_1(y_0 - y) + \alpha_2(x_0 - x) + \alpha_1(y_0 - y) \\ v(\sigma_1(x_0), y_0) - v(x, y) = A_2[\sigma_1(x_0) - x] + A_1(y_0 - y) \\ \qquad\qquad +\beta_2[\sigma_1(x_0) - x] + \beta_1(y_0 - y) \\ v(x_0, \sigma_2(y_0)) - v(x, y) = A_2(x_0 - x) + A_1[\sigma_2(y_0) - y] \\ \qquad\qquad +\gamma_2(x_0 - x) + \gamma_1[\sigma_2(y_0) - y]. \end{cases}$$

Hence, taking into account that $\alpha_j \to 0$, $\beta_j \to 0$, and $\gamma_j \to 0$ as $(x, y) \to (x_0, y_0)$, we get that the functions u and v are completely Δ-differentiable (as functions of the two real variables $x \in \mathbb{T}_1$ and $y \in \mathbb{T}_2$) and that

$$A_1 = \frac{\partial u(x_0, y_0)}{\Delta_1 x}, \quad -A_2 = \frac{\partial u(x_0, y_0)}{\Delta_2 y}, \quad A_2 = \frac{\partial v(x_0, y_0)}{\Delta_1 x}, \quad A_1 = \frac{\partial v(x_0, y_0)}{\Delta_2 y}.$$

Therefore the Cauchy–Riemann equations (3.6) hold and we have the formulas (3.7).

Now we show sufficiency. Assume that the functions u and v, where $f = u + iv$, are completely Δ-differentiable at the point (x_0, y_0) and that the Cauchy–Riemann equations (3.6) hold. Then we have

$$\begin{cases} u(x_0,y_0)-u(x,y)=A_1'(x_0-x)+A_2'(y_0-y)+\alpha_1'(x_0-x)+\alpha_2'(y_0-y) \\ u(\sigma_1(x_0),y_0)-u(x,y)=A_1'[\sigma_1(x_0)-x]+A_2'(y_0-y) \\ \qquad\qquad +\beta_{11}'[\sigma_1(x_0)-x]+\beta_{12}'(y_0-y) \\ u(x_0,\sigma_2(y_0))-u(x,y)=A_1'(x_0-x)+A_2'[\sigma_2(y_0)-y] \\ \qquad\qquad +\beta_{21}'(x_0-x)+\beta_{22}'[\sigma_2(y_0)-y] \end{cases}$$

and

$$\begin{cases} v(x_0,y_0)-v(x,y)=A_1''(x_0-x)+A_2''(y_0-y)+\alpha_1''(x_0-x)+\alpha_2''(y_0-y) \\ v(\sigma_1(x_0),y_0)-v(x,y)=A_1''[\sigma_1(x_0)-x]+A_2''(y_0-y) \\ \qquad\qquad +\beta_{11}''[\sigma_1(x_0)-x]+\beta_{12}''(y_0-y) \\ v(x_0,\sigma_2(y_0))-v(x,y)=A_1''(x_0-x)+A_2''[\sigma_2(y_0)-y] \\ \qquad\qquad +\beta_{21}''(x_0-x)+\beta_{22}''[\sigma_2(y_0)-y], \end{cases}$$

where α_j', β_{ij}' and α_j'', β_{ij}'' tend to zero as $(x,y)\to(x_0,y_0)$ and

$$A_1'=\frac{\partial u(x_0,y_0)}{\Delta_1 x}=\frac{\partial v(x_0,y_0)}{\Delta_2 y}=A_2''=:A_1$$

and

$$-A_2'=-\frac{\partial u(x_0,y_0)}{\Delta_2 y}=\frac{\partial v(x_0,y_0)}{\Delta_1 x}=A_1''=:A_2.$$

Therefore

$$\begin{aligned} f(z_0)-f(z)&=(A_1+iA_2)(z_0-z)+\alpha(z_0-z), \\ f(z_0^{\sigma_1})-f(z)&=(A_1+iA_2)(z_0^{\sigma_1}-z)+\beta(z_0^{\sigma_1}-z), \\ f(z_0^{\sigma_2})-f(z)&=(A_1+iA_2)(z_0^{\sigma_2}-z)+\gamma(z_0^{\sigma_2}-z), \end{aligned}$$

where

$$\begin{aligned} \alpha&=(\alpha_1'+i\alpha_1'')\frac{x_0-x}{z_0-z}+(\alpha_2'+i\alpha_2'')\frac{y_0-y}{z_0-z}, \\ \beta&=(\beta_{11}'+i\beta_{11}'')\frac{\sigma_1(x_0)-x}{z_0^{\sigma_1}-z}+(\beta_{12}'+i\beta_{12}'')\frac{y_0-y}{z_0^{\sigma_1}-z}, \\ \gamma&=(\beta_{21}'+i\beta_{21}'')\frac{x_0-x}{z_0^{\sigma_2}-z}+(\beta_{22}'+i\beta_{22}'')\frac{\sigma_2(y_0)-y}{z_0^{\sigma_2}-z}. \end{aligned}$$

Since

$$\begin{aligned} |\alpha| &\le |\alpha_1'+i\alpha_1''|\left|\frac{x_0-x}{z_0-z}\right|+|\alpha_2'+i\alpha_2''|\left|\frac{y_0-y}{z_0-z}\right| \\ &\le |\alpha_1'+i\alpha_1''|+|\alpha_2'+i\alpha_2''| \le |\alpha_1'|+|\alpha_1''|+|\alpha_2'|+|\alpha_2''|, \end{aligned}$$

we have $\alpha\to 0$ as $z\to z_0$. Similarly, $\beta\to 0$ and $\gamma\to 0$ as $z\to z_0$. Consequently, f is Δ-differentiable at z_0 and $f^{\Delta}(z_0)=A_1+iA_2$. ☐

Remark 3.3. It can be shown (see [4]) that, if the functions $u, v : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ are continuous and have the first order partial delta derivatives $\frac{\partial u(x,y)}{\Delta_1 x}$, $\frac{\partial u(x,y)}{\Delta_2 y}$, $\frac{\partial v(x,y)}{\Delta_1 x}$, $\frac{\partial v(x,y)}{\Delta_2 y}$ in some δ-neighborhood $U_\delta(x_0, y_0)$ of the point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ and if these derivatives are continuous at (x_0, y_0), then u and v are completely Δ-differentiable at (x_0, y_0). Therefore in this case, if in addition the Cauchy–Riemann equations (3.6) are satisfied, then $f(z) = u(x, y) + iv(x, y)$ is Δ-differentiable at $z_0 = x_0 + iy_0$.

Example 3.4. (i) The function $f(z)$ =constant on $\mathbb{T}_1 + i\mathbb{T}_2$ is Δ-analytic everywhere and $f^\Delta(z) = 0$.

(ii) The function $f(z) = z$ on $\mathbb{T}_1 + i\mathbb{T}_2$ is Δ-analytic everywhere and $f^\Delta(z) = 1$.

(iii) Consider the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy \quad \text{on} \quad \mathbb{T}_1 + i\mathbb{T}_2.$$

Hence $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$, and

$$\frac{\partial u(x,y)}{\Delta_1 x} = x + \sigma_1(x), \quad \frac{\partial u(x,y)}{\Delta_2 y} = -y - \sigma_2(y), \quad \frac{\partial v(x,y)}{\Delta_1 x} = 2y, \quad \frac{\partial v(x,y)}{\Delta_2 y} = 2x.$$

Therefore the Cauchy–Riemann equations become

$$x + \sigma_1(x) = 2x \quad \text{and} \quad -y - \sigma_2(y) = -2y,$$

which hold simultaneously if and only if $\sigma_1(x) = x$ and $\sigma_2(y) = y$ simultaneously. It follows that the function $f(z) = z^2$ is not Δ-analytic at each point of $\mathbb{Z} + i\mathbb{Z}$. So, the product of two Δ-analytic functions need not be Δ-analytic.

(iv) The function $f(z) = x^2 - y^2 + i2(x + 1/2)(y + 1/2)$ is Δ-analytic everywhere on $\mathbb{Z} + i\mathbb{Z}$. This function is not analytic anywhere on $\mathbb{R} + i\mathbb{R} = \mathbb{C}$.

(v) The function $f(z) = x^2 - y^2 + ix(2y + 1)$ is Δ-analytic everywhere on $\mathbb{R} + i\mathbb{Z}$.

Example 3.5. (i) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{R} + i\mathbb{R} = \mathbb{C}$ is the usual complex plane and the three conditions (3.3)–(3.5) of Definition 3.1 coincide and reduce to the classical definition of analyticity (differentiability) of functions of a complex variable [15].

(ii) Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$. Then $\mathbb{T}_1 + i\mathbb{T}_2 = \mathbb{Z} + i\mathbb{Z} = \mathbb{Z}[i]$ is the set of Gaussian integers. The neighborhood $U_\delta(z_0)$ of z_0 contains the single point z_0 for $\delta < 1$. Therefore, in this case, the condition (3.3) disappears, while the conditions (3.4) and (3.5) reduce to the single condition

$$\frac{f(z_0 + 1) - f(z_0)}{1} = \frac{f(z_0 + i) - f(z_0)}{i} \tag{3.8}$$

with $f^\Delta(z_0)$ equal to the left (and hence also to the right) hand side of (3.8). The condition (3.8) coincides with the definition of monodiffric functions of the first kind introduced earlier by Isaacs [13]. Note that in [13] Isaacs has defined also monodiffric functions of the second kind, in which the condition

$$\frac{f(z_0+1+i)-f(z_0)}{1+i}=\frac{f(z_0+i)-f(z_0+1)}{i-1}$$

is required instead of (3.8). A time scales counterpart of monodiffric functions of the second kind will be considered by the authors elsewhere.

4. CURVES IN THE TIME SCALE COMPLEX PLANE

Let $\mathbb{T}$ be a time scale with the forward jump, backward jump, and delta differentiation operators σ, ρ, and Δ, respectively. Given the points $a, b \in \mathbb{T}$ with $a \le b$, let $[a, b] = \{t \in \mathbb{T} : a \le t \le b\}$ be the closed interval in $\mathbb{T}$. Further, let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two other time scales and define $\mathbb{T}_1 + i\mathbb{T}_2$ as in (3.1). Let σ_i, ρ_i, and Δ_i be the forward jump, backward jump, and delta differentiation operators for $\mathbb{T}_i$, $i \in \{1, 2\}$, respectively.

Definition 4.1. A complex function

$$z=\lambda(t)=\varphi(t)+i\psi(t), \quad t\in[a,b]\subset\mathbb{T}, \tag{4.1}$$

where $\varphi : [a, b] \to \mathbb{T}_1$ and $\psi : [a, b] \to \mathbb{T}_2$ are continuous (in the time scale topology) functions, is said to define a (*time scale continuous*) *curve* Γ in the time scale complex plane $\mathbb{T}_1 + i\mathbb{T}_2$.

The values of the function λ are called the *points* of the curve, and the set of points of the curve, i.e., the range of λ, will often be referred to as simply the *curve* (when no ambiguity can arise). In particular, the points $z_0 = \lambda(a)$ and $z_1 = \lambda(b)$ are called the *initial* and *final points* of the curve, respectively, and z_0, z_1 are called the *end points* of the curve. The initial and final points of a curve may coincide, in which case the curve is said to be *closed*. The time scale variable t is called the *parameter* of the curve, and the equation (4.1), mapping the values of the parameter onto the points of the curve, is called the *parametric equation* of the curve. We can also think of Γ as an *oriented curve*, in the sense that a point $z' = \lambda(t') \in \Gamma$ is regarded as *distinct* from a point $z'' = \lambda(t'') \in \Gamma$ if $t' \ne t''$ and as *preceding* z'' if $t' < t''$. The oriented curve Γ is then said to be "traversed in the direction of increasing t". It will always be clear from the context whether Γ is a curve in the set-theoretic sense, i.e., the continuous image of a closed time scale interval, or an oriented curve as just described. Two curves Γ_1 and Γ_2 with equations

$$x=\varphi_1(t),\ y=\psi_1(t), \quad t\in[a,b]\subset\mathbb{T}$$

and

$$x=\varphi_2(\tau),\ y=\psi_2(\tau), \quad \tau\in[\alpha,\beta]\subset\tilde{\mathbb{T}},$$

respectively, are regarded as *identical* if the equation of one curve can be transformed into the equation of the other by means of a continuous (strictly) increasing change of parameter, i.e., if there is a continuous increasing function $\tau = h(t)$, $t \in [a,b]$, with the range $[\alpha,\beta]$, such that

$$\varphi_2(h(t)) = \varphi_1(t),\ \psi_2(h(t)) = \psi_1(t), \quad t \in [a,b].$$

(Then, of course, $\varphi_1(h^{-1}(\tau)) = \varphi_2(\tau)$ and $\psi_1(h^{-1}(\tau)) = \psi_2(\tau)$ for $\tau \in [\alpha,\beta]$, where h^{-1} is the continuous inverse of the function h.) Then we say that the two curves have the *same* direction. We say that the two curves have *opposite* directions if the function h above is decreasing. In this case, the initial point of Γ_1 is the same as the final point of Γ_2, and vice versa. The curve differing from Γ only by the direction in which it is traversed will be denoted by $-\Gamma$. Let the curve (4.1) be nonclosed, i.e., $\lambda(a) \neq \lambda(b)$. If the same point z corresponds to more than one parameter value in the (time scale) interval $[a,b]$, then we say that z is a *multiple point* (or *self-intersection point*) of the curve (4.1). A nonclosed curve with no multiple points is called a *simple curve* (or *Jordan curve*). If the curve (4.1) is closed, then it is called a *simple closed curve* if it has no multiple points on the half-open interval $[a,b)$.

Let Γ be a (time scale) continuous curve with equation (4.1). A partition of $[a,b]$ is any finite ordered set

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b], \quad \text{where} \quad a = t_0 < t_1 < \cdots < t_n = b. \tag{4.2}$$

Let us set $z_k = \lambda(t_k)$ for $k \in \{0,1,\ldots,n\}$ and

$$\ell(\Gamma,P) = \sum_{k=1}^{n} |z_k - z_{k-1}| = \sqrt{[\varphi(t_k) - \varphi(t_{k-1})]^2 + [\psi(t_k) - \psi(t_{k-1})]^2}. \tag{4.3}$$

Definition 4.2. The curve Γ is said to be *rectifiable with length* $\ell(\Gamma)$ if

$$\sup\{\ell(\Gamma,P):\ P \text{ is a partition of } [a,b]\} =: \ell(\Gamma) < \infty,$$

where the least upper bound is taken over all possible partitions (4.2). If the supremum does not exist, the curve is said to be *nonrectifiable.* In this case, Γ is considered to have no length at all (or, if preferred, infinite length).

The following theorem gives a sufficient condition for rectifiability of curves and a formula for evaluating their lengths.

Theorem 4.3. *Let the functions* φ *and* ψ *be continuous on* $[a,b]$ *and* Δ*-differentiable on* $[a,b)$*. If their* Δ*-derivatives* φ^Δ *and* ψ^Δ *are bounded and* Δ*-integrable over* $[a,b)$*, then the curve* Γ *defined by the parametric equation* (4.1) *is rectifiable and its length* $\ell(\Gamma)$ *can be evaluated by the formula*

$$\ell(\Gamma) = \int_a^b \left|\lambda^\Delta(t)\right| \Delta t = \int_a^b \sqrt{[\varphi^\Delta(t)]^2 + [\psi^\Delta(t)]^2}\,\Delta t.$$

For a proof of Theorem 4.3 see [6].

5. COMPLEX LINE DELTA AND NABLA INTEGRALS

Let Γ be a Δ-smooth curve (i.e., Γ satisfies the conditions of Theorem 4.3) with equation (4.1) and let

$$f(z) = u(x, y) + iv(x, y)$$

be a complex function, defined and continuous on Γ (this means that for each $z \in \Gamma$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z') - f(z)| < \varepsilon$ whenever $z' \in \Gamma$ and $|z' - z| < \delta$). Let P as in (4.2) be a partition of $[a, b]$ and let

$$z_k = \lambda(t_k) = \varphi(t_k) + i\psi(t_k), \quad k \in \{0, 1, \dots, n\}.$$

Take any $\tau_k \in [t_{k-1}, t_k)$ for $k \in \{1, 2, \dots, n\}$ and put

$$\xi_k = \lambda(\tau_k) = \varphi(\tau_k) + i\psi(t_k), \quad k \in \{1, 2, \dots, n\}.$$

Next, we introduce the integral sum

$$S = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}). \tag{5.1}$$

For every $\delta > 0$ there exists (see [8, Lemma 5.7] at least one partition P as in (4.2) such that for each $k \in \{1, 2, \dots, n\}$ either $t_k - t_{k-1} \le \delta$ or $t_k - t_{k-1} > \delta$ and $\sigma(t_{k-1}) = t_k$, where σ denotes the forward jump operator in $\mathbb{T}$. We denote by $\mathcal{P}_\delta([a, b])$ the set of all such partitions P of $[a, b]$.

Definition 5.1. We say that a complex number I is the *line delta integral* of the function f along the curve Γ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every integral sum S of f corresponding to a partition $P \in \mathcal{P}_\delta([a, b])$ independent of the way in which we choose $\tau_k \in [t_{k-1}, t_k)$ for $k \in \{1, 2, \dots, n\}$. We denote the number I, symbolically, by $\int_\Gamma f(z)\Delta z$ and write $\lim_{\delta \to 0} S = I$.

The following theorem gives conditions sufficient for the existence of complex line delta integrals.

Theorem 5.2. *Suppose that the curve Γ is given by the parametric equation (4.1), where φ and ψ are continuous on $[a, b]$ and Δ-differentiable on $[a, b)$. If φ^Δ and ψ^Δ are bounded and Δ-integrable over $[a, b)$ and if the function f is continuous on Γ, then the line delta integral of f along the curve Γ exists and can be computed by the formula*

$$\int_\Gamma f(z)\Delta z = \int_a^b f(\lambda(t))\, \lambda^\Delta(t)\Delta t. \tag{5.2}$$

Proof. We have

$$S = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} [u(\varphi(\tau_k), \psi(\tau_k)) + iv(\varphi(\tau_k), \psi(\tau_k))] [\varphi(t_k) + i\psi(t_k) - \varphi(t_{k-1}) - i\psi(t_{k-1})] \\
&= \sum_{k=1}^{n} \{u(\varphi(\tau_k), \psi(\tau_k)) [\varphi(t_k) - \varphi(t_{k-1})] - v(\varphi(\tau_k), \psi(\tau_k)) [\psi(t_k) - \psi(t_{k-1})]\} \\
&\quad + i \sum_{k=1}^{n} \{v(\varphi(\tau_k), \psi(\tau_k)) [\varphi(t_k) - \varphi(t_{k-1})] + u(\varphi(\tau_k), \psi(\tau_k)) [\psi(t_k) - \psi(t_{k-1})]\} .
\end{aligned}
$$

Therefore we see that the real and imaginary parts of S are both integral sums for real functions of the variables x and y, introduced in [6] for defining line delta integrals of the form

$$\int_\Gamma M(x,y)\Delta_1 x + N(x,y)\Delta_2 y,$$

where the first sum is constructed for the pair of functions u and $-v$ and for a given partition P of $[a,b]$ and the second sum for the pair v and u for the same partition. Under the conditions of the theorem on Γ and f, it follows from the above calculation by the sufficient conditions given in [6] for the existence of real line delta integrals that the line delta integral of f along Γ exists and satisfies

$$
\begin{aligned}
\int_\Gamma f(z)\Delta z &= \int_\Gamma u(x,y)\Delta_1 x - v(x,y)\Delta_2 y + i \int_\Gamma v(x,y)\Delta_1 x + u(x,y)\Delta_2 y \\
&= \int_a^b \left[u(\varphi(t), \psi(t))\varphi^\Delta(t) - v(\varphi(t), \psi(t))\psi^\Delta(t)\right] \Delta t \\
&\quad + i \int_a^b \left[v(\varphi(t), \psi(t))\varphi^\Delta(t) + u(\varphi(t), \psi(t))\psi^\Delta(t)\right] \Delta t \\
&= \int_a^b [u(\varphi(t), \psi(t)) + iv(\varphi(t), \psi(t))] \left[\varphi^\Delta(t) + i\psi^\Delta(t)\right] \Delta t \\
&= \int_a^b f(\lambda(t))\, \lambda^\Delta(t)\Delta t,
\end{aligned}
$$

i.e., (5.2) holds. □

Example 5.3. If $f(z) = 1$, then we have

$$\sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} (z_k - z_{k-1}) = z_n - z_0 = z' - z_0,$$

where z_0 is the initial and z' is the final point of the curve Γ. Thus $\int_\Gamma \Delta z = z' - z_0$. In particular, if Γ is closed, then $z' = z_0$ and $\int_\Gamma \Delta z = 0$.

Similarly to complex line delta integrals introduced above, we can define *complex line nabla integrals*. Let Γ be a curve defined by the parametric equation (4.1) and let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined on Γ. Having any partition P of $[a,b]$ of the form (4.2), put $z_k = \lambda(t_k)$ for $k \in \{0, 1, \ldots, n\}$ and choosing any $\tau_k \in (t_{k-1}, t_k]$, put $\xi_k = \lambda(\tau_k)$ for $k \in \{1, 2, \ldots, n\}$. Note that in contrast to the delta integral, now we choose the intermediate point τ_k in $(t_{k-1}, t_k]$ rather than in $[t_{k-1}, t_k)$. Then we form the integral sum $S' = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$ and define the line nabla integral $\int_\Gamma f(z)\nabla z$ to be the limit of S' as $\delta \to 0$ under the condition that S' corresponds to $P \in \mathcal{P}_\delta([a,b])$.

The following theorem can be proved similarly than Theorem 5.2.

Theorem 5.4. *Suppose that the curve* Γ *is given by the parametric equation* (4.1), *where* φ *and* ψ *are continuous on* $[a,b]$ *and* ∇*-differentiable on* $(a,b]$. *If* φ^∇ *and* ψ^∇ *are bounded and* ∇*-integrable over* $(a,b]$ *and if the function* f *is continuous on* Γ, *then the line nabla integral of* f *along the curve* Γ *exists and can be computed by the formula*

$$\int_\Gamma f(z)\nabla z = \int_a^b f(\lambda(t))\,\lambda^\nabla(t)\nabla t.$$

Now let us state some properties of complex line delta integrals. These properties can be verified either by using the formula (5.2) or by using the definition of a complex line delta integral as the limit of integral sums. In each case, we assume that Γ is a curve satisfying the conditions indicated in Theorem 5.2 and that f is defined and continuous on Γ.

Property 1. *Let* Γ *satisfy the conditions of Theorems 5.2 and 5.4. If we denote by* $-\Gamma$ *the curve* Γ *traversed in the opposite direction, then*

$$\int_\Gamma f(z)\Delta z = -\int_{-\Gamma} f(z)\nabla z.$$

Property 2. *Suppose that the curve* Γ *is composed from the curves* $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ *in such a way that the final point of* Γ_k *coincides with the initial point of* Γ_{k+1} *for* $k \in \{1, 2, \ldots, m-1\}$. *Then we have*

$$\int_\Gamma f(z)\Delta z = \sum_{k=1}^{m} \int_{\Gamma_k} f(z)\Delta z.$$

Property 3. *Let* $f_1, \ldots, f_m$ *be arbitrary complex functions which are defined and continuous on* Γ, *and let* $c_1, \ldots, c_m \in \mathbb{C}$. *Then*

$$\int_\Gamma \sum_{k=1}^{m} c_k f_k(z)\Delta z = \sum_{k=1}^{m} c_k \int_\Gamma f_k(z)\Delta z.$$

Property 4. *We have*

$$\left|\int_\Gamma f(z)\Delta z\right| \le \int_\Gamma |f(z)|\Delta \ell,$$

where in the right-hand side stands the real line delta integral of the first type (see [6]*) of the function* $|f|$ *along the curve* Γ. *Hence, as a corollary, we have*

$$\left|\int_\Gamma f(z)\Delta z\right| \le \left(\sup_\Gamma |f(z)|\right)\ell(\Gamma),$$

where $\ell(\Gamma)$ *denotes the length of the curve* Γ.

Property 5. *If the functions f_n for $n \in \mathbb{N}$ are continuous on Γ and if the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on Γ, then*

$$\int_\Gamma \sum_{n=1}^{\infty} f_n(z)\Delta z = \sum_{n=1}^{\infty} \int_\Gamma f_n(z)\Delta z.$$

Example 5.5. Any ordered collection of complex numbers $\{z_0, z_1, \ldots, z_n\}$ determines an oriented (time scale) curve Γ in $\mathbb{C}$, with the parametric equation

$$z = \lambda(t), \quad t \in [0, n] \subset \mathbb{Z},$$

where $\lambda(k) = z_k$ for $k \in [0, n] = \{0, 1, \ldots, n\}$. For any function $f : \Gamma \to \mathbb{C}$ we have

$$\int_\Gamma f(z)\Delta z = \sum_{k=1}^{n} f(\lambda(k-1))\,[\lambda(k) - \lambda(k-1)] = \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1})$$

and

$$\int_\Gamma f(z)\nabla z = \sum_{k=1}^{n} f(\lambda(k))\,[\lambda(k) - \lambda(k-1)] = \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}).$$

The curve $-\Gamma$ is given by the parametric equation

$$z = \lambda_1(t), \quad t \in [0, n] \subset \mathbb{Z},$$

where $\lambda_1(t) = \lambda(n - t)$. Therefore

$$\begin{aligned} \int_{-\Gamma} f(z)\nabla z &= \sum_{k=1}^{n} f(\lambda_1(k))\,[\lambda_1(k) - \lambda_1(k-1)] = \sum_{k=1}^{n} f(z_{n-k})(z_{n-k} - z_{n-k+1}) \\ &= \sum_{j=1}^{n} f(z_{j-1})(z_{j-1} - z_j) = -\sum_{j=1}^{n} f(z_{j-1})(z_j - z_{j-1}) = -\int_\Gamma f(z)\Delta z, \end{aligned}$$

and we have checked Property 1 of the integral given above, in this particular case.

Remark 5.6. We call the curve given by (4.1) *piecewise* Δ*-smooth* if φ and ψ are continuous on $[a, b]$ and there is a partition $a = c_0 < c_1 < \cdots < c_m = b$ of $[a, b]$ such that φ and ψ have bounded and Δ-integrable Δ-derivatives on each of the intervals $[c_{k-1}, c_k)$, $k \in \{1, 2, \ldots, m\}$. In case of a piecewise Δ-smooth curve Γ, it is natural to define complex line delta integrals along this curve as sums of line delta integrals along all Δ-smooth parts constituting the curve Γ. Then the equality (5.2) holds for piecewise Δ-smooth curves Γ as well. This equality is valid also in case when the function f is only piecewise continuous along the curve Γ. A similar remark can be made concerning the line nabla integrals.

6. A TIME SCALE VERSION OF CAUCHY'S INTEGRAL THEOREM

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and $\mathbb{T}_1 + i\mathbb{T}_2$ be the time scale complex plane defined in (3.1). Since $\mathbb{T}_1$ and $\mathbb{T}_2$ are closed subsets of $\mathbb{R}$, the set $\mathbb{T}_1 + i\mathbb{T}_2$ is a complete metric space with the metric defined in (3.2). Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}_1 + i\mathbb{T}_2$ the fundamental concepts such as open balls (disks), neigborhoods of points, open sets, closed sets, compact sets, boundary of a set, and so on. There is also the concept of continuous curve for general metric spaces and associated with it the concept of connectedness (arcwise connectedness). Namely, if $\mathcal{M}$ is a metric space, then any continuous mapping $h : [a, b] \to \mathcal{M}$ of the real line interval $[a, b]$ into the metric space $\mathcal{M}$ is called a (continuous) curve in $\mathcal{M}$. Above, in Section 4, we generalized the concept of continuous curve taking as $[a, b]$ an interval of a time scale instead of the reals $\mathbb{R}$. Accordingly, we can generalize the concept of arcwise connectedness to $\mathbb{T}_1 + i\mathbb{T}_2$.

Definition 6.1. Let $[a, b]$ be an interval in $\mathbb{T}_1$ with $a, b \in \mathbb{T}_1$ and $y_0 \in \mathbb{T}_2$. The set

$$\{z = x + iy_0 : \ x \in [a, b]\}$$

is called a horizontal line segment in $\mathbb{T}_1 + i\mathbb{T}_2$ and denoted by AB, where $A = a + iy_0$ and $B = b + iy_0$. Similarly, taking $x_0 \in \mathbb{T}_1$ and $[c, d] \subset \mathbb{T}_2$, we define a vertical line segment in $\mathbb{T}_1 + i\mathbb{T}_2$ as the set

$$\{z = x_0 + iy : \ y \in [c, d]\}$$

and denote it by CD, where $C = x_0 + ic$ and $D = x_0 + id$.

Definition 6.2. A finite sequence $P_1Q_1, P_2Q_2, \ldots, P_nQ_n$, each of whose term is a horizontal or vertical line segment in $\mathbb{T}_1 + i\mathbb{T}_2$, is said to form a *polygonal path* (or *broken line*) in $\mathbb{T}_1 + i\mathbb{T}_2$ with terminal points P_1 and Q_n if $Q_1 = P_2, Q_2 = P_3, \ldots, Q_{n-1} = P_n$. A set of points of $\mathbb{T}_1 + i\mathbb{T}_2$ is said to be *connected* if any two of its points are terminal points of a polygonal path of points contained in the set. A *component* of a set $\Omega \subset \mathbb{T}_1 + i\mathbb{T}_2$ is a nonempty maximal connected subset of Ω. A nonempty open connected set of points of $\mathbb{T}_1 + i\mathbb{T}_2$ is called a *domain.* A *closed domain* is a subset in $\mathbb{T}_1 + i\mathbb{T}_2$ being the closure of a domain in $\mathbb{T}_1 + i\mathbb{T}_2$.

Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half-closed bounded interval in $\mathbb{T}_2$. Let us introduce a "rectangle" in $\mathbb{T}_1 + i\mathbb{T}_2$ by

$$R = [a, b) + i[c, d) = \{x + iy : \ x \in [a, b), \ y \in [c, d)\} . \tag{6.1}$$

Let us set

$$L_1 = \{x + ic : \ x \in [a, b]\} , \quad L_2 = \{b + iy : \ y \in [c, d]\} ,$$

$$L_3 = \{x + id : \; x \in [a, b]\}, \quad L_4 = \{a + iy : \; y \in [c, d]\}.$$

Each of L_j for $j \in \{1, 2, 3, 4\}$ is an oriented line segment; for example, the positive orientation of L_1 arises according to the increase of x from a to b and the positive orientation of L_2 arises according to the increase of y from c to d. The set (closed curve)

$$\Gamma := L_1 \cup L_2 \cup (-L_3) \cup (-L_4)$$

is called the *positively oriented fence* of R. Positivity of orientation of Γ means that the rectangle R remains on the "left" side as we describe the fence curve Γ.

Definition 6.3. We say that $E \subset \mathbb{T}_1 + i\mathbb{T}_2$ is a *set of the type* ω if it is a connected set in $\mathbb{T}_1 + i\mathbb{T}_2$ being the union of a finite number of rectangles of the form (6.1) that are pairwise disjoint and adjoining to each other. Let $E = \bigcup_{k=1}^{m} R_k$ be a set of the type ω. Let Γ_k be the positively oriented fence of the rectangle R_k. Let us set $X = \bigcup_{k=1}^{m} \Gamma_k$. Further, let X_0 consist of a finite number of line segments each of which serves as a common part of fences of two adjoining rectangles belonging to $\{R_1, R_2, \ldots, R_m\}$. Then the set $\Gamma = X \setminus X_0$ forms a positively oriented closed "polygonal curve", which we call the *positively oriented fence of the set* E (the set E remains on the "left" as we describe the fence curve Γ).

Theorem 6.4 (Cauchy's Integral Theorem)**.** *Let $E \subset \mathbb{T}_1 + i\mathbb{T}_2$ be a set of the type ω and let Γ be its positively oriented fence. If a function f is delta analytic on $E \cup \Gamma$ and f^{Δ} is continuous on $E \cup \Gamma$, then*

$$\int_{\Gamma} f(z) d^* z = 0, \tag{6.2}$$

where the "star line integral" on the left side in (6.2) *denotes the sum of line delta integrals of f taken over the line segment constituents of Γ directed to the right or upwards and line nabla integrals of f taken over the line segment constituents of Γ directed to the left or downwards.*

Proof. Let us set $f(z) = u(x, y) + iv(x, y)$. Then, as was shown above in the proof of Theorem 5.2, we have

$$\int_{\Gamma} f(z) d^* z = \int_{\Gamma} u d^* x - v d^* y + i \int_{\Gamma} v d^* x + u d^* y. \tag{6.3}$$

Next, since f^{Δ} is continuous on $E \cup \Gamma$, the component functions u and v, along with their first partial delta derivatives, are continuous on $E \cup \Gamma$. Consequently, we can apply Green's formula (see [6] for Green's formula and [5] for double integrals)

$$\int_{\Gamma} M d^* x + N d^* y = \int \int_{E} \left(\frac{\partial N}{\Delta_1 x} - \frac{\partial M}{\Delta_2 y} \right) \Delta_1 x \Delta_2 y$$

to get from (6.3)

$$\int_\Gamma f(z)d^*z = \int\int_E \left(-\frac{\partial v}{\Delta_1 x} - \frac{\partial u}{\Delta_2 y}\right)\Delta_1 x \Delta_2 y + i\int\int_E \left(\frac{\partial u}{\Delta_1 x} - \frac{\partial v}{\Delta_2 y}\right)\Delta_1 x \Delta_2 y.$$

In view of the Cauchy–Riemann equations (3.6), the integrands of these two double delta integrals are zero throughout E. This completes the proof of (6.2). □

Example 6.5. Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and $z_0 = x_0 + iy_0$ be any point in $\mathbb{Z} + i\mathbb{Z}$. As the set $E \subset \mathbb{Z} + i\mathbb{Z}$ we take the single point rectangle

$$E = \{z_0\} = \{x + iy :\ x \in [x_0, x_0 + 1),\ y \in [y_0, y_0 + 1)\}\,. \tag{6.4}$$

The positively oriented fence Γ of E is the union of the oriented line segments (each of which consists of two points)

$$\Gamma_1 = \{z_0, z_0{+}1\},\quad \Gamma_2 = \{z_0{+}1, z_0{+}1{+}i\},\quad \Gamma_3 = \{z_0{+}1{+}i, z_0{+}i\},\quad \Gamma_4 = \{z_0{+}i, z_0\}.$$

For any function $f : \mathbb{Z} + i\mathbb{Z} \to \mathbb{C}$ we have (see Example 5.5 above)

$$\begin{aligned} \int_\Gamma f(z)d^*z &= \int_{\Gamma_1} f(z)\Delta z + \int_{\Gamma_2} f(z)\Delta z + \int_{\Gamma_3} f(z)\nabla z + \int_{\Gamma_4} f(z)\nabla z \\ &= f(z_0) + if(z_0 + 1) - f(z_0 + i) - if(z_0) \\ &= i\,[f(z_0 + 1) - f(z_0)] - [f(z_0 + i) - f(z_0)]\,. \end{aligned}$$

We see that the equality (6.2) is equivalent to (3.8), which in turn is equivalent to the delta analyticity of f at z_0. Since any bounded and connected set E in $\mathbb{Z} + i\mathbb{Z}$ is a finite union of disjoint rectangles of the form (6.4), we get that if f is delta analytic on E, then (6.2) holds, where Γ is the positively oriented fence of E. Thus we have checked Theorem 6.4 in this particular case.

REFERENCES

[1] B. Aulbach. *Analysis auf Zeitmengen*. Universität Augsburg, Augsburg, 1990. Lecture notes.

[2] B. Aulbach and S. Hilger. A unified approach to continuous and discrete dynamics. In *Qualitative Theory of Differential Equations (Szeged, 1988)*, volume 53 of *Colloq. Math. Soc. János Bolyai*, pages 37–56. North-Holland, Amsterdam, 1990.

[3] B. Aulbach and S. Hilger. Linear dynamic processes with inhomogeneous time scale. In *Nonlinear Dynamics and Quantum Dynamical Systems (Gaussig, 1990)*, volume 59 of *Math. Res.*, pages 9–20. Akademie Verlag, Berlin, 1990.

[4] M. Bohner and G. Sh. Guseinov. Partial differentiation on time scales. *Dynam. Systems Appl.*, 13:351–379, 2004.

[5] M. Bohner and G. Sh. Guseinov. Multiple integration on time scales. *Dynam. Systems Appl.*, 2005. To appear.

[6] M. Bohner and G. Sh. Guseinov. Line integrals and Green's formula on time scales. 2005. Submitted.

[7] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, Boston, 2001.

[8] M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.

[9] R. J. Duffin. Basic properties of discrete analytic functions. *Duke Math. J.*, 23:335–363, 1956.

[10] R. J. Duffin and E. L. Peterson. The discrete analogue of a class of entire functions. *J. Math. Anal. Appl.*, 21:619–642, 1968.

[11] J. Ferrand. Fonctions préharmoniques et fonctions préholomorphes. *Bull. Sci. Math.*, 68:152–180, 1944.

[12] S. Hilger. Analysis on measure chains — a unified approach to continuous and discrete calculus. *Results Math.*, 18:18–56, 1990.

[13] R. P. Isaacs. A finite difference function theory. *Univ. Nac. Tucumán. Revista A*, 2:177–201, 1941.

[14] R. P. Isaacs. Monodiffric functions. In *Construction and applications of conformal maps: Proceedings of a symposium*, volume 18 of *Nat. Bur. Standards Appl. Math. Series*. U.S. Government Printing Office, Washington, D.C., 1952.

[15] E. C. Titchmarsh. *The Theory of Functions*. Oxford University Press, London, second edition, 1939.

[16] D. Zeilberger. A new approach to the theory of discrete analytic functions. *J. Math. Anal. Appl.*, 57:350–367, 1977.

[17] D. Zeilberger and H. Dym. Further properties of discrete analytic functions. *J. Math. Anal. Appl.*, 58:405–418, 1977.