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An Oscillation Theorem for a Sturm–Liouville Eigenvalue Problem

By MARTIN BOHNER of Ulm

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Abstract. We consider a certain Sturm–Liouville eigenvalue problem with self–adjoint and non–separated boundary conditions. We derive an explicit formula for the oscillation number of any given eigenfunction.

1. Introduction

Let a, b, α_0, β_0, α_1, $\beta_1 \in \mathbb{R}$, $a < b$, let $q : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function, and let the conditions

$$(\mathrm{A}_1) \qquad \begin{cases} q \in C([a,b] \times \mathbb{R}), \\ q(t;\lambda_1) < q(t;\lambda_2) \quad \text{for all} \quad t \in (a,b), \ \lambda_1 < \lambda_2, \\ \lim\limits_{\lambda \to -\infty} q(t;\lambda) = -\infty \quad \text{for all} \quad t \in (a,b), \end{cases}$$

$$(\mathrm{A}_2) \qquad \alpha_0\beta_1 - \alpha_1\beta_0 = 1$$

be satisfied (which will be denoted by (A)). We consider the Sturm–Liouville eigenvalue problem (E), consisting of a linear differential equation of second order

$$(E_1) \qquad \ddot{x}(t) + q(t;\lambda)x(t) = 0 \quad (t \in [a,b])$$

subject to the two (under (A_2)) linearly independent and self–adjoint boundary conditions

$$(\mathrm{E}_2) \qquad \begin{cases} \alpha_0\dot{x}(a) + \beta_0 x(a) = x(b), \\ \alpha_1\dot{x}(a) + \beta_1 x(a) = -\dot{x}(b). \end{cases}$$

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(Note that the boundary conditions are non-separated and that any two linearly independent, self-adjoint, and non-separated boundary conditions may be equivalently transformed into (E_2) under (A_2), as is shown in [BA].) Assuming (A), there exists, as is well-known, (see e.g. [BI]) an at most countably infinite number of real and isolated eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ of (E). This number is infinite e.g. whenever $\lim_{\lambda\to\infty} q(\cdot\,;\lambda) = \infty$ holds on some nonempty subinterval of $[a,b]$, and it is finite e.g. whenever $\lim_{\lambda\to\infty} q(\cdot\,;\lambda) \leq \kappa \in \mathbb{R}$ holds on $[a,b]$ (similar statements are true when assuming $q \in C([a,b] \times I)$ for some real and open interval I; see e.g. [IN, p. 231]). The main result of this paper is an oscillation formula ((OF) in Theorem 3.1 below) for the problem (E) under (A), which calculates the oscillation number Z_p, i.e., the exact number of zeros of a given eigenfunction x_p corresponding to an eigenvalue λ_p on the interval $(a,b]$ for $p \in \mathbb{N}$.

In 1909, G. D. BIRKHOFF ([BI]) already discussed this problem, but his results were not correct, as was shown by G. BAUR ([BA]) in 1988. The main difference between those works and the present paper is that here we make use of an oscillation theorem for general Hamiltonian problems which has been developed by G. BAUR, W. KRATZ, and A. PEYERIMHOFF in [KR-PE, BA-KR, KR]. With this new method one can see that one of the 13 cases in [BA] actually never occurs; and the remaining 12 cases are unified by our oscillation formula

$$\text{(OF)} \qquad Z_p = p - \operatorname{ind}\mu_p - \operatorname{ind}\nu_p\,,$$

where μ_p, $\nu_p \in \mathbb{R}$ are constants, and where $\operatorname{ind} x = 1$ if $x < 0$ and 0 otherwise. Of course, the new method also yields the classical oscillation theorem of Sturm, where the boundary conditions need to be separated.

2. Notation and auxiliary results

Throughout we denote by Ker, Im, rank, def, ind, resp. the kernel, image, rank, defect (i.e., the dimension of the kernel), negative index (i.e., the number of negative eigenvalues) resp. of a matrix. Our main tool will be a general oscillation theorem for self-adjoint differential systems due to W. KRATZ ([KR, Theorem 1]). For convenience we restate this theorem (Theorem 2.1 below) for our special case; its proof is slightly different but essentially the same as in [KR]. To begin with, we provide the necessary notation:

Let $\lambda_p \in \mathbb{R}$ be an eigenvalue of (E), let x_p be a corresponding eigenfunction with $x_p(b) \neq 0$, and let $x_p^* = -\frac{x_p}{x_p(b)}$. We use $x(t,\lambda)$ to denote the solution of the initial value problem

$$\begin{cases} \ddot{x}(t) + q(t,\lambda)x(t) &= \;\; 0 \\ x(a) &= \;\; \alpha_0 \\ \dot{x}(a) &= -\beta_0 \end{cases}$$

(Note that $x_p^*(t)\,\dot{x}(t,\lambda_p) - \dot{x}_p^*(t)\,x(t,\lambda_p) \equiv 1$ holds on $[a,b]$.) Furthermore, we define the following matrices

$$\Lambda_p = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -x(b,\lambda_p) \\ -\dot{x}_p^*(b) & 0 & 1+\dot{x}(b,\lambda_p) \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_0 & 0 \\ 0 & \alpha_1 & 1 \end{pmatrix},$$

and

$$X_p = \begin{pmatrix} 0 & 0 & -1 \\ -x_p^*(a) & -x_p^*(a) & -\alpha_0 \\ 0 & -1 & x(b,\lambda_p) \end{pmatrix}.$$

Now, the orthogonal decomposition of $\operatorname{Im} R^T$ into $\operatorname{Im} X_p$ and its orthogonal complement leads to a unique matrix S_p^* and another matrix S_p such that $R^T = X_pS_p + S_p^*$ and $X_p^TS_p^* = 0$ hold. Then we consider a matrix T_p with $\operatorname{Im} T_p = \operatorname{Ker} S_p^*$ and finally we define $Q_p = T_p^T\Lambda_pS_pT_p$. With this setting, Theorem 1 from [KR] reads as follows:

Theorem 2.1. *Assume* (A). *Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of* (E) *and let x_p be an eigenfunction with $x_p(b) \ne 0$ corresponding to the eigenvalue λ_p. Then we have*

$$z_p + \operatorname{ind} Q_p = n_p + \operatorname{rank} T_p - 2,$$

where z_p denotes the number of zeros of x_p in (a,b), and where n_p denotes the number of eigenvalues (counting multiplicities) of (E) *that are less than λ_p.*

3. The oscillation formula

We shall now construct the matrices Q_p and T_p, compute $\operatorname{ind} Q_p$ and $\operatorname{rank} T_p$, and apply Theorem 2.1 to prove the following theorem which is the main result of this paper.

Theorem 3.1. *Assume* (A). *Let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues (counting multiplicities) of* (E). *An eigenfunction x_p corresponding to an eigenvalue λ_p, $p \in \mathbb{N}$, vanishes exactly*

(OF) $$Z_p = p - \operatorname{ind}\mu_p - \operatorname{ind}\nu_p$$

times on $(a,b]$, where the constants μ_p and ν_p are defined by

$$\begin{cases} \mu_p = -\alpha_0 x_p(a)\,x_p(b) - (1-\operatorname{def}\alpha_0)\operatorname{def} x_p(a)\,; \\ \nu_p = (-1)^{p-1}(\alpha_0 - \beta_0\operatorname{def}\alpha_0)\,. \end{cases}$$

The remainder of this section is devoted to the proof of our main result.

Lemma 3.2. *With the assumptions and notation of Theorem* 2.1 *we have*

(OF*) $$z_p = 1 + n_p - \operatorname{ind}\mu_p - \operatorname{ind} x(b,\lambda_p)\,.$$

Proof. For convenience we use the abbreviations

$$N_p = \operatorname{def} x_p(a) = \begin{cases} 1 & \text{if} \quad x_p(a) = 0, \\ 0 & \text{if} \quad x_p(a) \neq 0, \end{cases}$$

and

$$N_p^* = \begin{cases} 0 & \text{if} \quad x_p(a) = 0, \\ \dfrac{1}{x_p^*(a)} & \text{if} \quad x_p(a) \neq 0. \end{cases}$$

Using the matrices

$$S_p = \frac{1}{1+N_p\alpha_0^2}\begin{pmatrix} N_p^*\alpha_0 + (1-N_p)x(b,\lambda_p) & -N_p^*\alpha_0 & -N_p^*\alpha_0\dot{x}_p^*(b) \\ -x(b,\lambda_p) & -N_p\alpha_0^2 x(b,\lambda_p) & N_p\alpha_0^2\dot{x}(b,\lambda_p) - 1 \\ -1 & -N_p\alpha_0^2 & -N_p\alpha_0^2\dot{x}_p^*(b) \end{pmatrix},$$

$$S_p^* = \frac{1}{1+N_p\alpha_0^2}\begin{pmatrix} N_p\alpha_0^2 & -N_p\alpha_0^2 & -N_p\alpha_0^2\dot{x}_p^*(b) \\ -N_p\alpha_0 & N_p\alpha_0 & N_p\alpha_0\dot{x}_p^*(b) \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_p = \begin{pmatrix} 1-N_p & N_p & N_p\dot{x}_p^*(b) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

one can easily verify the relations $X_pS_p + S_p^* = R^T$, $X_p^TS_p^* = 0$, and $\operatorname{Im} T_p = \operatorname{Ker} S_p^*$ (use the formulae $N_p^*x_p^*(a) = 1 - N_p$, $N_px_p^*(a) = 0$, $1 + \dot{x}(b,\lambda_p) = -\dot{x}_p^*(b)x(b,\lambda_p)$, and $\alpha_1 = x_p^*(a) + \alpha_0\dot{x}_p^*(b)$). Now, we have $\operatorname{rank} T_p = 3 - N_p$ and

$$Q_p = \frac{1}{1+N_p\alpha_0^2}\begin{pmatrix} N_p^*\alpha_0 + (1-N_p)x(b,\lambda_p) & -N_p^*\alpha_0 & -N_p^*\alpha_0\dot{x}_p^*(b) \\ -N_p^*\alpha_0 & N_p^*\alpha_0 + N_px(b,\lambda_p)\left(1+\alpha_0^2\right) & \dot{x}_p^*(b)\left\{N_p^*\alpha_0 + N_px(b,\lambda_p)\left(1+\alpha_0^2\right)\right\} \\ -N_p^*\alpha_0\dot{x}_p^*(b) & \dot{x}_p^*(b)\left\{N_p^*\alpha_0 + N_px(b,\lambda_p)\left(1+\alpha_0^2\right)\right\} & \left\{\dot{x}_p^*(b)\right\}^2\left\{N_p^*\alpha_0 + N_px(b,\lambda_p)\left(1+\alpha_0^2\right)\right\} \end{pmatrix},$$

$$\operatorname{ind} Q_p = \operatorname{ind}\left\{\left(1+N_p\alpha_0^2\right)\begin{pmatrix} N_p^*+N_p & N_p^* & 0 \\ 0 & 1 & 0 \\ 0 & -\dot{x}_p^*(b) & 1 \end{pmatrix} Q_p \begin{pmatrix} N_p^*+N_p & 0 & 0 \\ N_p^* & 1 & -\dot{x}_p^*(b) \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= \operatorname{ind}\begin{pmatrix} \left\{N_p^*\right\}^2 x(b,\lambda_p) & 0 & 0 \\ 0 & N_p^*\alpha_0 + N_px(b,\lambda_p)\left(1+\alpha_0^2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$
\begin{aligned}
&= \operatorname{ind}\left[\left\{N_p^*\right\}^2 x\left(b,\lambda_p\right)\right] + \operatorname{ind}\left[N_p^*\alpha_0 + N_p x\left(b,\lambda_p\right)\left(1+\alpha_0^2\right)\right] \\
&= \operatorname{ind} x\left(b,\lambda_p\right) + \operatorname{ind}\mu_p - N_p .
\end{aligned}
$$

An application of Theorem 2.1 yields (OF*). □

It is now possible to transform (OF*) into the "nicer" formula (OF). To do so, we need to provide some facts about the function $x(\,\cdot\,,\lambda_p)$ that occurs in (OF*).

Lemma 3.3. *Assume* (A). *Let $\lambda_1^* < \lambda_2^* < \lambda_3^* < \cdots$ denote the eigenvalues of the problem*

$$
(\mathrm{E}^*) \qquad \begin{cases} \ddot{x}(t) + q(t,\lambda)x(t) = 0 \\ \alpha_0\dot{x}(a) + \beta_0 x(a) = 0 \\ x(b) = 0 . \end{cases}
$$

Then we have

(i) *λ^* is an eigenvalue of* (E^*) *iff $x(b,\lambda^*) = 0$;*

(ii) *if $\lambda \in \left(\lambda_{p-1}^*, \lambda_p^*\right]$, then $x(\,\cdot\,,\lambda)$ has exactly $p-1$ zeros on (a,b);*

(iii) *putting $\lambda_0^* = -\infty$, the comparison result $\lambda_{p-1}^* \le \lambda_p \le \lambda_p^*$ holds for all $p \in \mathbb{N}$ (where $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ are the eigenvalues of problem (E));*

(iv) *if $x(\,\cdot\,,\lambda)$ has exactly q zeros on (a,b), and if $\nu_q = (-1)^{q-1}(\alpha_0 - \beta_0 \operatorname{def}\alpha_0)$, then*

$$
\begin{aligned}
\operatorname{sgn} x(b,\lambda) &= \operatorname{sgn}\nu_{q+1} \quad \text{if} \quad x(b,\lambda) \neq 0, \quad \text{and} \\
\operatorname{sgn} \dot{x}(b,\lambda) &= \operatorname{sgn}\nu_{q} \quad \text{if} \quad x(b,\lambda) = 0 .
\end{aligned}
$$

Proof. (i), (ii), and (iii) are well-known facts (see e.g. [BA, KR-PE]), and (iv) follows from the simplicity of the zeros of $x(\,\cdot\,,\lambda)$. □

Proof of Theorem 3.1. Put $\lambda_0 = \infty$ and let $p \in \mathbb{N}$.

Suppose $x_p(b) \neq 0$. If $\lambda_p \in \left(\lambda_{p-1}^*, \lambda_p^*\right)$, then $x\left(b,\lambda_p\right) \neq 0$ and $x\left(\,\cdot\,,\lambda_p\right)$ has exactly $p-1$ zeros on (a,b). Hence $\left(\text{note } \lambda_{p-1} \le \lambda_{p-1}^* < \lambda_p\right)$ $Z_p = 1 + (p-1) - \operatorname{ind}\mu_p - \operatorname{ind}\nu_p$. Otherwise, we have $\lambda_p \in \left\{\lambda_{p-1}^*, \lambda_p^*\right\}$. Now, $x\left(b,\lambda_p\right) = 0$ and $\dot{x}\left(b,\lambda_p\right) = -1$ $\left(\text{since } x_p^*(b)\,\dot{x}\left(b,\lambda_p\right) - \dot{x}_p^*(b)\,x\left(b,\lambda_p\right) = 1\right)$. Also, λ_p is a double eigenvalue since both $x_p(\,\cdot\,)$ and $x\left(\,\cdot\,,\lambda_p\right)$ solve (E). We have that $n_p = p-1 = p-1-(1-\operatorname{ind}\nu_{p-1}) = p-1-\operatorname{ind}\nu_p$ if $\lambda_p = \lambda_p^* > \lambda_{p-1}^* \ge \lambda_{p-1}$ $\left(\text{note that } x\left(\,\cdot\,,\lambda_p\right) \text{ vanishes } p-1 \text{ times on } (a,b)\right)$ and $n_p = p-2 = p-1-\operatorname{ind}\nu_{p-2} = p-1-\operatorname{ind}\nu_p$ if $\lambda_p = \lambda_{p-1}^* < \lambda_p^* \le \lambda_{p+1}$, $p \neq 1$ $\left(\text{observe that } x\left(\,\cdot\,,\lambda_p\right) \text{ has now } p-2 \text{ zeros in } (a,b)\right)$.

Now, suppose that $x_p(b) = 0$. Then $x\left(\,\cdot\,,\lambda_p\right)$ and $x_p(\,\cdot\,)$ are linearly dependent and we have $\mu_p = 0$ as well as $\dot{x}\left(b,\lambda_p\right) = -1$. (For the proof of the latter statement pick the solution y of (E_1) where $\lambda = \lambda_p$ with $y(a) = -\alpha_1$ and $\dot{y}(a) = \beta_1$. Now we have $y(b) - \dot{x}\left(b,\lambda_p\right) = 2$ (see e.g. [BI]) and $\dot{x}\left(b,\lambda_p\right) y(b) = -1$ which yields $y(b) = 1$ and $\dot{x}\left(b,\lambda_p\right) = -1$.) If $\lambda_p = \lambda_p^*$, then $x\left(\,\cdot\,,\lambda_p\right)$ has $p-1$ zeros on (a,b), i.e., $Z_p = p$ and $\operatorname{ind}\nu_p = 1 - \operatorname{ind}\nu_{p-1} = 0$. If $\lambda_p = \lambda_{p-1}^*$, $p \neq 1$, then $x\left(\,\cdot\,,\lambda_p\right)$ vanishes $p-2$ times on (a,b), i.e., $Z_p = p-1$ and $\operatorname{ind}\nu_p = \operatorname{ind}\nu_{p-2} = 1$.

This proves that our oscillation formula (OF) is valid in any case. □

Remark 3.4. Observe that the case $\alpha_0 < 0$, $\mu_1 < 0$ can never occur (see e.g. [BA, p. 192]), since an application of (OF) would yield the impossible number of $Z_1 = 1 - 1 - 1 = -1$ zeros of an eigenfunction x_1 corresponding to the eigenvalue λ_1 of (E) on $(a, b]$.

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Universität Ulm
Abteilung Mathematik V
Helmholtzstr. 18
D-88069 Ulm
Germany
e-mail:
bohner@mathematik.uni-ulm.de