Research paper

Almost periodic solutions of Cohen–Grossberg neural networks with time-varying delay and variable impulsive perturbations

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\begin{abstract}
In this paper, we consider the problem of existence of almost periodic solutions of impulsive Cohen–Grossberg neural networks with time-varying delays. The impulses are not at fixed moments, but are realized when the integral curves of solutions meet given hypersurfaces, i.e., the investigated model is with variable impulsive perturbations. Sufficient conditions for perfect stability of almost periodic solutions are derived. The main results are obtained by employing the Lyapunov–Razumikhin method and a comparison principle. In addition, the obtained results are extended to the uncertain case, and robust stability of almost periodic solutions is also investigated. An example is considered to demonstrate the effectiveness of our results.
\end{abstract}

\section{Introduction}

Neural networks have been intensively studied in the literature due to their important numerous applications in fields such as classification, parallel computation, associative memory, nonlinear optimization problems, pattern recognition, automatic control, processing signals, etc. [1,2]. So-called Cohen–Grossberg neural networks (CGNNs) form an important class of competitive neural networks with regard to their potential to be advantageous in global pattern formation and partial memory storage [3].

As is well known [4], in practice, due to the finite speeds of the switching and transmission of signals in a network, time delays unavoidably exist in a working network, and they may lead to oscillation, instability, bifurcation, or chaos of networks. Thus, CGNNs with fixed and time-varying delays have drawn an increasing attention. See [5–9] and the references therein. The topic of delayed CGNNs does not lose its actuality today but rather attracts more interest [10–12].

On the other hand, impulsive differential equations serve as mathematical models to investigate the dynamics of processes that are subject to sudden changes in their states [13–16]. Since impulsive factors and delays can greatly affect the dynamics of neural networks models, their effects have been extensively studied in the literature [17–19]. For CGNNs under impulsive perturbations, see [20,21] and the references therein.

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It is recognized that the qualitative analysis of neural network models is very important for their practical design [22]. Stability and attractivity are the most investigated qualitative properties, and many important and interesting stability results for CGNNs have been reported, see [9,23–26]. In the existing results, the authors mainly investigated the qualitative behavior of equilibria and periodic solutions. But, in reality, due to many environmental factors, it is natural to consider almost periodic neural network’s attractors [27]. Indeed, in real–world processes, various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, and one has to consider the environment to be almost periodic. Almost periodicity is a more realistic, more general and complicated concept than periodicity. Due to its significance and the great opportunities for applications, the existence of almost periodic processes has become a hot research topic, especially for almost periodic neural networks. It is also well known that existence and nonexistence of almost periodic solutions to a given system are important parts of oscillation theory, which is an intrinsic feature of many dynamical systems. We refer the reader to Couchouron and co-authors [28–33] for some recent developments in this direction and potential applications of almost periodicity. Recently, there exist some results for existence and stability of almost periodic solutions of CGNNs with delays (without impulses) [34–37] and of CGNNs with impulsive perturbations (without delays) [38,39]. Almost periodic solutions for delayed impulsive CGNNs are hardly considered and investigated [40–42]. However, to the best of our knowledge, in all published papers on almost periodicity of impulsive CGNNs the authors considered only fixed moments of impulsive perturbations.

It is well known that variable impulsive perturbations are more general, and impulsive mathematical models under such perturbations have numerous important applications. In the investigation of such systems, there arise a number of difficulties related to the phenomena “beating” of the solutions, bifurcation, loss of the property of autonomy, etc. See [43–45] and the references therein. Recently, the effect of variable impulsive perturbations has been considered for some classes of neural networks [46–48]. The qualitative results in the mentioned references on variable-time impulsive systems are mainly proved by the B-equivalence method [49,50] that requires a reduction to a system with fixed moments of impulsive perturbations. However, although it is of great importance in real life applications, the generalization to almost periodicity of delayed CGNNs with variable impulsive perturbations is not yet developed, and this is the main aim of our paper. Also, different from other authors, we do not reduce the model to a system with fixed moments of impulsive perturbations. Instead, we are using the comparison principle coupled with the Lyapunov method, which has great power in applications since no knowledge for the solutions is required. In addition, we extend our results to the uncertain case. Indeed, in practical implementation of neural networks, the qualitative behavior can often be destroyed by some uncertain factors due to modeling errors, external disturbances, and parameter fluctuations [51–53].

It is worth to emphasize that some authors investigated the effect of uncertain terms on the stability behavior of CGNNs. In [54], a set of sufficient conditions ensuring robust global exponential convergence of the equilibrium of CGNNs with time delays is given. In [55], a fixed-time master-slave synchronization of CGNNs with parameter uncertainties and time-varying delays is investigated. The authors of the paper [56] analyzed robust stability for a class of switched CGNNs with mixed time-varying delays and proposed some sufficient conditions that guarantee the investigated neural network models to be globally asymptotically stable for all admissible parametric uncertainties. However, in order to establish robust stability or synchronization criteria, all authors in [54–56] consider zero solutions or equilibria of CGNNs and do not assume that the neural networks are affected by some impulsive factors. Compared with all previous works, we consider almost periodic solutions and variable impulsive perturbations in our robust stability analysis. The main contributions of this paper can be highlighted in the following aspects.

1. By means of nonsmooth analysis and the framework of Lyapunov method, the existence of an almost periodic solution is established for addressed CGNNs with time-varying delays and variable impulsive perturbations.
2. Based on the comparison principle and Razumikhin technique for delayed systems and suitable Lyapunov function method, some sufficient conditions for perfect exponential stability of the almost periodic solution of CGNNs under consideration are introduced.
3. We consider parametric uncertainty in our dynamic analysis. In most of the CGNNs with time delays in the available literature, the existence of uncertain factors and variable impulsive perturbations are not taken into consideration simultaneously. This shows the novelty of our proposed result.
4. Novel robust stability criteria are proposed for CGNNs with time-varying delays and variable impulsive perturbations.

The remaining part of the paper is organized as follows. In Section 2, we introduce some notations, definitions, and preliminaries which will be used later. In Section 3, sufficient conditions are obtained that ensure existence and perfect stability of a unique almost periodic solution of CGNNs with time-varying delays and variable impulsive perturbations. Section 4 is devoted to the uncertain case. By employing the inequality technique, we propose robust stability criteria for almost periodic solutions of CGNNs with time-varying delays, variable impulsive perturbations, and parametric uncertainties. In Section 5, an example is given to show the effectiveness of the obtained results. Finally, some conclusion remarks are drawn in Section 6.

2. Preliminaries

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space endowed with the norm $||x|| = \sum_{i=1}^{n} x_i^2$ and let $\mathbb{R}_+ = [0, \infty)$. Throughout the paper, $\mathbb{Z}$ means the set of all integer numbers, and $C(A, B)$ denotes the class of all continuous functions $\varphi: A \to B$, where $A$ and $B$ are appropriate sets.
The goal of this paper is to investigate the existence of almost periodic solutions of a class of Cohen–Grossberg neural networks with time-varying delay and variable impulsive perturbations in the form

\[
\begin{cases}
\dot{x}_i(t) = -a_i(x_i(t)) \left[ b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) \right. \\
\left. - \sum_{j=1}^{n} d_{ij}(t) g_j(x_j(t - \theta_j(t))) - l_i(t) \right], \quad t \neq \tau_k(x(t)),
\end{cases}
\]

(2.1)

where

\[
\begin{align*}
a_i, c_{ij}, d_{ij}, f_j, g_j, \theta_j, l_i & \in C(\mathbb{R}, \mathbb{R}^+), \quad i, j = 1, \ldots, n, \\
b_i \in C(\mathbb{R}^2, \mathbb{R}^+), \quad P_i \in \mathbb{R}, \quad i = 1, \ldots, n, \\
0 & \leq \theta_j(t) \leq \nu, \quad \nu > 0, \quad t > \theta_j(t), \quad j = 1, \ldots, n, \\
\tau_k : \mathbb{R}^n & \to \mathbb{R}, \quad k \in \mathbb{Z}.
\end{align*}
\]

System (2.1) is a generalization of existing models of impulsive neural networks with time-varying delay, and our results will improve and generalize some known results obtained in [5–12,20,21,23–27,34–42].

Let, for simplicity \( X(t,x) = (X_1(t,x), X_2(t,x), \ldots, X_n(t,x)) \), where

\[
X_i(t,x) = -a_i(x_i(t)) \left[ b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) \right. \\
\left. - \sum_{j=1}^{n} d_{ij}(t) g_j(x_j(t - \theta_j(t))) - l_i(t) \right], \quad i = 1, 2, \ldots, n,
\]

and let \( P = \text{diag}(P_1, P_2, \ldots, P_n) \). Let \( t_0 \in \mathbb{R} \) and \( \varphi_0 : [-\nu, 0) \to \mathbb{R}^n \). For the future considerations, we denote by \( x(t) \), where

\[
x(t) = x(t; t_0, \varphi_0) = (x_1(t; t_0, \varphi_0), \ldots, x_n(t; t_0, \varphi_0))^T,
\]

the solution of (2.1) that satisfies the initial conditions

\[
\begin{align*}
x(t; t_0, \varphi_0) &= \varphi_0(t - t_0), \quad t_0 - \nu \leq t \leq t_0, \\
x(t_0^+; t_0, \varphi_0) &= \varphi_0(0)
\end{align*}
\]

(2.2)

and by \( J^+(t_0, \varphi) \) the maximal interval of the type \([t_0, \beta)\) in which the solution \( x(\cdot; t_0, \varphi) \) is defined. The solution \( x(t) \) of (2.1), (2.2) is a piecewise continuous function [15,16,29,43] with points of discontinuity of the first kind at which it is left continuous, i.e., at the moments \( \tau_k \) when the integral curve of the solution \( x \) meets the hypersurfaces

\[
\sigma_k = \{(t, x) \in [t_0, \infty) \times \mathbb{R}^n : t = \tau_k(x)\},
\]

where it is continuous from the left, we have

\[
x_i(t^-_k) = x_i(t^-_k), \quad x_i(t^+_k) = x_i(t^-_k) + P_i x_i(t^-_k).
\]

The points \( t_1, t_2, \ldots, t_k < t_{k+1} < \ldots \) are the impulsive moments. Let us note that, in general, \( k \neq l_k \). Let \( J \subset \mathbb{R}^+ \) be an interval. We define the following classes of functions.

\[
\begin{align*}
PC(J, \mathbb{R}^n) &= \{ \varphi : J \to \mathbb{R}^n : \varphi \text{ is continuous everywhere except} \\
& \text{at the points } \tilde{t} \in J \text{ at which} \\
& \varphi(\tilde{t}^-) \text{ and } \varphi(\tilde{t}^+) \text{ exist and } \varphi(\tilde{t}^-) = \varphi(\tilde{t}^+) \}, \\
PCB(J, \mathbb{R}^n) &= \{ \varphi \in PC(J, \mathbb{R}^n) : \varphi \text{ is bounded on } J \}.
\end{align*}
\]

To guarantee the absence of the phenomenon “beating” of the solutions [16,43], existence, uniqueness, and continuability of the solution \( x = x(\cdot; t_0, \varphi_0) \) of the IVP (2.1), (2.2) on the interval \([t_0, \infty)\) for \( \varphi_0 \in PCB([-\nu, 0], \mathbb{R}^n) \) and \( t_0 \in \mathbb{R} \), we assume

1. \( \tau_0(x) \equiv t_0 \) for \( x \in \mathbb{R}^n \), the functions \( \tau_k \) are continuous, and \( t_0 < \tau_1(x) < \tau_2(x) < \ldots < \tau_k(x) \to \infty \) as \( k \to \infty \) uniformly on \( \mathbb{R}^n \).
2. The functions \( a_i, b_i, c_{ij}, d_{ij}, f_j, g_j, \theta_j, l_i \) are continuous on their domains, \( i, j = 1, 2, \ldots, n \).
3. \( l_k < l_{k+1} < \ldots < l_{k+p} < \ldots \) where \( l_k \) is the number of hypersurfaces met by the integral curve \((t, x(t))\) of (2.1) at the moment \( t_k \) where \( k, l_k, p \in \mathbb{Z} \).
4. The integral curves of (2.1) meet each hypersurface \( \sigma_1, \sigma_2, \ldots \) at most once.
We equip the space $PC$ with the norm $\| \cdot \|_v$ defined by

$$\| \varphi \|_v = \sup_{-\nu \leq \varepsilon \leq 0} \| \varphi (\varepsilon) \| \quad \text{for} \quad \varphi \in PC.$$ 

In the case $\nu = \infty$, we have $\| \varphi \|_{\infty} = \sup_{\varepsilon \in (-\infty, 0]} \| \varphi (\varepsilon) \|$. We note that $PC$ with the norm $\| \cdot \|_v$ is a Banach space.

Before we present our main definitions for almost periodicity, we give some essential notation as follows.

$$B = \left\{ t_k : t_k \in (-\infty, \infty), \ t_k < t_{k+1}, \ l_k \in \mathbb{Z}, \ \lim_{k \to \infty} t_k = \pm \infty \right\}$$

is the set of all unbounded and strictly increasing sequences with distance

$$\rho \left( \left\{ t_k^{(1)} \right\}, \left\{ t_k^{(2)} \right\} \right).$$

For $T, \tilde{T} \in B$, let $s(T \cup \tilde{T}) : B \to B$ be a map such that the set $s(T \cup \tilde{T})$ forms a strictly increasing sequence. For $D \subset \mathbb{R}$, let $D_{\varepsilon} = \{ t + \varepsilon : t \in D \}$ and $\Theta_{\varepsilon} (D) = \cap \{ D_{\varepsilon} \}$ for $\varepsilon > 0$. By $\Psi = (\psi, T)$, we shall denote an element from the space $PC(\mathbb{R}, \mathbb{R}) \times B$.

For every sequence of real numbers $\{ s_p \}$, $p \in \mathbb{N}$, $\phi_{s_p}$ mean the sets $\{ \psi (t + s_p), \ T - s_p \} \subset PC(\mathbb{R}, \mathbb{R}) \times B$, where

$$T - s_p = \left\{ t_k - s_p : k \in \mathbb{Z}, \ p \in \mathbb{N} \right\}.$$ 

**Definition 2.1** (See [29]). The set of sequences

$$\{ t_k^l \}, \quad t_k^l = t_{k+l} - t_k, \quad k, l \in \mathbb{Z},$$

called uniformly almost periodic if from each infinite sequence of shifts

$$\{ t_k - s_p \}, \quad k \in \mathbb{Z}, \quad p \in \mathbb{N}, \quad s_p \in \mathbb{R},$$

we can choose a convergent subsequence in $B$.

**Definition 2.2** (See [29]). The sequence $\{ \Psi_p \}$,

$$\Psi_p = (\psi_p, T_p) \in PC(\mathbb{R}, \mathbb{R}) \times B.$$

converges to $\Psi = (\psi, T) \in PC(\mathbb{R}, \mathbb{R}) \times B$ if for any $\varepsilon > 0$, there exists $p_0 > 0$ such that for $p \geq p_0$, we have

$$\rho(T, T_p) < \varepsilon, \quad \| \psi_p(t) - \psi(t) \| < \varepsilon \quad \text{uniformly for} \quad t \in \mathbb{R} \setminus \Theta_{\varepsilon} (s(T_p \cup T)).$$

**Definition 2.3** (See [29]). The function $\psi \in PC(\mathbb{R}, \mathbb{R})$ is said to be almost periodic piecewise continuous with points of discontinuity of the first kind $t_k, \{ t_k \} \in B$, if for every sequence of real numbers $\{ s_m \}$, there exists a subsequence $\{ s_p \}$, $s_p = s_m^p$, such that $\Psi_{s_p}$ is compact in $PC(\mathbb{R}, \mathbb{R}) \times B$.

**Remark 2.4.** For some basic concepts and theorems on almost periodic functions, and for the significance of the almost periodicity, we refer the reader to [28–33] and the references therein.

The following conditions will guarantee existence and uniqueness of the nodes and are essential for the future process [38].

1. **(H1)** The functions $a_i, \ i = 1, 2, \ldots, n$, are bounded and there exist positive constants $\underline{a}_i$ and $\overline{a}_i$ such that $1 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$ for $t \in \mathbb{R}$.

2. **(H2)** The functions $b_i, \ i = 1, 2, \ldots, n$, are continuous in the second variable, almost periodic in the sense of Bohr in the first variable and uniformly in the second variable, and there exist almost periodic continuous positive functions $B_i$ such that

$$\frac{b_i(t, x) - b_i(t, y)}{x - y} \geq B_i(t) \quad \text{for} \quad x, y \in \mathbb{R}, \ x \neq y.$$ 

3. **(H3)** The functions $c_{ij}, d_{ij}, l_i, i, j = 1, 2, \ldots, n$, are almost periodic in the sense of Bohr.

4. **(H4)** There exist positive constants $L_i, M_i, H_i^{(1)}, H_i^{(2)}, i = 1, 2, \ldots, n$, with

$$|f_i(x_1) - f_i(x_2)| \leq L_i |x_1 - x_2|, \quad |g_i(x_1) - g_i(x_2)| \leq M_i |x_1 - x_2|, \quad |f_i(x)| \leq H_i^{(1)}, \quad |g_i(x)| \leq H_i^{(2)}$$

for all $x, x_1, x_2 \in \mathbb{R}, \ x_1 \neq x_2$, and $f_i(0) = g_i(0) = 0$.

5. **(H5)** The function $\psi_0 \in PC$ is almost periodic.

6. **(H6)** The set of sequences $\{ t^l_k \}$, $t^l_k = t^l_{k+l} - t^l_k, \ k, l \in \mathbb{Z}$ is uniformly almost periodic, and $\inf t^l_k > 0$. 
Let the conditions (H1)-(H6) hold, and let \( s'_i \) be an arbitrary sequence of real numbers. Then, there exists a subsequence \( \{s_p \} \), \( s_p = s'_{i_p} \), such that (2.1) moves into the system
\[
\begin{align*}
\dot{x}_i(t) &= -a_i(x_i(t)) \left[ b^i(t, x_i(t)) - \sum_{j=1}^{n} c^i_j(t) f_j(x_j(t)) \right] \\
- \sum_{j=1}^{n} d^i_j(t) g_j(x_j(t - \theta^i_j(t))) - l^i_j(t), & \quad t \neq \tau^i_k(x(t)), \\
\tau^i_k(x(t^+)) &= x_i(t) + p_k x_i(t), & \quad t = \tau^i_k(x(t)),
\end{align*}
\]
and the set of systems in the form (2.3) shall be denoted by \( \mathcal{H}(2.1) \). Again, we use the notation \( X^i(t, x) = (X^i_1(t, x), X^i_2(t, x), \ldots, X^i_n(t, x)) \), where
\[
X^i_1(t, x) = -a_i(x_i(t)) \left[ b^i(t, x_i(t)) - \sum_{j=1}^{n} c^i_j(t) f_j(x_j(t)) \right] \\
- \sum_{j=1}^{n} d^i_j(t) g_j(x_j(t - \theta^i_j(t))) - l^i_j(t), & \quad i = 1, 2, \ldots, n.
\]
In the further considerations, we will use the Lyapunov function method. That is why the sets
\[
\mathcal{G}_k = \{ (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \tau_{k-1}(x) < t < \tau_k(x), \; \tau_{k-1}(y) < t < \tau_k(y) \}
\]
for \( k \in \mathbb{Z} \),
\[
\mathcal{G} = \bigcup_{k=-\infty}^{\infty} \mathcal{G}_k,
\]
and the following definitions will be important.

**Definition 2.5.** A function \( V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \) belongs to class \( V_0 \) if the following conditions are fulfilled.
1. \( V \) is continuous in \( \mathcal{G} \) and \( V(t, 0, 0) = 0 \) for \( t \in \mathbb{R} \).
2. \( V \) satisfies locally a Lipschitz condition with respect to its second and third arguments on each of the sets \( \mathcal{G}_k \).
3. For each \( k \in \mathbb{Z} \) and \( (t^+_k, x^+_0, y^+_0) \in \mathcal{G}_k \), there exist the finite limits
   \[
   \begin{align*}
   V(t^+_k, x^+_0, y^+_0) &= \lim_{(u, x, y) \to (t^+_k, x^+_0, y^+_0)} V(t, x, y), \\
   V(t^+_k, x^+_0, y^+_0) &= \lim_{(u, x, y) \to (t^+_k, x^+_0, y^+_0)} V(t, x, y),
   \end{align*}
   \]
   and the equality \( V(t^+_k, x^+_0, y^+_0) = V(t^+_k, x^+_0, y^+_0) \) is valid.

**Definition 2.6** (See [29]). Let \( V \in V_0 \). For \( t \neq \tau_k(x(t)) \) and \( t \neq \tau_k(y(t)) \), \( k \in \mathbb{Z} \), we consider the upper right-hand derivative of \( V \) with respect to (2.1), defined by
\[
D^+ V(t, \phi(0), \psi(0)) = \limsup_{\chi \to 0^+} \frac{V(t + \chi, x(t + \chi; t_0, \phi), x(t + \chi; t_0, \psi)) - V(t, \phi(0), \psi(0))}{\chi},
\]
where \( (t, \phi, \psi) \in \mathbb{R}_+ \times PC \times PC \).

In the next sections, we shall use the following lemma whose proof is similar to the proofs of the comparison theorems in [16,29].

**Lemma 2.7.** Assume that the function \( V \in V_0 \) is such that for \( t \in [t_0, \infty) \), \( \phi, \psi \in PC \),
\[
\begin{align*}
V(t^+, \phi(0) + \Delta \phi, \psi(0) + \Delta \psi) \leq V(t, \phi(0), \psi(0)), & \quad t \in [\tau_k(\phi), t_k(\psi) : k \in \mathbb{Z}],
\end{align*}
\]
and for \( \mu \in \mathbb{R} \), the inequality
\[
D^+ V(t, \phi(0), \psi(0)) \leq \mu V(t, \phi(0), \psi(0)),
\]
for \( t \notin [\tau_k(\phi), t_k(\psi) : k \in \mathbb{Z}] \)
is valid whenever \( V(t + \xi, \phi(\xi), \psi(\xi)) \leq V(t, \phi(0), \psi(0)) \) for \( -\nu \leq \xi \leq 0 \). Then,
\[
V(t, x(t; t_0, \phi_0), x(t; t_0, \psi_0)) \leq \sup_{-\nu \leq \xi \leq 0} V(t^+_0, \phi_0(\xi), \psi_0(\xi)) \exp(\mu(t - t_0))
\]
for \( t \in [t_0, \infty) \).
3. Existence and stability of almost periodic solutions

In this section, we study the existence and stability of almost periodic solutions for (2.1). We introduce the symmetric matrix \( \Lambda = (\lambda_{ij})_{1 \leq i, j \leq n} \), where

\[
\lambda_{ij} = \begin{cases} 
-B_i^+ a_i + |c_{ii}^+| L_i a_i + \frac{1}{2} \sum_{k=1}^{n} |d_{ik}^+| M_{ik} & \text{for } i = j, \\
\frac{|c_{ii}^+| L_i a_i + |c_{jj}^+| L_j a_j}{2} & \text{for } i \neq j.
\end{cases}
\]

\( (s_p) \) is an arbitrary sequence of real numbers,

\[
B_i^- = \inf_{t \in \mathbb{R}} B_i(t), \quad c_{ij}^+ = \sup_{t \in \mathbb{R}} c_{ij}(t), \quad d_{ij}^+ = \sup_{t \in \mathbb{R}} d_{ij}(t),
\]

and \( \lambda_1 < 0 \) is the greatest eigenvalue of \( \Lambda \).

**Theorem 3.1.** Assume the following.

1. Conditions \((H_1) - (H_2)\) are fulfilled.
2. The inequalities

\[
\frac{a_i}{|1 + p_i|} < 1, \quad i = 1, 2, \ldots, n
\]

hold.
3. There exists a solution \( x(t; t_0, \varphi_0) \) of (2.1) such that

\[
\|x(t; t_0, \varphi_0)\| < C \quad \text{where} \quad t \geq t_0, \quad C > 0.
\]

If \( \lambda_2 < -2\lambda_1 \), where \( \lambda_2 := \max_{1 \leq i \leq n} (M_i a_i \sum_{j=1}^{n} |d_{ij}^+|) \), then (2.1) has a unique almost periodic solution \( \omega \) such that

a. \( \|\omega(t)\| \leq C_1, \quad C_1 < C \),

b. \( \mathcal{H}(\omega, t_0) \subset \mathcal{H}(2.1) \).

**Proof.** Let \( t_0 \in \mathbb{R} \) and \( \{s_p\} \) be any sequence of real numbers such that \( s_p \to \infty \) as \( p \to \infty \) and \( \{s_p\} \) moves (2.1) to a system at \( \mathcal{H}(2.1) \). We define a Lyapunov function

\[
V(x, y) = \frac{1}{2} \sum_{i=1}^{n} \left[ v_i(x_i, y_i) \right]^2,
\]

where

\[
v_i(x_1, x_2) = \text{sgn}(x_1 - x_2) \int_{x_2}^{x_1} \frac{d\eta}{a_i(\eta)}, \quad i = 1, 2, \ldots, n.
\]

In the next considerations, we will use that

\[
\left| \frac{x_1 - x_2}{a_i} \right| \leq v_i(x_1, x_2) \leq \frac{\left| x_1 - x_2 \right|}{a_i}.
\]

(3.2)

For any \( \alpha \in \mathbb{R} \), let \( p_0 = p_0(\alpha) \) be the smallest value of \( p \) such that \( s_{p_0} + \alpha \geq t_0 \). Since \( \|x(t; t_0, \varphi_0)\| \leq C_1, \quad C_1 < C \) for all \( t \geq t_0 \), we have

\[
\|x(t + s_p; t_0, \varphi_0)\| \leq C_1 \quad \text{for} \quad t \geq \alpha, \quad p \geq p_0.
\]

Let \( U \subset (\beta, \infty) \) be compact. Then, for any \( \varepsilon > 0 \), choose an integer \( n_0(\varepsilon, \beta) \geq p_0(\beta) \) so large that for \( l \geq p \geq n_0(\varepsilon, \beta) \) and \( t \in \mathbb{R}, \quad t \neq \tau_k(\varphi_0), \quad k \in \mathbb{Z} \), we have

\[
\sup_{-\varepsilon \leq \xi \leq 0} \left\{ \sum_{i=1}^{n} \left[ \text{sgn}(\varphi_{s_0}(\xi + s_p) - \varphi_{s_0}(\xi + s_1)) \int_{\varphi_{s_0}(\xi + s_1)}^{\varphi_{s_0}(\xi + s_p)} \frac{d\eta}{a_i(\eta)} \right]^2 \right\} < \varepsilon e^{-2\lambda_1 \frac{\alpha}{a_i} (t-t_0)}.
\]

(3.3)

Let \( t = \tau_k(\chi) \). Then, by (3.1) and Condition 2, we obtain

\[
v_i(x_i(t^+ + s_p), x_i(t^+ + s_1)) \leq \int_{x_i(t^+ + s_1)}^{x_i(t^+ + s_p)} \frac{d\eta}{a_i(\eta)} = \int_{x_i(t^+ + s_1)}^{x_i(t^+ + s_p)} \frac{d\eta}{a_i(\eta)}.
\]
Let now \( \eta = \xi (1 + P) \). From Condition 2 and (3.2), we get
\[
V_i(x_i(t^+ + s_p), x_i(t^+ + s_l)) \leq \left| \int_{x_i(t^+ + s)}^{x_i(t^+ + s + s_p)} \frac{1 + P}{a_i((1 + P)\xi)^2} d\xi \right|
\leq \frac{a_i}{a_i} |1 + P| V_i(x_i(t + s_p), x_i(t + s_l))
\leq V_i(x_i(t + s_p), x_i(t + s_l)).
\]

Then
\[
V(x(t^+ + s_p), x(t^+ + s_l)) = \frac{1}{2} \sum_{i=1}^{n} \left[ V_i(x_i(t^+ + s_p), x_i(t^+ + s_l)) \right]^2
\leq \frac{1}{2} \sum_{i=1}^{n} \left[ V_i(x_i(t + s_p), x_i(t + s_l)) \right]^2
= V(x(t + s_p), x(t + s_l)), \quad t = \tau_k(x).
\] (3.4)

Let \( t \geq t_0 \) and \( \tau_{k-1}(x) < t < \tau_k(x) \). Then, for the total derivative of \( V \), we have
\[
\frac{d}{dt} V(x(t + s_p), x(t + s_l)) = \sum_{i=1}^{n} V_i(x_i(t + s_p), x_i(t + s_l)) \dot{x}_i(x_i(t + s_p), x_i(t + s_l)).
\] (3.5)

On the other hand, from (3.1) and (2.1), we get
\[
\dot{v}_i(x_i(t + s_p), x_i(t + s_l)) = sgn(x_i(t + s_p) - x_i(t + s_l)) \left[ \frac{\dot{x}_i(t + s_p)}{a_i(x_i(t + s_p))} - \frac{\dot{x}_i(t + s_l)}{a_i(x_i(t + s_l))} \right]
= sgn(x_i(t + s_p) - x_i(t + s_l)) \left[ -(b_i(t + s_p, x_i(t + s_p)) - b_i(t + s_l, x_i(t + s_l))) \right.
+ \sum_{j=1}^{n} \left( c_{ij}(t + s_p) f_j(x_j(t + s_p)) - c_{ij}(t + s_l) f_j(x_j(t + s_l)) \right)
+ \sum_{j=1}^{n} d_{ij}(t + s_p) g_j(x_j(t + s_p))
- \sum_{j=1}^{n} d_{ij}(t + s_l) g_j(x_j(t + s_l)) + l_i(t + s_p) - l_i(t + s_l) \right].
\] (3.6)

Then, from (H2), (3.2), and the almost periodicity of \( b_i(\cdot, x) \), we get
\[
sgn(x_i(t + s_p) - x_i(t + s_l)) \left[ -(b_i(t + s_p, x_i(t + s_p)) - b_i(t + s_l, x_i(t + s_l))) \right]
\leq -(b_i(t + s_p, x_i(t + s_p)) - b_i(t + s_l, x_i(t + s_l))) sgn(x_i(t + s_p) - x_i(t + s_l))
+ (b_i(t + s_l, x_i(t + s_l)) - b_i(t + s_p, x_i(t + s_p))) sgn(x_i(t + s_p) - x_i(t + s_l))
\leq -B_i^0 \theta_i(x_i(t + s_p), x_i(t + s_l)) + \varepsilon.
\] (3.7)

Next, from (3.2), (H3), and (H4), we have
\[
sgn(x_i(t + s_p) - x_i(t + s_l)) \left\{ \sum_{j=1}^{n} c_{ij}(t + s_p) f_j(x_j(t + s_p)) - \sum_{j=1}^{n} c_{ij}(t + s_l) f_j(x_j(t + s_l)) \right\}
\leq \sum_{j=1}^{n} \left| c_{ij}(t + s_p) - c_{ij}(t + s_l) \right| \left| f_j(x_j(t + s_p)) \right|
+ \sum_{j=1}^{n} \left| c_{ij}(t + s_l) \right| \left| f_j(x_j(t + s_p)) - f_j(x_j(t + s_l)) \right|
\leq \varepsilon \sum_{j=1}^{n} H_j^{(1)} + \sum_{j=1}^{n} \left| c_{ij} \right| L_j \|v_j(x_j(t + s_p), x_j(t + s_l))\|.\] (3.8)
Analogously as above, we obtain

\[
\text{sgn}(x(t + s_p) - x(t + s_l)) \left\{ \sum_{j=1}^{n} d_{ij} (t + s_p) g_j (x_j(t + s_p - \theta_j(t + s_p))) - \sum_{j=1}^{n} d_{ij} (t + s_l) g_j (x_j(t + s_l - \theta_j(t + s_l))) \right\}
\]

\[
\leq \sum_{j=1}^{n} |d_{ij}(t + s_p) - d_{ij}(t + s_l)||g_j(x(t + s_p - \theta_j(t + s_p)))| + \sum_{j=1}^{n} |d_{ij}(t + s_l)||g_j(x(t + s_p - \theta_j(t + s_p))) - g_j(x(t + s_l - \theta_j(t + s_l)))| \leq \epsilon \sum_{j=1}^{n} H_j^{(2)}
\]

\[
+ \sum_{j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j(x_j(t + s_p - \theta_j(t + s_p)), x_j(t + s_l - \theta_j(t + s_l))).
\]

(3.9)

On the other hand, using the inequality \(2ab \leq a^2 + b^2\), we have

\[
\sum_{j=1}^{n} \sum_{j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j(x(t + s_p), x(t + s_l) + \sum_{j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j^2(x(t + s_p), x(t + s_l))
\]

\[
\leq \frac{1}{2} \sum_{i,j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j^2(x(t + s_p), x(t + s_l))
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j^2(x(t + s_p - \theta_j(t + s_p)), x(t + s_l - \theta_j(t + s_l))).
\]

(3.10)

Then, from (3.5)–(3.10) and the inequalities

\[v_i \leq (v_1^2 + v_2^2 + \ldots + v_n^2)^{\frac{1}{2}} \leq \sqrt{V},\]

we get

\[D^+ V(x(t + s_p), x(t + s_l))
\]

\[
\leq \sum_{i=1}^{n} \left( -B_i a_i + |c_{ii}^+| L_i \overline{a}_i + \frac{1}{2} \sum_{j=1}^{n} |d_{ij}||M_j\overline{a}_j v_j^2(x(t + s_p), x(t + s_l))
\]

\[
+ \sum_{i,j=1}^{n} |c_{ij}^+| L_i \overline{a}_j v_j(x(t + s_p), x(t + s_l) + v_j(x(t + s_p), x(t + s_l))
\]

\[
+ \max_{1 \leq i \leq n} \left( M_i \overline{a}_i \sum_{j=1}^{n} |d_{ij}|| \frac{1}{2} \sum_{i=1}^{n} v_i^2(x_i(t + s_p - \theta_i(t + s_p)), x_i(t + s_l - \theta_i(t + s_l)))
\]

\[
+ \epsilon \left( n + \sum_{j=1}^{n} (H_j^{(1)} + H_j^{(2)}) \right) \sum_{i=1}^{n} v_i(x_i(t + s_p), x_i(t + s_l))
\]

\[
\leq 2\lambda_1 V(x(t + s_p), x(t + s_l)) + \lambda_2 \sup_{t-v \leq s \leq t} V(x(s + s_p), x(s + s_l))
\]

\[
+ \epsilon H \sqrt{V(x(t + s_p), x(t + s_l)), \quad t \neq \tau_k(x)}.
\]

(3.11)

where

\[H = n + \sum_{j=1}^{n} (H_j^{(1)} + H_j^{(2)}).\]

From the above, whenever

\[V(x(\chi + s_p, \cdot), x(\chi + s_l, \cdot)) \leq V(x(t + s_p, \cdot), x(t + s_l, \cdot)), \quad t - v \leq \chi \leq t, \quad t \geq 0\]
using (3.4), (3.11), and Lemma 2.7, for $t \in [t_0, \infty)$, we get
\[
V(x(t + s_p), x(t + s_i)) \leq \left( \sup_{-\psi \leq \xi \leq 0} V(\psi_0(\xi + s_p), \psi_0(\xi + s_i)) + \frac{2\varepsilon H}{2\lambda_1 + \lambda_2} \right) e^{\frac{2\varepsilon H(\xi - t_0)}{2\lambda_1 + \lambda_2}} \leq \Gamma \varepsilon^2,
\]
where $\Gamma$ is a positive constant. Therefore,
\[
\|x(t + s_p) - x(t + s_i)\| \leq a \varepsilon^2 (x(t + s_p), x(t + s_i)) < a \varepsilon^2.
\]
where
\[
a := \max_{t \in \mathbb{R}} a_t.
\]
From the last inequality, there exists a function $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $x(t + s_p) - \omega(t) \to 0$ for $p \to \infty$, and since $\beta$ is arbitrary, it follows that $\omega(t)$ is defined uniformly on $t \in \mathbb{R}$. In the same way, we can prove that
\[
\lim_{p \to \infty} x(t + s_p)
\]
exists uniformly on all compact subsets of $\mathbb{R}$. In the sequel, we will consider the limit
\[
\lim_{p \to \infty} x(t + s_p) = \tilde{\omega}(t)
\]
and then
\[
\begin{align*}
\dot{\omega}(t) &= \lim_{p \to \infty} \left( X(t + s_p, x(t + s_p)) - X(t + s_p, \omega(t)) + X(t + s_p, \omega(t)) \right) \\
&= \mathcal{X}(t, \omega(t)), \quad t \neq t_0^i := \lim_{p \to \infty} (t_0^i + s_p).
\end{align*}
\]
(3.12)
Next, for $t = t_0^i$, we get
\[
\begin{align*}
\omega(t_0^i) - \tilde{\omega}(t_0^i) &= \lim_{p \to \infty} \left( x(t_0^i + s_p + 0) - x(t_0^i + s_p - 0) \right) \\
&= \lim_{p \to \infty} P(x(t_0^i + s_p)) = P(\omega(t_0^i)).
\end{align*}
\]
(3.13)
From (3.12) and (3.13), it follows that the function $\omega$ is a solution of (2.1).

Now we have to prove almost periodicity of $\omega$, and for this, we need the sequence $\{s_p\}$ moving (2.1) to $\mathcal{H}(2.1)$. We will use (3.1). Let us consider
\[
V(\omega(\sigma), \omega(\sigma + s_p - s_i)) = \frac{1}{2} \sum_{l=1}^{n} |v_l(\omega(\sigma), \omega(\sigma + s_p - s_i))|^2.
\]
Then, in the same way as above, we have
\[
V((\omega(t + s_p), \omega(t + s_i)) < \Gamma \varepsilon^2.
\]
From the last inequality, for $l \geq p \geq p_0(\varepsilon)$, we have
\[
\|\omega(t + s_p) - \omega(t + s_i)\| < a \varepsilon^2.
\]
(3.14)
On the other hand, from the definition of the sequence $\{s_p\}$, we obtain
\[
\rho(t_0^i + s_p, t_0^i + s_i) < \varepsilon \quad \text{for} \quad l \geq p \geq p_0(\varepsilon).
\]
Then, from (3.14), it follows that the sequence $\omega(t + s_p)$ converges uniformly to $\omega(t)$. The assertions a. and b. follow immediately, and the proof is complete. □

Remark 3.2. Cohen–Grossberg neural networks play a major role in numerous biological and engineering applications (especially in global pattern formation and partial memory storage), so there exists a great number of publications on this topic. Theorem 3.1 extends the existing results on almost periodicity in [34–37] considering impulsive perturbations. The proposed results in Theorem 3.1 also generalize the results in [38–42], taking into account variable impulsive perturbations. If the impulses are realized at fixed times, then the results in [38,40–42] can be received as corollaries of our result.

Next, we will investigate stability of almost periodic solutions of (2.1), and we introduce the following definition which is analogous to the definition given in [29].

Definition 3.3. The solution $\dot{x}$ of (2.1) with initial function $\dot{\varphi}_0 \in \mathcal{PC}$ is

a. equi-bounded if
\[
(\forall \lambda > 0) (\forall t_0 \in \mathbb{R}) (\exists \beta > 0) (\forall \dot{\varphi}_0 \in \mathcal{PC} : \|\dot{\varphi}_0\|_\psi < \lambda) (\forall t \geq t_0): \|\dot{x}(t)\| < \beta.
\]
b. exponentially stable if for $\tilde{\phi}_0^1 \in \mathbb{C}$, $\|\tilde{\phi}_0 - \tilde{\phi}_0^1\|_v < \lambda (\lambda > 0)$, and $t_0 \in \mathbb{R}$, there exists a constant $c > 0$ such that

$$
\left\| \tilde{x}(t; t_0, \tilde{\phi}_0) - \tilde{x}^s(t; t_0, \tilde{\phi}_0^1) \right\| \leq m(\tilde{\phi}_0 - \tilde{\phi}_0^1) \exp(-c(t - t_0)), \quad t \geq t_0,
$$

where $m(0) = 0$, $m(\varphi) \geq 0$, and $m(\varphi)$ is Lipschitzian with respect to $\varphi \in \mathbb{C}$, $\|\varphi\|_v < \lambda$.

c. perfectly exponentially stable if it is exponentially stable and the number $\beta$ in a. is independent of $t_0 \in \mathbb{R}$.

**Theorem 3.4.** Assume that the conditions of Theorem 3.1 hold. Then the almost periodic solution of (2.1) is perfectly exponentially stable.

**Proof.** Let $\tilde{\omega}$ be an arbitrary solution of (2.3). Set

$$
\dot{\tilde{\omega}}(t) = \tilde{\omega}(t) - \omega(t),
$$

$$
X^s(t, \tilde{\omega}(t)) = X^s(t, \tilde{\omega}(t) + \omega(t)) - X^s(t, \omega(t)).
$$

Now, we consider the system

$$
\begin{aligned}
\frac{d}{dt} \tilde{\omega} &= X^s(t, \tilde{\omega}(t)), \quad t \neq \tau_k^s(\tilde{\omega}), \\
\tilde{\omega}(t^+) &= \tilde{\omega}(t) + \bar{P} \tilde{\omega}(t), \quad t = \tau_k^s(\tilde{\omega}), \quad k \in \mathbb{Z},
\end{aligned}
$$

(3.15)

and let $W(t, \tilde{\omega}) = V(t, \omega(t), \omega(t) + \tilde{\omega})$. Then, from Lemma 2.7, it follows that the zero solution $\tilde{\omega}(t) \equiv 0$ of (3.15) is perfectly exponentially stable, and consequently $\omega$ is perfectly exponentially stable. □

**Remark 3.5.** In previous stability results for CGNNs [4,24–26,38], the authors investigated the behavior of equilibria and periodic solutions. Theorem 3.4 generalizes the existing stability results considering almost periodic solutions, which are more realistic and general. Indeed, the concept of almost periodicity includes as special cases these of periodic and constant solutions. Hence, the results of this paper enrich and complement the earlier publications.

4. Uncertain CGNNs

In this section, we study the existence and robust stability of almost periodic solutions for uncertain CGNNs with time-varying delay and variable impulsive perturbations. We consider the uncertain CGNN with time-varying delay and variable impulsive perturbations corresponding to (2.1) given by

$$
\begin{aligned}
\dot{x}_i(t) &= -a_i(x_i(t)) + \bar{a}_i(x_i(t)) \left( b_i(t, x_i(t)) + \bar{b}_i(t, x_i(t)) \right) \\
&\quad - \sum_{j=1}^n \left( c_{ij}(t) + \bar{c}_{ij}(t) \right) (f_j(x_j(t)) + \bar{f}_j(x_j(t))) \\
&\quad - \sum_{j=1}^n \left( d_{ij}(t) + \bar{d}_{ij}(t) \right) (g_j(x_j(t - \theta_j(t))) + \bar{g}_j(x_j(t - \theta_j(t)))) \\
&\quad - \bar{l}_i(t) - \bar{l}_i(t), \quad t \neq \tau_k(x(t)), \\
x_i(t^+) &= x_i(t) + \bar{P} x_i(t) + \bar{P} x_i(t), \quad t = \tau_k(x(t)), \\
i = 1, \ldots, n, \quad k \in \mathbb{Z},
\end{aligned}
$$

(4.1)

where $\bar{a}_i, \bar{b}_i, \bar{c}_{ij}, \bar{d}_{ij}, \bar{f}_j, \bar{g}_j, \bar{l}_i \in \mathbb{C}(\mathbb{R}, \mathbb{R})$, $\bar{P}_i \in \mathbb{R}$, $i, j = 1, \ldots, n$, represent the uncertainty of the system [5]. In the case when all of these functions and constants are zeros, we recover the “nominal system” (2.1).

**Definition 4.1.** The almost periodic solutions of (2.1) are said to be perfectly robustly exponentially stable if for any functions $\bar{a}_i, \bar{b}_i, \bar{c}_{ij}, \bar{d}_{ij}, \bar{f}_j, \bar{g}_j, \bar{l}_i$, $i, j = 1, \ldots, n$, the almost periodic solutions of (4.1) are perfectly exponentially stable.

For the next theorem, we need the following conditions.

$(\bar{H}_1)$ For $a_i^+ = \sup_{x \in \mathbb{R}} a_i(x)$, $i = 1, 2, \ldots, n$, we have

$$
a_i^+ \in [\bar{a}_i - \underline{a}_i, \bar{a}_i - \underline{a}_i].
$$

$(\bar{H}_2)$ The functions $\bar{b}_i$, $i = 1, 2, \ldots, n$, are continuous in the second variable, almost periodic in the sense of Bohr in the first variable and uniformly in the second variable, and

$$
\frac{b_i(t, x) + \bar{b}_i(t, x) - (b_i(t, y) + \bar{b}_i(t, y))}{x - y} \geq \bar{b}_i(t) \quad \text{for} \quad x, y \in \mathbb{R}, \quad x \neq y.
$$

$(\bar{H}_3)$ The functions $\bar{c}_{ij}, \bar{l}_i, \bar{d}_{ij}, i, j = 1, 2, \ldots, n$, are almost periodic in the sense of Bohr, and $\sup_{t \in \mathbb{R}} \bar{d}_{ij}(t) = \bar{d}_{ij}$. 
(H₄) There exist positive constants $\tilde{L}_i, \tilde{M}_i, \tilde{H}^{(1)}_i, \tilde{H}^{(2)}_i$, $i = 1, 2, \ldots, n$, with
\[
|\tilde{f}_i(x_1(t)) - \tilde{f}_i(x_2(t))| \leq \tilde{L}_i |x_1(t) - x_2(t)|, \quad |\tilde{g}_i(x_1(t)) - \tilde{g}_i(x_2(t))| \leq \tilde{M}_i |x_1(t) - x_2(t)|,
\]
\[
|\tilde{f}_i(x(t))| \leq \tilde{H}^{(1)}_i, \quad |\tilde{g}_i(x(t))| \leq \tilde{H}^{(2)}_i
\]
for all $x, x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, and $\tilde{f}_i(0) = \tilde{g}_i(0) = 0$.

(\text{H}_5) The unknown constants $\tilde{P}_i$, $i = 1, 2, \ldots, n$, are such that
\[
\tilde{P}_i \in [-P, -P] \quad \text{and} \quad \tilde{a}_i |1 + P + \tilde{P}_i| < \tilde{a}_i.
\]

We will use the matrix $\hat{\lambda} = (\hat{\lambda}_{ij})_{1 \leq i, j \leq n}$, where
\[
\hat{\lambda}_{ij} = \begin{cases}
-B_i^{-1} \bar{d}_i + \sum_{k=1}^{n} d_k^+ \bar{d}_k^+ (M_k + \bar{M}_k) \sum_{j=1}^{n} |x_j^+| & \text{for } i = j, \\
\frac{1}{2} |c_{ii}^+(L_i + \bar{L}_i)\bar{a}_i + |c_{jj}^+(L_j + \bar{L}_j)\bar{a}_j| & \text{for } i \neq j,
\end{cases}
\]
\{(s_p)\} is an arbitrary sequence of real numbers, and $\hat{\lambda}_1 < 0$ is the greatest eigenvalue of $\hat{\lambda}$.

**Theorem 4.2.** Assume the following.

1. The conditions of Theorem 3.1 hold.
2. Conditions (H₁)–(H₅) are fulfilled.

If $\hat{\lambda}_2 < -2\hat{\lambda}_1$, where $\hat{\lambda}_2 := \max_{1 \leq i \leq n} ((M_i + \bar{M}_i)\bar{a}_i \sum_{j=1}^{n} |x_{ij}^+|)$, then the almost periodic solutions of (2.1) are perfectly robustly exponentially stable.

**Proof.** The fact that (4.1) has a perfectly exponentially stable almost periodic solution follows in the same way as in the proofs of Theorem 3.1 and Theorem 3.4. Then, from Definition 4.1, it follows that this solution is perfectly robustly exponentially stable. \(\square\)

**Remark 4.3.** Due to modeling inaccuracies, environment noise, or external disturbance, parameters are often fluctuating within some scopes in engineering applications, and parametric uncertainty often breaks the stability of models. Thus, it is meaningful to consider parametric uncertainty in the dynamic analysis of CGNNs. Theorem 4.2 extends the existing results on almost periodic solutions of delayed impulsive CGNNs [40–42] to the uncertain case which is another contribution to the theory. Also, as far as we know, there are few results about robust stability of such neural networks [54–56]. But these papers investigated zero solutions or equilibria. Since we consider almost periodic solutions and variable impulsive perturbations in our robust stability analysis, the results of this paper enrich and complement the earlier publications.

5. An example

In order to illustrate the feasibility of our established criteria in the preceding sections, we provide a concrete example. Consider the 2-D impulsive Cohen–Grossberg-type neural network with time-varying delays
\[
\begin{align*}
\dot{x}_1(t) &= -a_1(x_1(t)) \left[ b_1(t, x_1(t)) - \sum_{j=1}^{2} c_{1j}(t) f_j(x_j(t)) \right. \\
& \quad \left. - \sum_{j=1}^{2} d_{1j}(t) g_j(x_j(t) - \theta_j(t)) \right] - l_1(t), \quad t \neq \tau_k(x(t)), \\
\Delta x_1(t) &= \begin{pmatrix} -1/4 & 0 \\ 0 & -2/3 \end{pmatrix} x(t), \quad t = \tau_k(x(t)), \\
\dot{x}_2(t) &= -a_2(x_2(t)) \left[ b_2(t, x_2(t)) - x_2(t) \right] - l_2(t), \quad t \neq \tau_k(x(t)), \\
\Delta x_2(t) &= \begin{pmatrix} 0 & 1 \\ -1/3 & 0 \end{pmatrix} x(t), \quad t = \tau_k(x(t)),
\end{align*}
\]
where
\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad l_1(t) = \cos t, \quad l_2(t) = \sin t,
\]
\[
a_1(x) = 3 + 0.2 \sin(x\sqrt{2}), \quad a_2(x) = 4 - 0.1 \cos(x\sqrt{3}),
\]
\[
b_1(t, x) = b_2(t, x) = x(3 + \sin(t\sqrt{2})),
\]
\[
f_j(x) = g_j(x) = \frac{|x + 1| - |x - 1|}{2}, \quad 0 \leq \theta_j(t) \leq 1,
\]
\[
c_{11}(t) = 0.8 - 0.2 \sin(t\sqrt{3}), \quad c_{12}(t) = 0.1 - 0.1 \cos(t\sqrt{3}) - 0.3 \cos t.
\]
c_{21}(t) = 0.4 - 0.1 \cos(t \sqrt{2}) - 0.1 \cos t, \quad c_{22}(t) = 0.2 - 0.3 \sin(t \sqrt{2}),

d_{31}(t) = 0.2 \sin(t \sqrt{3}), \quad d_{12}(t) = 0.3 \cos t + 0.1 \cos(t \sqrt{3}),

d_{23}(t) = 0.1 \cos t + 0.1 \cos(t \sqrt{2}), \quad d_{22}(t) = 0.3 \sin(t \sqrt{2}).

the functions \( \tau_k(x) = 6 \arctan(x) + 3\pi + 20k, \ k = 1, 2, \ldots \) are continuous and satisfy

\[ t_0 < \tau_1(x) < \tau_2(x) < \ldots < \tau_k(x) \to \infty \ \text{as} \ k \to \infty \ \text{uniformly on} \ \mathbb{R}^2, \]

and the impulsive moments are such that \( t_k \in B, \) the set of sequences \( \{t_{kh}\}, \ t_{kh} = t_k + \bar{l} - t_k, \ k, l \in \mathbb{Z}, \) is uniformly almost periodic, \( \inf_k t_{kh}^1 = \theta = 1. \) Hence, one gets

\[ a_1 = 2.8, \quad \bar{a}_1 = 3.2, \quad a_2 = 3.9, \quad \bar{a}_2 = 4.1, \]

\[ B_1 = B_2 = 4, \quad L_1 = L_2 = M_1 = M_2 = 1, \]

and

\[ \Lambda = (\lambda_{ij})_{2 \times 2} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} -7.7 & 1.985 \\ 1.985 & -13.3 \end{pmatrix}. \]

It is easy to verify that the conditions of Theorem 3.1 are satisfied for \( \lambda_2 = 4.51. \) We also have that the constants \( P_i, \ i = 1, 2, \) are such that

\[ \frac{\bar{a}_1}{a_1} |1 + P_1| = \frac{9.6}{11.2} < 1, \quad \frac{\bar{a}_2}{a_2} |1 + P_2| = \frac{41}{11.7} < 1. \]

Since \( 4.51 = \lambda_2 < -2 \lambda_1 = 14.1356, \) according to Theorem 3.1, there exists a unique almost periodic solution \( \omega(t) \) of (5.1), and according to Theorem 3.4, the almost periodic solution \( \omega(t) \) of (5.1) is perfectly exponentially stable. Computer simulations of the perfectly exponentially stable behavior of the almost periodic solution are given in Figs. 1 and 2.
Now, we consider the uncertain CGNN system

\[
\begin{align*}
\dot{x}_i(t) &= - (a_i(x_i(t)) + \tilde{a}_i(x_i(t))) \left[ b_i(t, x_i(t)) + \tilde{b}_i(t, x_i(t)) \right] \\
&\quad - \sum_{j=1}^{n} \left( c_{ij}(t) + \tilde{c}_{ij}(t) \right) \left( f_j(x_j(t)) + \tilde{f}_j(x_j(t)) \right) \\
&\quad - \sum_{j=1}^{n} \left( d_{ij}(t) + \tilde{d}_{ij}(t) \right) \left( g_j(x_j(t - \theta_j(t))) + \tilde{g}_j(x_j(t - \theta_j(t))) \right) \\
&\quad - \tilde{l}_i(t) - \tilde{l}_i(t), \quad t \neq \tau_k(x(t)), \\
x_i(t^+) &= x_i(t) + \tilde{P}_i x_i(t) + \tilde{P}_i x_i(t), \quad t = \tau_k(x(t)),
\end{align*}
\]

(5.2)

where the functions $\tilde{a}_i, \tilde{b}_i, \tilde{c}_{ij}, \tilde{d}_{ij}, \tilde{f}_j = \tilde{g}_j, \tilde{l}_i$ and the constants $\tilde{P}_i, i, j = 1, 2,$ represent the uncertainty of the system. If the uncertain functions are bounded so that all conditions of Theorem 4.2 are satisfied and the uncertain constants $\tilde{P}_i$ are such that

\[
2.8 \left| \frac{3}{4} + \tilde{P}_i \right| < 3.2, \quad 3.9 \left| \frac{2}{3} + \tilde{P}_i \right| < 4.1,
\]

then the almost periodic solutions of (5.2) are perfectly robustly exponentially stable.

**Remark 5.1.** The considered example demonstrates the effectiveness of our results. In addition, we show that Theorem 4.2 presents criteria under which, if the uncertain terms are variable but bounded, then the model is capable to generate perfectly robustly exponentially stable almost periodic processes.

6. Concluding remarks

In this paper, the existence of almost periodic processes in delayed CGNNs with variable impulsive perturbations is investigated. The effect of uncertain parameters is also considered. With this research, we extend and improve some existing results for almost periodic solutions of CGNNs to the uncertain case [40–42]. In addition, compared with all previous works on uncertain CGNNs (see, for example, [54–56]), we consider almost periodic solutions and variable impulsive perturbations in our robust stability analysis.

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References


