PERGAMON

Applied Mathematics Letters 12 (1999) 101–106

Applied Mathematics Letters

A Sturmian Theorem for Recessive Solutions of Linear Hamiltonian Difference Systems

M. Bohner
Department of Mathematics, San Diego State University
5500 Campanile Drive, San Diego, CA 92182-7720, U.S.A.
bohner@saturn.sdsu.edu

O. Došlý
Department of Mathematics, Masaryk University
Janáčkovo Nám. 2a, CZ-66295 Brno, Czech Republic
dosly@math.muni.cz

W. Kratz
Abteilung Mathematik V, Universität Ulm
Helmholtzstr. 18, D-89069 Ulm, Germany
kratz@mathematik.uni-ulm.de

(Received and accepted November 1997)

Communicated by R. P. Agarwal

Abstract—The main result of the paper is a Sturmian-type separation theorem for the recessive solutions of linear Hamiltonian difference systems. The assumptions on the Hamiltonian systems allow to include the study of Sturm-Liouville difference equations of higher order. As an application of the main result, some Sturmian-type comparison results are obtained.

Keywords—Discrete Hamiltonian system, Conjoined basis, Focal point, Recessive solution, Sturmian theorem.

1. INTRODUCTION

We consider the Linear Hamiltonian Difference System (LHDS)

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^\top u_k \tag{H}$$

with $\Delta x_k := x_{k+1} - x_k$, where A_k, B_k, C_k are real $n \times n$-matrices, B_k and C_k are symmetric, and where $I - A_k$ is invertible for $k \in \mathbb{Z}$. In the last years, considerable efforts have been made to find analogies between oscillation properties of solutions of the difference system (H) and of its continuous counterpart

$$\dot{x} = A(t)x + B(t)u, \qquad \dot{u} = C(t)x - A^\top(t)u, \tag{C}$$

The author is greatful to the Alexander von Humbold Foundation for supporting this work.

see [1] and the references in it. An important step was made in the papers [2,3], where the so-called Reid Roundabout Theorem and the Sturmian-Type Theory are derived for the LHDS without assuming that the matrices B_k are invertible. Similar statements for (H) with positive definite (in particular invertible) B_k's were proven in [4–6]. But note that systems with invertible B_k's do not cover important systems and equations like the higher-order Sturm-Liouville equations.

In this paper, we derive a certain extremal property of the so-called *recessive solutions* of (H), which is known in the continuous case as the zero-maximality of the *principle solutions* of (C) (at ∞). This result states, under suitable conditions, the following. If a principle solution of (C) has a focal point in some interval $[T, \infty)$, then every conjoined basis of (C) must also have a focal point in that interval. In this paper, we show that the recessive solutions of (H) have a similar extremal property. In our discrete case, the situation is more complicated because of the fact that in addition to singularities one has to also consider "sign changes". Therefore, we have to use methods which are different from the continuous ones.

2. NOTATION AND AUXILIARY RESULTS

First, we recall basic concepts and notation that is needed in the formulation of our main result. For a symmetric (and real) matrix D the inequality $D \geq 0$ ($D > 0$) means that D is nonnegative (positive) definite. By $V^{\dagger}$ we denote the Moore-Penrose generalized inverse of a matrix V. Ker and Im denote the kernel and the image of the matrix indicated, respectively. The symbol Δ stands for the usual forward difference operator, i.e., $\Delta w_k = w_{k+1} - w_k$.

We consider together with (H) its matrix version (again denoted by (H)), namely

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^{\top} U_k, \tag{H}$$

where a solution (X, U) consists of real $n \times n$-matrices X_k, U_k, and where we assume throughout that A_k, B_k, C_k are real $n \times n$-matrices satisfying

$$B_k, C_k \text{ are symmetric}, \qquad I - A_k \text{ is invertible for all } k, \tag{A}$$

we put $\tilde{A}_k = (I - A_k)^{-1}$. Similarly, as in the continuous case, a solution (X, U) of (H) is said to be a *conjoined basis* of (H) if, for $k \in \mathbb{Z}$, $X_k^{\top} U_k \equiv U_k^{\top} X_k$ and rank $(X_k^{\top}, U_k^{\top}) \equiv n$. For a conjoined basis (X, U) of (H) we say that an interval $(k, k+1]$ contains a *focal point* of X or of (X, U) [2, Definition 3] or [7, Definition 3] if

$$\operatorname{Ker} X_{k+1} \not\subset \operatorname{Ker} X_k \quad \text{or} \quad D_k := X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \not\geq 0. \tag{1}$$

Note that if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ (which trivially happens if X_{k+1} is invertible), then the matrix D_k is symmetric (see, e.g., [2, Theorem 1]). The second condition in (1) may be interpreted as a "sign change" which is mentioned in the introduction. The LHDS (H) is said to be *disconjugate* on an (integer) interval $[k, l] \cap \mathbb{Z}$ if the solution (X, U) of (H) with the initial conditions $X_k = 0$, $U_k = I$ has no focal point in $(k, l+1]$ (see [2, Definition 4 and Theorem 2]). We say that (H) is *eventually disconjugate* if there exists $M \in \mathbb{N}$ such that (H) is disconjugate on $[M, \infty)$ (i.e., on $[M, l] \cap \mathbb{Z}$ for all $l > M$), and (H) is said to be *eventually controllable* (or controllable for large k (see [7, Definition 5]) if there exist $M, \kappa \in \mathbb{N}$ such that the only vector solution of (H) with $x_m = \cdots = x_{m+\kappa} = 0$ for some $m \geq M$ is the trivial solution $x = u = 0$ (see [7, Section 4]).

Next, if (X, U) is any conjoined basis of (H) such that the matrices X_k are invertible on some interval $[N, \infty) \cap \mathbb{Z}$, then every solution $(\bar{X}, \bar{U})$ of (H) can be expressed in the form

$$\begin{gathered} \bar{X}_k = X_k (S_k K + L), \qquad \bar{U}_k = U_k (S_k K + L) + X_k^{\top -1} K, \\ \text{where } S_k = \sum_{j=N}^{k-1} X_{j+1}^{-1} \tilde{A}_j B_j X_j^{\top -1} = \sum_{j=N}^{k-1} X_j^{-1} D_j X_j^{\top -1} \end{gathered} \tag{2}$$

for $k \geq N$, with $D_j = X_j X_{j+1}^{-1} \tilde{A}_j B_j$ as in (1), and where K and L are constant matrices, more precisely

$$L = X_m^{-1} \bar{X}_M, \qquad K \equiv X_j^\top \bar{U}_j - U_j^\top \bar{X}_j.$$

One may verify directly that $(\bar{X}, \bar{U})$ given by (2) solves (H) (see [5, Propostion 2.2]), and it is a conjoined basis if and only if $K^\top L = L^\top K$ and rank $(K^\top, L^\top) = n$.

Now we summarize properties of recessive solutions of (H). A conjoined basis $(\tilde{X}, \tilde{U})$ of (H) is said to be a *recessive solution* (at ∞) if there exists $M \in \mathbb{Z}$ and another conjoined basis (X, U) such that $\tilde{X}_k$ and X_k are invertible for all $k \geq M$, $X_k^\top \tilde{U}_k - U_k^\top \tilde{X}_k$ is invertible, and such that $\lim_{k\to\infty} X_k^{-1} \tilde{X}_k = 0$ (see [7, Section 4] or [1, Chapter 5]). If the LHDS (H) is eventually disconjugate and eventually controllable, then, as is shown in [7, Section 4], a conjoined basis $(\tilde{X}, \tilde{U})$ is a recessive solution if and only if there exists $N \in \mathbb{Z}$ such that $\tilde{X}_k$ is invertible, $\tilde{D}_k := \tilde{X}_k \tilde{X}_{k+1}^{-1} \tilde{A}_k B_k \geq 0$ for all $k \geq N$, $\tilde{S}_k := \sum_{j=N}^{k-1} \tilde{X}_j^{-1} \tilde{D}_j^{\top -1} \tilde{X}_j^{\top -1}$ is invertible for large k, and such that $\lim_{k\to\infty} \tilde{S}_k^{-1} = 0$ (or equivalently, $\lim_{k\to\infty} \tilde{S}_k = \infty$, i.e., all eigenvalues tend to infinity as $k \to \infty$). Moreover, a recessive solution exists under these assumptions, and it is unique up to an invertible right factor according to the following lemma. For a special case of this result see [1, Chapter 5].

LEMMA 1. *Assume (A), suppose that (H) is eventually controllable and eventually disconjugate, and let (X,U) and $(\tilde{X}, \tilde{U})$ be recessive solutions of (H). Then there exists an invertible matrix C such that*

$$\tilde{X}_k = X_k C \quad \textit{and} \quad \tilde{U}_k = U_k C, \qquad \textit{for all } k \in \mathbb{Z}.$$

PROOF. By our assumptions and the discussion above, there exists $N \in \mathbb{Z}$ such that

$$X_k \text{ and } \tilde{X}_k \text{ are invertible, } D_k \geq 0,\ \tilde{D}_k \geq 0,$$

for all $k \geq N$, and such that

$$\lim_{k\to\infty} S_k^{-1} = \lim_{k\to\infty} \tilde{S}_k^{-1} = 0,$$

where $D_k, \tilde{D}_k, S_k, \tilde{S}_k$ are defined accordingly. By the representation formulae (2) we have that

$$\begin{aligned} \tilde{X}_k &= X_k (S_k K + L), & \tilde{U}_k &= U_k (S_k K + L) + X_k^{\top -1} K, \\ X_k &= \tilde{X}_k \left(\tilde{S}_k \tilde{K} + \tilde{L}\right), & U_k &= \tilde{U}_k \left(\tilde{S}_k \tilde{K} + \tilde{L}\right) + \tilde{X}_k^{\top -1} \tilde{K}, \end{aligned}$$

for $k \geq N$, where

$$L = X_N^{-1} \tilde{X}_N = \tilde{L}^{-1}, \qquad K = -\tilde{K}^\top, \qquad L^\top K = K^\top L.$$

Hence, $R = KL^{-1}$ is symmetric. Since $S_k^{-1} \to 0+$ as $k \to \infty$, it follows from [8, Theorem 1] that

$$\lim_{k\to\infty} R \left(R + S_k^{-1}\right)^{-1} S_k^{-1} = 0.$$

Moreover, the representation formulae for X_k and $\tilde{X}_k$ imply that

$$R \left(R + S_k^{-1}\right)^{-1} S_k^{-1} = K (S_k K + L)^{-1} = K \tilde{X}_k^{-1} X_k = K \left(\tilde{S}_k \tilde{K} + \tilde{L}\right) = -\tilde{K}^\top \tilde{L} - \tilde{K}^\top \tilde{S}_k \tilde{K} \to 0.$$

Since $\lim_{k\to\infty} \tilde{S}_k = \infty$, we can conclude that $\tilde{K} = 0 = K$. Hence, $\tilde{X}_k = X_k C$, $\tilde{U}_k = U_k C$ with the invertible matrix $C = L$. ∎

Linear Hamiltonian systems are closely related to Riccati matrix equations, namely: if (X, U) is a solution of (H) such that X_k is invertible for $k \in [M, N+1] \cap \mathbb{Z}$, then the quotient $Q_k := U_k X_k^{-1}$ solves the Riccati matrix difference equation

$$Q_{k+1} = C_k + \left(I - A_k^\top\right) Q_k (I + B_k Q_k)^{-1} (I - A_k), \tag{R}$$

for $M \le k \le N$. Moreover, Q_k is symmetric, and $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k = (I + B_k Q_k)^{-1} B_k$. If (H) is eventually controllable and eventually disconjugate and if $(\tilde{X}, \tilde{U})$ is a recessive solution, then the quotient $Q_k^- := \tilde{Q}_k = \tilde{U}_k \tilde{X}_k^{-1}$ is unique (the invertible right factor cancels out!), and it is called the *eventually minimal* (or distinguished) solution of (R) (at ∞). The terminology eventually minimal is justified by the fact that any other symmetric solution Q of (R), which exists for $k \ge l$ with some l, satisfies $Q_k \ge Q_k^-$ for all $k \ge m$ with $m \ge l$ sufficiently large (see [7, Section 4]). Finally, we cite the following lemma [9, Lemma 7] which we will use.

LEMMA 2. *Assume (A) and suppose that $Q_k, \tilde{Q}_k, Q_{k+1}, \tilde{Q}_{k+1}$ are symmetric and satisfy (R) for some fixed $k \in \mathbb{Z}$ with $(I + B_k Q_k)^{-1} B_k \ge 0$. Then $Q_{k+1} \ge \tilde{Q}_{k+1}$ implies $Q_k \ge \tilde{Q}_k$.*

3. MAIN RESULT

Now we can formulate the main result of this paper, a Sturmian-type separation theorem for recessive solutions.

THEOREM 1. *Assume (A), and suppose that (H) is eventually controllable and eventually disconjugate. Let $(\tilde{X}, \tilde{U})$ be a recessive solution of (H) such that $\tilde{X}_k$ is invertible and $\tilde{D}_K := \tilde{X}_k \tilde{X}_{k+1}^{-1} \tilde{A}_k B_k \ge 0$ for $k \ge N+1$. If either $\tilde{X}_N$ is singular or $\tilde{D}_N \not\ge 0$, then, for every conjoined basis (X, U) of (H), either X_k is singular or $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k \not\ge 0$ for some $k \ge N$.*

PROOF. Suppose, by contradiction, that X_k is invertible and that $D_k \ge 0$ for all $k \ge N$. Since $\tilde{Q}_k = \tilde{U}_k \tilde{X}_k^{-1}$ is the eventually minimal solution of (R), we get that $\tilde{Q}_k \le Q_k$ for all sufficiently large k. Hence, by Lemma 2, $\tilde{Q}_k \le Q_k$ for all $k \ge N+1$. Therefore,

$$\tilde{D}_N - D_N = B_N \tilde{A}_N^{\top} \left(Q_{N+1} - \tilde{Q}_{N+1} \right) \tilde{A}_N B_N \ge 0,$$

and this implies $\tilde{D}_N \ge 0$ because $D_N \ge 0$.

We put

$$\bar{X}_k = X_k (I + S_k), \qquad \bar{U}_k = U_k (I + S_k) + X_k^{\top -1} \tag{3}$$

with $S_k = \sum_{j=N}^{k-1} X_{j+1} \tilde{A}_j B_j X_j^{\top -1}$ for $k \ge N$. Then, by (2) with $K = L = I$, $(\bar{X}, \bar{U})$ is a conjoined basis of (H). Since $S_{k+1} - S_k = X_k^{-1} D_k X_k^{\top -1} \ge 0$ for $k \ge N$, $S_N = 0$, we have that $S_k \ge 0$, and therefore, $\bar{X}_k$ is invertible for $k \ge N$. Hence, again by (2) (use that $\bar{X}_j^{\top} U_j - \bar{U}_j^{\top} X_j \equiv -I$), we have that

$$X_k = \bar{X}_k \left(I - \bar{S}_k \right), \qquad U_k = \bar{U}_k \left(I - \bar{S}_k \right) - \bar{X}_k^{\top -1}, \tag{4}$$

for $k \ge N$, where $\bar{S}_k = \sum_{j=N}^{k-1} \bar{X}_{j+1}^{-1} \tilde{A}_j B_j \bar{X}_j^{\top -1}$. It follows from (3) and (4) that

$$\left(I - \bar{S}_k \right) \left(I + S_k \right) = I,$$

so that

$$I - \bar{S}_k = (I + S_k)^{-1} \ge (I + S_{k+1})^{-1} = I - \bar{S}_{k+1}, \qquad \text{for } k \ge N.$$

Thus, $0 \le \bar{S}_k \le \bar{S}_{k+1} = I - (I + S_{k+1})^{-1} \le I, \bar{D}_k \ge 0$ for $k \ge N$, and therefore, the limit $\bar{S}_\infty := \lim_{k \to \infty} \bar{S}_k$ exists with $0 \le \bar{S}_\infty \le I$. Moreover, $\bar{S}_\infty > 0$, because otherwise $\bar{S}_k c = 0$ for all $k \ge N$ with some $c \in \mathbb{R}^n \setminus \{0\}$. Then, (x, u) with $x_k = \bar{X}_k \bar{S}_k c$ and $u_k = (\bar{U}_k \bar{S}_k + \bar{X}_k^{\top -1}) c$ is a nontrivial solution of (H) (again by (2)) with $x_k = 0$ for all $k \ge N$, contradicting our assumption that (H) is eventually controllable. Now, using (2) again, we consider the conjoined basis $(\hat{X}, \hat{U})$ defined by

$$\hat{X}_k = \bar{X}_k \left(\bar{S}_\infty - \bar{S}_k \right), \qquad \hat{U}_k = \bar{U}_k \left(\bar{S}_\infty - \bar{S}_k \right) - \bar{X}_k^{\top -1} \tag{5}$$

for $k \ge N$. Because (H) is eventually controllable, it follows with the same argument as above that there exists $M \ge N$ such that $\hat{X}_k$ is invertible for all $k \ge M$. Therefore, for $k \ge M$,

$$\bar{X}_k = \hat{X}_k \left(\hat{S}_k + L \right), \qquad \bar{U}_k = \hat{U}_k \left(\hat{S}_k + L \right) + \hat{X}_k^{\top -1}, \tag{6}$$

where $L = \hat{X}_M^{-1}\bar{X}_M$ and $\hat{S}_k = \sum_{M+1}^{k-1} \hat{X}_{j+1}^{-1}\tilde{A}_j B_j \hat{X}_j^{\top -1}$. By (5) and (6),

$$\hat{S}_k + L = \left(\bar{S}_\infty - \bar{S}_k\right)^{-1} \to \infty, \qquad \text{as } k \to \infty,$$

and $\hat{S}_{k+1} - \hat{S}_k \geq 0$. Hence, for $k \geq M, \hat{X}_k$ is invertible, $\hat{D}_k \geq 0$, and we have that $\hat{S}_k^{-1} \to 0$ so that $(\hat{X}, \hat{U})$ is a recessive solution of (H). Thus, by Lemma 1, $\tilde{X}_k = \hat{X}_k C$ and $\tilde{U}_k = \hat{U}_k C$ for an invertible matrix C. Now, for $k = N$, (5) yields that $\tilde{X}_N = \bar{X}_N \bar{S}_\infty C$ is invertible (use that $\bar{S}_\infty > 0$). Altogether we have shown that $\tilde{X}_N$ is invertible and that $\tilde{D}_N \geq 0$ which contradicts our assumption so that the proof is complete. Note that $(\bar{X}, \bar{U})$ is a so-called *dominant solution* of (H) (see [1;7, Section 4]). ∎

4. APPLICATIONS

By using inequalities for Riccati difference equations given in [9] we obtain from our Theorem 1 the following "comparison results". We consider another LHDS (and denote the corresponding Riccati equation by $(\hat{\text{R}})$)

$$\Delta X_k = \hat{A}_k X_{k+1} + \hat{B}_k U_k, \qquad \Delta U_k = \hat{C}_k X_{k+1} - \hat{A}_k^\top U_k, \tag{$\hat{\text{H}}$}$$

where $\hat{A}_k, \hat{B}_k, \hat{C}_k$ satisfy the same assumptions (A) as A_k, B_k, C_k in (H). Moreover, we denote

$$\mathcal{H}_k := \begin{pmatrix} -C_k & A_k^\top \\ A_k & B_k \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{H}}_k := \begin{pmatrix} -\hat{C}_k & \hat{A}_k^\top \\ \hat{A}_k & \hat{B}_k \end{pmatrix}$$

which are real and symmetric by (A). We need the following result [9, Theorem 1].

LEMMA 3. *Assume (A) for (H) and $(\hat{\text{H}})$, and let (X, U) and $(\hat{X}, \hat{U})$ be conjoined bases of (H) and $(\hat{\text{H}})$, respectively. Suppose that $Q_k, \hat{Q}_k$ are symmetric with*

$$X_k^\top Q_k X_k = U_k^\top X_k, \quad \hat{X}_k^\top \hat{Q}_k \hat{X}_k = \hat{U}_k^\top \hat{X}_k, \qquad \text{for } k \in J^* = [0, N+1] \cap \mathbb{Z},$$

and assume for $k \in J^*$

$$\mathcal{H}_k \geq \hat{\mathcal{H}}_k, \qquad \hat{B}_k \geq \hat{B}_k B_k^\dagger \hat{B}_k, \qquad \operatorname{Ker} B_k \subset \operatorname{Ker} \hat{B}_k. \tag{7}$$

If $\operatorname{Im} \hat{X}_0 \subset \operatorname{Im} \hat{X}_0$, $\hat{X}_0^\top(\hat{Q}_0 - Q_0)\hat{X}_0 \geq 0$, and if (X,U) has no focal point in $(0, N+1]$, then $(\hat{X}, \hat{U})$ has no focal point in $(0, N+1]$ either, and $\hat{X}_k^\top(\hat{Q}_k - Q_k)\hat{X}_k \geq 0$ for all $k \in J^*$.

Now our applications of Theorem 1 are as follows.

COROLLARY 1. *Assume (A) for (H) and $(\hat{\text{H}})$, suppose that (H) and $(\hat{\text{H}})$ are eventually controllable and that (H) is eventually disconjugate, and assume that (7) holds for all $k \geq M$. Let (X, U) and $(\hat{X}, \hat{U})$ be recessive solutions of (H) and $(\hat{\text{H}})$, respectively, and suppose that $\hat{X}_k$ is invertible, $\hat{D}_k \geq 0$ for all $k \geq N+1$ with $N \geq M$, and either $\hat{X}_N$ is singular or $\hat{D}_N \not\geq 0$. Then either X_k is singular or $D_k \not\geq 0$ for some $k \geq N$.*

PROOF. Observe first that (7) and eventual disconjugacy of (H) imply eventual disconjugacy of $(\hat{\text{H}})$ by Lemma 3, so that a recessive solution of $(\hat{\text{H}})$ exists. Next, suppose, by contradiction, that X_k is invertible and $D_k \geq 0$ for all $k \geq N$ and let $(\tilde{X}, \tilde{U})$ be the conjoined basis of $(\hat{\text{H}})$ with $\tilde{X}_N = I$, $\tilde{U}_N = Q_N := U_N X_N^{-1}$. Then, by Lemma 3, $(\tilde{X}, \tilde{U})$ has no focal point in (N, ∞). Hence, $\tilde{X}_k$ is invertible and $\tilde{D}_k \not\geq 0$ for all $k \geq N$. But, by Theorem 1, $\tilde{X}_k$ is singular or $\tilde{D}_k \not\geq 0$ for some $k \geq N$, which is a contradiction. ∎

Combining the previous corollary with Theorem 1 we obtain another comparison result.

COROLLARY 2. *Assume (A) for (H) and $(\hat{\text{H}})$, suppose that (H) and $(\hat{\text{H}})$ are eventually controllable and that (H) is eventually disconjugate, and assume that (7) holds for all $k \geq M$. Moreover, let $(\hat{X}, \hat{U})$ be a recessive solution of $(\hat{\text{H}})$ such that $\hat{X}_k$ is invertible, $\hat{D}_k \geq 0$ for all $k \geq N+1$ with $N \geq M$, and either $\hat{X}_N$ is singular or $\hat{D}_N \not\geq 0$. Then, for every conjoined basis (X, U) of (H), either X_k is singular or $D_k \not\geq 0$ for some $k \geq N$.*

REFERENCES

1. C.D. Ahlbrandt and A.C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Academic Publishers, Boston, (1996).
2. M. Bohner, Linear Hamiltonian difference systems: Disconjugacy and Jacobi-type conditions, *J. Math. Anal. Appl.* **199**, 804–826, (1996).
3. M. Bohner, Riccati matrix difference equations and linear Hamiltonian difference systems, *Dynamics of Continuous, Discrete, and Impulsive Systems* **2** (2), 147–159, (1996).
4. L. Erbe and P. Yan, Disconjugacy for linear Hamiltonian difference systems, *J. Math Anal. Appl.* **167**, 335–367, (1992).
5. L. Erbe and P. Yan, Qualitative properties of Hamiltonian difference systems, *J. Math. Anal. Appl.* **171**, 334–345, (1992).
6. L. Erbe and P. Yan, On the discrete Riccati equation and its applications to discrete Hamiltonian systems, *Rocky Mountain J. Math.* **25** (1), 167–178, (1995).
7. M. Bohner and O. Došlý, Disconjugacy and transformations for symplectic systems, *Rocky Mountain J. Math.*, (1995).
8. W. Kratz, A substitute of l'Hospital's rule for matrices, *Proc. Amer. Math. Soc.* **99**, 395–402, (1987).
9. M. Bohner, O. Došlý and W. Kratz, Inequalities for solutions of Riccati matrix difference equations, *J. Math. Anal. Appl.* **221**, 262–286, (1998).
10. W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademie Verlag, Berlin, (1995).