A Sturmian Theorem for Recessive Solutions of Linear Hamiltonian Difference Systems

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(Received and accepted November 1997)
Communicated by R. P. Agarwal

Abstract—The main result of the paper is a Sturmian-type separation theorem for the recessive solutions of linear Hamiltonian difference systems. The assumptions on the Hamiltonian systems allow to include the study of Sturm-Liouville difference equations of higher order. As an application of the main result, some Sturmian-type comparison results are obtained. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Discrete Hamiltonian system, Conjoined basis, Focal point, Recessive solution, Sturmian theorem.

1. INTRODUCTION

We consider the Linear Hamiltonian Difference System (LHDS)

\[ \Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k \]  

(H)

with \( \Delta x_k := x_{k+1} - x_k \), where \( A_k, B_k, C_k \) are real \( n \times n \)-matrices, \( B_k \) and \( C_k \) are symmetric, and where \( I - A_k \) is invertible for \( k \in \mathbb{Z} \). In the last years, considerable efforts have been made to find analogies between oscillation properties of solutions of the difference system (H) and of its continuous counterpart

\[ \dot{x} = A(t)x + B(t)u, \quad \dot{u} = C(t)x - A^T(t)u, \]  

(C)

The author is greatful to the Alexander von Humbold Foundation for supporting this work.
see [1] and the references in it. An important step was made in the papers [2,3], where the so-called Reid Roundabout Theorem and the Sturmian-Type Theory are derived for the LHDS without assuming that the matrices $B_k$ are invertible. Similar statements for (H) with positive definite (in particular invertible) $B_k$'s were proven in [4–6]. But note that systems with invertible $B_k$'s do not cover important systems and equations like the higher-order Sturm-Liouville equations.

In this paper, we derive a certain extremal property of the so-called recessive solutions of (H), which is known in the continuous case as the zero-maximality of the principle solutions of (C) (at $\infty$). This result states, under suitable conditions, the following. If a principle solution of (C) has a focal point in some interval $[T, \infty)$, then every conjoined basis of (C) must also have a focal point in that interval. In this paper, we show that the recessive solutions of (H) have a similar extremal property. In our discrete case, the situation is more complicated because of the fact that in addition to singularities one has to also consider “sign changes”. Therefore, we have to use methods which are different from the continuous ones.

2. NOTATION AND AUXILIARY RESULTS

First, we recall basic concepts and notation that is needed in the formulation of our main result. For a symmetric (and real) matrix $D$ the inequality $D \geq 0$ ($D > 0$) means that $D$ is nonnegative (positive) definite. By $V^\dagger$ we denote the Moore-Penrose generalized inverse of a matrix $V$. $\text{Ker}$ and $\text{Im}$ denote the kernel and the image of the matrix indicated, respectively. The symbol $\Delta$ stands for the usual forward difference operator, i.e., $\Delta w_k = w_{k+1} - w_k$.

We consider together with (H) its matrix version (again denoted by (H)), namely

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^\dagger U_k,$$

where a solution $(X, U)$ consists of real $n \times n$-matrices $X_k, U_k$, and where we assume throughout that $A_k, B_k, C_k$ are real $n \times n$-matrices satisfying

$$B_k, C_k \text{ are symmetric}, \quad I - A_k \text{ is invertible for all } k,$$

we put $\tilde{A}_k = (I - A_k)^{-1}$. Similarly, as in the continuous case, a solution $(X, U)$ of (H) is said to be a conjoined basis of (H) if, for $k \in \mathbb{Z}$, $X_k^\dagger U_k \equiv U_k^\dagger X_k$ and rank $(X_k^\dagger, U_k^\dagger) = n$. For a conjoined basis $(X, U)$ of (H) we say that an interval $(k, k + 1]$ contains a focal point of $X$ or of $(X, U)$ [2, Definition 3] or [7, Definition 3] if

$$\text{Ker} X_{k+1} \not\subset \text{Ker} X_k \quad \text{or} \quad D_k := X_k X_{k+1}^\dagger \tilde{A}_k B_k \not\geq 0.$$

Note that if $\text{Ker} X_{k+1} \subset \text{Ker} X_k$ (which trivially happens if $X_{k+1}$ is invertible), then the matrix $D_k$ is symmetric (see, e.g., [2, Theorem 1]). The second condition in (1) may be interpreted as a “sign change” which is mentioned in the introduction. The LHDS (H) is said to be disconjugate on an (integer) interval $[k, l] \cap \mathbb{Z}$ if the solution $(X, U)$ of (H) with the initial conditions $X_k = 0, U_k = I$ has no focal point in $(k, l + 1]$ (see [2, Definition 4 and Theorem 2]). We say that (H) is eventually disconjugate if there exists $M \in \mathbb{N}$ such that (H) is disconjugate on $[M, \infty)$ (i.e., on $[M, l] \cap \mathbb{Z}$ for all $l > M$), and (H) is said to be eventually controllable (or controllable for large $k$ (see [7, Definition 5]) if there exist $M, \kappa \in \mathbb{N}$ such that the only vector solution of (H) with $x_m = \cdots = x_{m+k} = 0$ for some $m \geq M$ is the trivial solution $x = u = 0$ (see [7, Section 4]).

Next, if $(X, U)$ is any conjoined basis of (H) such that the matrices $X_k$ are invertible on some interval $[N, \infty) \cap \mathbb{Z}$, then every solution $(\tilde{X}, \tilde{U})$ of (H) can be expressed in the form

$$\tilde{X}_k = X_k (S_k K + L), \quad \tilde{U}_k = U_k (S_k K + L) + X_k^\dagger K,$$

where $S_k = \sum_{j=N}^{k-1} X_j^{-1} \tilde{A}_j B_j X_j^\dagger = \sum_{j=N}^{k-1} X_j^{-1} D_j X_j^\dagger$.
for $k \geq N$, with $D_j = X_j X_{j+1}^{-1} \tilde{A}_j B_j$ as in (1), and where $K$ and $L$ are constant matrices, more precisely

$$L = X_{-N}^{-1} \tilde{X}_M, \quad K = X_j^T \tilde{U}_j - U_j^T \tilde{X}_j.$$  

One may verify directly that $(\tilde{X}, \tilde{U})$ given by (2) solves (H) (see [5, Proposition 2.2]), and it is a conjoined basis if and only if $K^T L = L^T K$ and $\text{rank}(K^T, L^T) = n$.

Now we summarize properties of recessive solutions of (H). A conjoined basis $(\tilde{X}, \tilde{U})$ of (H) is said to be a recessive solution (at $\infty$) if there exists $M \in \mathbb{Z}$ and another conjoined basis $(X, U)$ such that $\tilde{X}_k$ and $X_k$ are invertible for all $k \geq M$, $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k$ is invertible, and such that $\lim_{k \to \infty} X_k^{-1} \tilde{X}_k = 0$ (see [7, Section 4] or [1, Chapter 5]). If the LHS of (H) is eventually disconjugate and eventually controllable, then, as is shown in [7, Section 4], a conjoined basis $(\tilde{X}, \tilde{U})$ is a recessive solution if and only if there exists $N \in \mathbb{Z}$ such that $\tilde{X}_k$ is invertible, $\tilde{D}_k := \tilde{X}_k \tilde{X}_{k+1}^{-1} \tilde{A}_k B_k \geq 0$ for all $k \geq N$, $\tilde{S}_k := \sum_{j=N}^{k-1} \tilde{X}_j^{-1} \tilde{D}_j^{-1} \tilde{X}_j^T$ is invertible for large $k$, and such that $\lim_{k \to \infty} \tilde{S}_k^{-1} = 0$ (or equivalently, $\lim_{k \to \infty} \tilde{S}_k = \infty$, i.e., all eigenvalues tend to infinity as $k \to \infty$). Moreover, a recessive solution exists under these assumptions, and it is unique up to an invertible right factor according to the following lemma. For a special case of this result see [1, Chapter 5].

**Lemma 1.** Assume (A), suppose that (H) is eventually controllable and eventually disconjugate, and let $(X, U)$ and $(\tilde{X}, \tilde{U})$ be recessive solutions of (H). Then there exists an invertible matrix $C$ such that

$$\tilde{X}_k = X_k C \quad \text{and} \quad \tilde{U}_k = U_k C, \quad \text{for all } k \in \mathbb{Z}.$$  

**Proof.** By our assumptions and the discussion above, there exists $N \in \mathbb{Z}$ such that $X_k$ and $\tilde{X}_k$ are invertible, $D_k \geq 0$, $\tilde{D}_k \geq 0$, for all $k \geq N$, and such that

$$\lim_{k \to \infty} S_k^{-1} = \lim_{k \to \infty} \tilde{S}_k^{-1} = 0,$$

where $D_k, \tilde{D}_k, S_k, \tilde{S}_k$ are defined accordingly. By the representation formulae (2) we have that

$$\tilde{X}_k = X_k (S_k K + L), \quad \tilde{U}_k = U_k (S_k K + L) + X_k^{-1} K,$$

$$X_k = \tilde{X}_k \left( \tilde{S}_k \tilde{K} + \tilde{L} \right), \quad U_k = \tilde{U}_k \left( \tilde{S}_k \tilde{K} + \tilde{L} \right) + \tilde{X}_k^{-1} \tilde{K},$$

for $k \geq N$, where

$$L = X_N^{-1} \tilde{X}_N = \tilde{L}^{-1}, \quad K = -\tilde{K}^T, \quad L^T K = K^T L.$$  

Hence, $R = KL^{-1}$ is symmetric. Since $S_k^{-1} \to 0+$ as $k \to \infty$, it follows from [8, Theorem 1] that

$$\lim_{k \to \infty} R (R + S_k^{-1})^{-1} S_k^{-1} = 0.$$  

Moreover, the representation formulae for $X_k$ and $\tilde{X}_k$ imply that

$$R (R + S_k^{-1})^{-1} S_k^{-1} = K (S_k K + L)^{-1} = K \tilde{X}_k^{-1} X_k = K \left( \tilde{S}_k \tilde{K} + \tilde{L} \right) = \tilde{K}^T \tilde{L} - \tilde{K}^T \tilde{S}_k \tilde{K} \to 0.$$  

Since $\lim_{k \to \infty} \tilde{S}_k = \infty$, we can conclude that $\tilde{K} = 0 = K$. Hence, $\tilde{X}_k = X_k C, \tilde{U}_k = U_k C$ with the invertible matrix $C = L$. \[\blacksquare\]

Linear Hamiltonian systems are closely related to Riccati matrix equations, namely: if $(X, U)$ is a solution of (H) such that $X_k$ is invertible for $k \in [M, N+1] \cap \mathbb{Z}$, then the quotient $Q_k := U_k X_k^{-1}$ solves the Riccati matrix difference equation

$$Q_{k+1} = C_k + (I - A_k) Q_k (I + B_k Q_k)^{-1} (I - A_k),$$

(R)
for $M \leq k \leq N$. Moreover, $Q_k$ is symmetric, and $D_k = X_kX_{k+1}^{-1}\tilde{A}_kB_k = (I + B_kQ_k)^{-1}B_k$. If (H) is eventually controllable and eventually disconjugate and if $(\bar{X}, \bar{U})$ is a recessive solution, then the quotient $Q_k^- := \bar{Q}_k = \bar{U}_k\bar{X}_k^{-1}$ is unique (the invertible right factor cancels out!), and it is called the eventually minimal (or distinguished) solution of (R) at $\infty$. The terminology eventually minimal is justified by the fact that any other symmetric solution $Q$ of (R), which exists for $k \geq l$ with some $l$, satisfies $Q_k \geq Q_k^-$ for all $k \geq m$ with $m \geq l$ sufficiently large (see [7, Section 4]). Finally, we cite the following lemma [9, Lemma 7] which we will use.

**Lemma 2.** Assume (A) and suppose that $Q_k, \bar{Q}_k, Q_{k+1}, \bar{Q}_{k+1}$ are symmetric and satisfy (R) for some fixed $k \in \mathbb{Z}$ with $(I + B_kQ_k)^{-1}B_k \geq 0$. Then $Q_{k+1} \geq \bar{Q}_{k+1}$ implies $Q_k \geq \bar{Q}_k$.

### 3. MAIN RESULT

Now we can formulate the main result of this paper, a Sturmian-type separation theorem for recessive solutions.

**Theorem 1.** Assume (A), and suppose that (H) is eventually controllable and eventually disconjugate. Let $(\bar{X}, \bar{U})$ be a recessive solution of (H) such that $\bar{X}_k$ is invertible and $\bar{D}_K := \bar{X}_k\bar{X}_{k+1}^{-1}\bar{A}_kB_k \geq 0$ for $k \geq N + 1$. If either $\bar{X}_N$ is singular or $\bar{D}_N \geq 0$, then, for every conjoined basis $(X, U)$ of (H), either $X_k$ is singular or $D_k = X_kX_{k+1}^{-1}A_kB_k \geq 0$ for some $k \geq N$.

**Proof.** Suppose, by contradiction, that $X_k$ is invertible and that $D_k \geq 0$ for all $k \geq N$. Since $\bar{Q}_k = \bar{U}_k\bar{X}_k^{-1}$ is the eventually minimal solution of (R), we get that $Q_k \leq Q_k$ for all sufficiently large $k$. Hence, by Lemma 2, $\bar{Q}_k \leq Q_k$ for all $k \geq N + 1$. Therefore,

$$\bar{D}_N - D_N = B_N \bar{A}_N^T (Q_{N+1} - \bar{Q}_{N+1}) \bar{A}_N B_N \geq 0,$$

and this implies $\bar{D}_N \geq 0$ because $D_N \geq 0$.

We put

$$\bar{X}_k = X_k(I + S_k), \quad \bar{U}_k = U_k(I + S_k) + X_k^{-1}$$

with $S_k = \sum_{j=k}^{k+1} X_j \bar{A}_j B_j X_j^{-1}$ for $k \geq N$. Then, by (2) with $K = L = I$, $(\bar{X}, \bar{U})$ is a conjoined basis of (H). Since $S_{k+1} - S_k = X_k^{-1}D_kX_k^{-1} \geq 0$ for $k \geq N$, $S_N = 0$, we have that $\bar{S}_k \geq 0$, and therefore, $\bar{X}_k$ is invertible for $k \geq N$. Hence, again by (2) (use that $\bar{X}_j^T U_j - U_j^T \bar{X}_j \equiv -I$), we have that

$$X_k = \bar{X}_k(I - \bar{S}_k), \quad U_k = \bar{U}_k(I - \bar{S}_k) - \bar{X}_k^{-1}$$

for $k \geq N$, where $\bar{S}_k = \sum_{j=N}^{k-1} \bar{X}_j \bar{A}_j B_j \bar{X}_j^{-1}$. It follows from (3) and (4) that

$$(I - \bar{S}_k)(I + S_k) = I,$$

so that

$$I - \bar{S}_k = (I + S_k)^{-1} \geq (I + S_{k+1})^{-1} = I - \bar{S}_{k+1}, \quad \text{for } k \geq N.$$

Thus, $0 \leq \bar{S}_k \leq \bar{S}_{k+1} = I - (I + S_{k+1})^{-1} \leq I, \bar{D}_k \geq 0$ for $k \geq N$, and therefore, the limit $\bar{S}_\infty := \lim_{k \to \infty} \bar{S}_k$ exists with $0 \leq \bar{S}_\infty \leq I$. Moreover, $\bar{S}_\infty > 0$, because otherwise $\bar{S}_k c = 0$ for all $k \geq N$ with some $c \in \mathbb{R}^n \setminus \{0\}$. Then, $(x, u)$ with $x_k = \bar{X}_k \bar{S}_k c$ and $u_k = (\bar{U}_k \bar{S}_k + \bar{X}_k^{-1}) c$ is a nontrivial solution of (H) (again by (2)) with $x_k = 0$ for all $k \geq N$, contradicting our assumption that (H) is eventually controllable. Now, using (2) again, we consider the conjoined basis $(\tilde{X}, \tilde{U})$ defined by

$$\tilde{X}_k = \bar{X}_k(I - \bar{S}_k), \quad \tilde{U}_k = \bar{U}_k(I - \bar{S}_k) - \bar{X}_k^{-1}$$

for $k \geq N$. Because (H) is eventually controllable, it follows with the same argument as above that there exists $M \geq N$ such that $\tilde{X}_k$ is invertible for all $k \geq M$. Therefore, for $k \geq M$,

$$\tilde{X}_k = \tilde{X}_k(I - \tilde{S}_k + L), \quad \tilde{U}_k = \tilde{U}_k(I - \tilde{S}_k + L) + \tilde{X}_k^{-1},$$

so that

$$I - \tilde{S}_k = (I + S_k)^{-1} \geq (I + S_{k+1})^{-1} = I - \tilde{S}_{k+1}, \quad \text{for } k \geq N.$$
where \( L = \dot{X}_M^{-1} \dot{Y}_M \) and \( \dot{S}_k = \sum_{j=1}^{k-1} \dot{X}_{j+1}^{-1} \dot{A}_j B_j \dot{X}_j^{-1} \). By (5) and (6),

\[
\dot{S}_k + L = (S_{\infty} - \dot{S}_k)^{-1} \rightarrow \infty, \quad \text{as } k \rightarrow \infty,
\]

and \( \dot{S}_{k+1} - \dot{S}_k \geq 0 \). Hence, for \( k \geq M, \dot{X}_k \) is invertible, \( \dot{D}_k \geq 0 \), and we have that \( \dot{S}_k^{-1} \rightarrow 0 \) so that \( (\dot{X}, \dot{U}) \) is a recessive solution of (H). Thus, by Lemma 1, \( \dot{X}_k = \dot{X}_k C \) and \( \dot{U}_k = \dot{U}_k C \) for an invertible matrix \( C \). Now, for \( k = N, (5) \) yields that \( \dot{X}_N = \dot{X}_N S_{\infty} C \) is invertible (use that \( \dot{S}_{\infty} > 0 \)). Altogether we have shown that \( \dot{X}_N \) is invertible and that \( \dot{D}_N \geq 0 \) which contradicts our assumption so that the proof is complete. Note that \( (\dot{X}, \dot{U}) \) is a so-called dominant solution of (H) (see [1;7, Section 4]).

4. APPLICATIONS

By using inequalities for Riccati difference equations given in [9] we obtain from our Theorem 1 the following “comparison results”. We consider another LHDS (and denote the corresponding Riccati equation by \( \dot{R} \))

\[
\Delta X_k = \dot{A}_k X_{k+1} + \dot{B}_k U_k, \quad \Delta U_k = \dot{C}_k X_{k+1} - \dot{A}_k^T U_k,
\]

where \( \dot{A}_k, \dot{B}_k, \dot{C}_k \) satisfy the same assumptions (A) as \( A_k, B_k, C_k \) in (H). Moreover, we denote

\[
\mathcal{H}_k := \begin{pmatrix} -C_k & A_k^T \\ \dot{A}_k & \dot{B}_k \end{pmatrix} \quad \text{and} \quad \dot{\mathcal{H}}_k := \begin{pmatrix} -\dot{C}_k & \dot{A}_k^T \\ \dot{A}_k & \dot{B}_k \end{pmatrix}
\]

which are real and symmetric by (A). We need the following result [9, Theorem 1].

**Lemma 3.** Assume (A) for (H) and \( \dot{R} \), and let \((X, U)\) and \((\dot{X}, \dot{U})\) be conjoined bases of (H) and \( \dot{R} \), respectively. Suppose that \( Q_k, \dot{Q}_k \) are symmetric with

\[
X_k^T Q_k X_k = U_k^T X_k, \quad \dot{X}_k^T \dot{Q}_k \dot{X}_k = \dot{U}_k^T \dot{X}_k, \quad \text{for } k \in J^* = [0, N+1] \cap \mathbb{Z},
\]

and assume for \( k \in J^* \)

\[
\mathcal{H}_k \geq \dot{\mathcal{H}}_k, \quad \dot{B}_k \geq \dot{B}_k B_k^T \dot{B}_k, \quad \text{Ker} B_k \subset \text{Ker} \dot{B}_k.
\]

If \( \text{Im} \dot{X}_0 \subset \text{Im} \dot{X}_0, \dot{X}_0^T (\dot{Q}_0 - Q_0) \dot{X}_0 \geq 0 \), and if \((X, U)\) has no focal point in \([0, N+1]\), then \((\dot{X}, \dot{U})\) has no focal point in \([0, N+1]\) either, and \( \dot{X}_k^T (\dot{Q}_k - Q_k) \dot{X}_k \geq 0 \) for all \( k \in J^* \).

Now our applications of Theorem 1 are as follows.

**Corollary 1.** Assume (A) for (H) and \( \dot{R} \), suppose that (H) and \( \dot{R} \) are eventually controllable and that (H) is eventually disconjugate, and assume that (7) holds for all \( k \geq M \). Let \((X, U)\) and \((\dot{X}, \dot{U})\) be recessive solutions of (H) and \( \dot{R} \), respectively, and suppose that \( \dot{X}_k \) is invertible, \( \dot{D}_k \geq 0 \) for all \( k \geq N + 1 \) with \( N \geq M \), and either \( \dot{X}_N \) is singular or \( \dot{D}_N \neq 0 \). Then either \( X_k \) is singular or \( D_k \neq 0 \) for some \( k \geq N \).

**Proof.** Observe first that (7) and eventual disconjugacy of (H) imply eventual disconjugacy of \( \dot{R} \) by Lemma 3, so that a recessive solution of \( \dot{R} \) exists. Next, suppose, by contradiction, that \( X_k \) is invertible and \( D_k \geq 0 \) for all \( k \geq N \) and let \((\dot{X}, \dot{U})\) be the conjoined basis of \( \dot{R} \) with \( \dot{X}_N = I, \dot{U}_N = Q_N := U_N X_N^{-1} \). Then, by Lemma 3, \((\dot{X}, \dot{U})\) has no focal point in \((N, \infty)\). Hence, \( \dot{X}_k \) is invertible and \( \dot{D}_k \neq 0 \) for all \( k \geq N \). But, by Theorem 1, \( \dot{X}_k \) is singular or \( \dot{D}_k \neq 0 \) for some \( k \geq N \), which is a contradiction.

Combining the previous corollary with Theorem 1 we obtain another comparison result.

**Corollary 2.** Assume (A) for (H) and \( \dot{R} \), suppose that (H) and \( \dot{R} \) are eventually controllable and that (H) is eventually disconjugate, and assume that (7) holds for all \( k \geq M \). Moreover, let \((\dot{X}, \dot{U})\) be a recessive solution of \( \dot{R} \) such that \( \dot{X}_k \) is invertible, \( \dot{D}_k \geq 0 \) for all \( k \geq N + 1 \) with \( N \geq M \), and either \( \dot{X}_N \) is singular or \( \dot{D}_N \neq 0 \). Then, for every conjoined basis \((X, U)\) of (H), either \( X_k \) is singular or \( D_k \neq 0 \) for some \( k \geq N \).
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